K-THEORY: FROM MINIMAL GEODESICS TO SUSY FIELD THEORIES

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In memory of Raoul Bott, friend and mentor.

Abstract. There are many models for the K-theory spectrum known today, each one having its own history and applications. The purpose of this note is to give an elementary description of eight such models (and their completions) and relate all of them by canonical maps, most of which are homeomorphisms (rather then just homotopy equivalences). The first model are Raoul Bott's iterated spaces of minimal geodesic in orthogonal groups. This model actually predates the invention of K-theory but completely calculates the coefficients of this generalized cohomology theory. Our last model are certain spaces of super symmetric (1|1)-dimensional field theories introduced by the second two authors for the purpose of generalizing them to cocycles for elliptic cohomology, in terms of certain super symmetric (2|1)-dimensional field theories.

Introduction

At the first Arbeitstagung 1957 in Bonn, Alexander Grothendieck presented his version of the Riemann-Roch theorem in terms of a group (now known as *Grothendieck group*) constructed from (isomorphism classes of) algebraic vector bundles over algebraic manifolds. He used the letter K to abbreviate 'Klassen', the German word for (isomorphism) classes. Michael Atiyah and Friedrich Hirzebruch instantly realized that the same construction can be applied to all complex vector bundles over a topological space X, obtaining a commutative ring $K(X)$, where addition and multiplication come from direct sum respectively tensor product of vector bundles. For example, every complex vector bundle over the circle is trivial and hence $K(S^1) = \mathbb{Z}$. Moreover, it is also easy to see that

$$K(S^2) = \mathbb{Z}[L]/(L-1)^2,$$

is generated by Hopf's line bundle L over the 2-sphere.

At the second Arbeitstagung in 1958, Raoul Bott explained his celebrated periodicity theorem which can be expressed as the computations $K(S^{2n-1}) \cong K(S^1)$ and $K(S^{2n}) \cong K(S^2)$ for all $n \in \mathbb{N}$. Bott also proved a real periodicity theorem, where one studies the Grothendieck group of (isomorphisms classes of) real vector bundles over X. After dividing by the subgroup generated by trivial bundles (denoted by a tilde over the K-groups) one obtains

$$\widetilde{KO}(S^n) \cong \begin{cases} \mathbb{Z} & \text{for} \quad n \equiv 0 \mod 4 \\ \mathbb{Z}/2 & \text{for} \quad n \equiv 1,2 \mod 8 \\ 0 & \text{else} \end{cases}$$

It was again Atiyah and Hirzebruch who realized that Bott's periodicity theorem could be used to define *generalized cohomology theories* $K^n(X)$ and $KO^n(X)$ that are 2- respectively 8-periodic and with $K^0 = K, KO^0 = KO$. These satisfy the same Eilenberg-Steenrod axioms (functoriality, homotopy invariance and Mayer-Vietoris principle) as the ordinary cohomology groups $H^n(X)$ but if one takes X to be a point then one obtains non-trivial groups for $n \neq 0$. In fact, the above computation yields by the suspension isomorphism

$$K^{-n}(pt) \cong \widetilde{K}^{-n}(S^0) \cong \widetilde{K}^0(S^n) \cong \begin{cases} \mathbb{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

and similarly for real K-theory. Several classical problems in topology were solved using this new cohomology theory, for example the maximal number of independent vector fields on the n-sphere was determined explicitly. A modern way to express any generalized cohomology theory is to write down a spectrum, i.e. a sequence of spaces with certain structure maps. For ordinary cohomology, these would be the Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$ and for K-theory one can use Bott's spaces of iterated minimal geodesics in orthogonal groups, see below.

The purpose of this note is to give an elementary description of eight models (and their completions) of the spaces in the K-theory spectrum and relate all of them by canonical maps, most of which are homeomorphisms (rather then just homotopy equivalences). We will exclusively work with real K-theory KO but all statements and proofs carry over to the complex case, and, with a little more care, also to Atiyah's Real K-theory.

Before we start a more precise discussion, we'll give a list of the models that will be described in this paper. Recall that a CW-complex E_n is said to *represent* the functor KO^n if there are natural isomorphisms of abelian groups

$$[X, E_n] \cong \widetilde{KO}^n(X)$$

for CW-complexes X. By Brown's representation theorem, such E_n exist and are unique up to homotopy equivalence. The suspension isomorphism $\widetilde{KO}^n(E_n) \cong \widetilde{KO}^{n+1}(\Sigma E_n)$ in real K-theory then takes the identity map on E_n to a map $\Sigma E_n \to E_{n+1}$ whose adjoint must be a homotopy equivalence

$$\epsilon_n : E_n \xrightarrow{\simeq} \Omega E_{n+1}.$$

The sequence $\{E_n, \epsilon_n\}_{n\in\mathbb{Z}}$ of spaces and structure maps is an Ω-spectrum representing the cohomology theory KO. The *homotopy groups* of the spectrum, $\pi_n KO := \pi_n E_0$, are then given by the connected components $\pi_0 E_{-n} \cong \pi_0 \Omega^n E_0 \cong \pi_n E_0$ which explains partially why we describe these negatively indexed spaces below.

To fix notation, let C_n be the Clifford algebra associated to the *positive* definite inner product on $\mathbb{R}^n$. It has generators $e_1, ..., e_n$ satisfying the relations

$$e_i^2 = -\mathbb{1}, \quad e_i e_j + e_j e_i = 0, \quad i \neq j$$

and it turns into a C^*-algebra via $e_i^* = -e_i$. For $n \geq 0$, we define C_{-n} to be the Clifford C^*-algebra for the negative definite inner product, so the operators e_i are self-adjoint and $e_i^2 = \mathbb{1}$. We also fix a separable Hilbert space H_n with a $*$-representation of C_{n-1} such that all (i.e. one or two) irreducible C_{n-1}-modules appear infinitely often. We get a $\mathbb{Z}/2$-graded C_n-module $\mathcal{H}_n := H_n \otimes_{C_{n-1}} C_n$ and denote the grading involution by α.

Theorem 1. *The following spaces represent the $(-n)$-th space in the real K-theory spectrum. Moreover, for each value of n where they are defined, the spaces in the first six items are homeomorphic, and the three spaces appearing in (7) are also homeomorphic to each other.*

1. *Bott spaces $\mathcal{B}_n$, where $\mathcal{B}_1$ is the infinite orthogonal group and for $n > 1$, the space $\mathcal{B}_n$ is the space of minimal geodesics in $\mathcal{B}_{n-1}$.*
2. *Milnor spaces $\mathcal{M}_n$ of "C_{n-1}-module structures" on H_n. More precisely, these are unitary structures J on H_n, such that $J - e_{n-1}$ has finite rank and $Je_i = -e_i J$ for $1 \leq i \leq n-2$.*
3. *Spaces Inf_n^{fin} of "infinitesimal generators", i.e. odd, self-adjoint unbounded C_n-linear operators on $\mathcal{H}_n$ with finite rank resolvent.*
4. *Configuration spaces Conf_n^{fin} of finite dimensional (ungraded) mutually perpendicular C_n-submodules V_λ of $\mathcal{H}_n$, labelled by finitely many $\lambda \in \mathbb{R}$, and with $V_{-\lambda} = \alpha(V_\lambda)$.*
5. *Classifying spaces $\mathcal{Q}_n$ of (internal space) categories that arise from certain C_n-module categories by Quillen's $S^{-1}S$-construction.*
6. *Spaces $\mathcal{SGO}_n^{fin}$ of super semigroups of self-adjoint C_n-linear (finite rank) operators on $\mathcal{H}_n$.*
7. *The corresponding completions $\mathrm{Conf}_n, \mathrm{Inf}_n$ and $\mathcal{SGO}_n$, where the finite rank condition is replaced by compactness.*
8. *Spaces $\mathcal{EFT}_n$ of super symmetric Euclidean field theories of dimension $(1|1)$ and degree n.*
9. *Atiyah-Singer spaces $\mathcal{F}_n$ of certain skew-adjoint Fredholm operators on H_n, anti-commuting with the C_{n-1}-action.*

The above theorem only gives very rough descriptions of the spaces involved, detailed definitions for item (k) can be found in Section k below. Section k also contains the proof that the spaces in (k) are homeomorphic (respectively homotopy equivalent) to spaces appearing previously.

To our best knowledge, the homeomorphisms between the spaces in (2), (4), (5) and (6) are new, even though it was well known that the spaces are homotopy equivalent for abstract reasons (since they represent the same Ω-spectrum). Moreover, our maps relating the spaces in (3) and (9) seem to be new and slightly easier than the original ones.

Remark 2. The spaces in (3) to (8) are defined for all $n \in \mathbb{Z}$ and the theorem holds for all these n. The Bott and Milnor spaces only make sense for $n \geq 1$ and the same seems to be true for the spaces in (9). This comes from the fact that the Atiyah-Singer spaces $\mathcal{F}_n$ are defined in terms of the ungraded Hilbert space H_n and for $n \leq 0$ our translation to $\mathcal{H}_n$ doesn't work well. However, this can be circumvented by never mentioning the Clifford algebra C_{n-1} in the definitions and working with the ungraded algebra C_n^{ev} instead. Then the spaces $\mathcal{F}_n$ are defined for all $n \in \mathbb{Z}$ and our theorem holds. We chose the formulation above to better connect with reference [AS].

We now give a rough outline of where the spaces come from and how they are related. The spaces $\mathcal{B}_n$ were defined by Raoul Bott in his classic paper [B] on "the stable homotopy of the classical groups" which was the starting point of all of K-theory. The Milnor spaces are defined in Milnor's book [Mi] on Morse theory, using the notation $\Omega_{n-1} = \mathcal{M}_n$. It seems very likely that Atiyah, Bott and Shapiro had studied these spaces before Milnor but unfortunately, we don't know the precise history. We will recall in Section 2 how one can easily compute (iterated) spaces of minimal geodesics in the orthogonal group in terms of Clifford module structures on H_n.

The configuration spaces $\mathrm{Conf}_n^{\mathrm{fin}}$ are the easiest to work with because one can geometrically picture its elements well. It came as a surprise to us that these simple spaces are actually the geometric realizations $\mathcal{Q}_n$ of certain (internal space) Quillen categories. Given a configuration $\{V_\lambda\}$ in $\mathrm{Conf}_n^{\mathrm{fin}}$, we can interpret it as the eigenspaces and eigenvalues of an odd, self-adjoint, C_n-linear operator $\mathcal{D} \in \mathrm{Inf}_n^{\mathrm{fin}}$ that is given by

$$\mathcal{D}(v) := \lambda \cdot v \quad \forall\, v \in V_\lambda.$$

Since the original configuration is finite, it is clear that $\mathcal{D}$ has a finite rank resolvent and also a finite dimensional domain. We are therefore lead to expand the usual definition of a self-adjoint (unbounded) operator as follows: The domain is not necessarily dense but the operator is self-adjoint on the closure of its domain. As a consequence, it is very natural to study completions Conf_n where there is a discrete set of labels and hence the corresponding operator $\mathcal{D} \in \mathrm{Inf}_n$ may have dense domain (and has compact resolvent). The resulting spaces Inf_n are equipped with the generalized norm topology and the fact that one can retract the completed spaces back to their finite rank subspaces goes back to (at least) Segal [Se] but we reprove this fact here.

The operator $\mathcal{D}$ can be used as the infinitesimal generator of the super semigroup

$$(t, \theta) \mapsto e^{-t\mathcal{D}^2 + \theta\mathcal{D}}$$

of (finite rank respectively compact) operators on $\mathcal{H}_n$. These are the elements of $\mathcal{SGO}_n^{\mathrm{fin}}$ respectively $\mathcal{SGO}_n$. Here $(t, \theta) \in \mathbb{R}_{>0}^{1|1}$ parametrize a certain super semigroup whose super Lie algebra is free on one odd generator.

The homeomorphism from $\mathrm{Conf}_n^{\mathrm{fin}}$ to $\mathcal{M}_n$ is basically given by applying the inverse of the Cayley transform to the operator $\mathcal{D}$. If one applies this transformation to elements in Conf_n one obtains interesting completions of the Milnor and Bott spaces.

The spaces $\mathcal{EFT}_n$ were introduced by two of us in [ST] as super semigroups of *Hilbert-Schmidt* operators, this condition coming from the requirement that a field theory leads to vectors (here given by the operators) in a Hilbert space. In the meantime, we have developed a very general notion of super symmetric field theories and thus the definition of $\mathcal{EFT}_n$ should be changed accordingly. However, the homeomorphism type of the space does not change and thus we will not discuss this new aspect in this paper.

If one starts with a closed n-dimensional spin manifold M then the C_n-linear Dirac operator D_M (called *Atiyah-Singer operator* in [LM, p.140]) is an example of a (non-finite) element in Inf_n, where $\mathcal{H}_n$ are the L^2-sections of the C_n-linear spinor bundle on M. One can think of this operator as the infinitesimal generator of a super symmetric $(1|1)$-dimensional quantum field theory, with Euclidean (rather than Minkowski) signature. Actually, physicists would call it super symmetric quantum mechanics on M, not a field theory, since space is 0-dimensional.

The spaces $\mathcal{F}_n$ first appeared in the article [AS] by Atiyah-Singer and are probably the most common model for K-theory. They make all the wonderful applications to analysis possible. From our point of view, the connection is easiest to make with the space Inf_n^{inf}: Starting with a skew-adjoint Fredholm operator T_0 on H_n that anti-commutes with $e_1, \dots, e_{n-1}$, we can turn it into an odd, self-adjoint, C_n-linear Fredholm operator

$$T = T_0 \otimes e_n \quad \text{or equivalently} \quad T = \begin{pmatrix} 0 & T_0^* \\ T_0 & 0 \end{pmatrix}$$

on $\mathcal{H}_n \cong H_n \oplus H_n$. It is easy to see that the map $T_0 \mapsto T$ is a homeomorphism and it is important to note that the skew-symmetry of T_0 is equivalent to the relation $Te_n = e_nT$. This correspondence actually extends to the well known case $n = 0$ where one starts with all Fredholm operators on H_0 and gets all odd, self-adjoint Fredholm operators on $H_0 \oplus H_0$.

A Fredholm operator has an essential gap around zero and hence one can push the essential spectrum outside zero all the way into $\pm\infty$ by a homotopy. This turns a bounded operator into an unbounded and is the basic step in the homotopy equivalence that takes a Fredholm operator T to an infinitesimal generator $\mathcal{D}$. In the analytic literature, one can sometimes find concrete formulas in terms of functional calculus (which just describes the movement of the spectrum of T) like

$$T = \frac{\mathcal{D}}{1 + \mathcal{D}^2}$$

Such precise formulas are not important from our point of view but the following subtlety arises in the operator $\mathcal{D}$: its eigenspace at ∞, by definition the orthogonal complement of the domain of $\mathcal{D}$, is decomposed into the parts at $+\infty$ respectively $-\infty$. Such a datum is not present in general elements of $\mathrm{Inf}_n^{\mathrm{fin}}$ and it reflects the fact that we started with a *bounded* operator. Roughly speaking, this represents no problem up to homotopy if both these parts at $\pm\infty$ are infinite dimensional. This uses Kuiper's theorem and is the only non-elementary aspect of this paper.

Taking into account the C_n-action, this is related to the following well known subtlety in the Atiyah-Singer spaces of Fredholm operators. If $n \not\equiv 3 \mod 4$, the spaces $\mathcal{F}_n$ are given by operators T_0 (or equivalently T) as above. However, Atiyah-Singer showed that for $n \equiv 3 \mod 4$ the space of C_{n-1}-antilinear skew-adjoint Fredholm operators on H_n has two boring, contractible components $\hat{\mathcal{F}}_n^{\pm}$ consisting of operators T_0 such that

$$e_1 \cdot e_2 \cdots e_{n-1} \cdot T_0$$

is essentially positive (respectively negative). Recall that an operator is *essentially positive* if it is positive on a closed invariant subspace of finite codimension. So in the precise version for $\mathcal{F}_n$ in [AS], these two components $\hat{\mathcal{F}}_n^{\pm}$ are disregarded. It turns out that the above functional calculus leads to a map of all C_{n-1}-antilinear skew-adjoint Fredholm operators to our spaces $\mathrm{Inf}_n^{\mathrm{fin}}$ but this map is a quasifibration (with contractible fibres) only on the component $\mathcal{F}_n$. Hence our spaces automatically remove the need for thinking about the above subtleties that arise from bounded operators and are interestingly only visible in the presence of special Clifford actions.

We end this introduction by explaining the easiest description (that we know) of a symmetric ring spectrum that represents K-theory. Let $\mathcal{H}_{-1}$ be a C_{-1}-module as before Theorem 1, in particular it contains a submodule

$V \cong C_{-1}$ as C_{-1}-modules. Then $\mathcal{H}_{-n} := \mathcal{H}_{-1}^{\otimes n}$ has the desired properties and there are corresponding spaces $E_n := \mathrm{Inf}_{-n}^{\mathrm{fin}}$ of operators as in (5) of Theorem 1. One can also use the completed version Inf_{-n} instead. E_n contains a canonical base point, namely the operator whose domain is zero (and thus all eigenvalues are at ∞).

Theorem 3. *For $n \geq 0$, the spaces E_n form a symmetric ring spectrum representing real K-theory. The relevant structures are given as follows.*

- *The symmetric group Σ_n acts by permuting the n factors of $\mathcal{H}_{-n}$.*
- *The multiplication maps $E_n \wedge E_m \longrightarrow E_{n+m}$ are given by the formula*

$$(D_n, D_m) \mapsto D_n \otimes \mathbb{1} + \mathbb{1} \otimes D_m$$

- *The Σ_n-equivariant structure maps $\mathbb{R}^n \longrightarrow E_n$ are given by sending $v \in \mathbb{R}^n$ to its Clifford action on the C_{-n}-module $V^{\otimes n}$ (and ∞ on the orthogonal complement in $\mathcal{H}_{-n}$). As $v \mapsto \infty$, Clifford multiplication also goes to ∞ and hence the structure maps can be extended to S^n, sending the point at ∞ to the base point in E_n. These operators are odd and* self-adjoint *which explains the negative sign of $-n$.*

This result is a reformulation of a theorem of Michael Joachim [Jo], so we shall not give a proof. By using complex Hilbert spaces and Clifford algebras, all our results translate to complex K-theory. In fact, keeping track of the involution of complex conjugation, one also gets Atiyah's Real K-theory which contains both, real K-theory (via taking fixed points) and complex K-theory (by forgetting the conjugation map).

1. Bott spaces of minimal geodesics

The beginning of the story is Raoul Bott's classic paper [B] on "The stable homotopy of the classical groups". For a Riemannian manifold M, let $v = (P, Q, h)$ denote a 'base point in M' which is actually a pair of points $P, Q \in M$, together with a fixed homotopy class h of paths connecting P and Q. If $P = Q$ then h is just an element in $\pi_1(M, P)$. Bott considers the space M^v of minimal geodesics from P to Q in the homotopy class h. Let $|v|$ be the first positive integer which occurs as the index of some geodesic with base point v. Then Bott proved the following theorem in [B]:

Theorem 4 (Bott). *If M is a symmetric space, so is M^v. Moreover, the based loop space Ω_v can be built, up to homotopy, by starting with M^v and attaching cells of dimension $\geq |v|$:*

$$\Omega_v \simeq M^v \cup e^{|v|} \cup \text{ (higher dimensional cells)} \quad \text{written as} \quad M^v \xrightarrow{|v|} M$$

For example, if $M = S^n$ with the round metric and P, Q are not antipodal, then there is a unique minimal geodesic from P to Q and $|v| = n-1$ along the geodesic that goes around 1.5 times. This follows because this second shortest geodesic reaches the point $-Q$ that has an $(n-1)$-dimensional variation of geodesics to Q. In the notation above, one gets

$$pt = M^v \xrightarrow{(n-1)} M = S^n$$

which implies that ΩS^n is $(n-2)$-connected, or equivalently, that S^n is $(n-1)$-connected, not such a great result. However, if one considers non-minimal geodesics, one can say much more. In fact, the indices of geodesics from P to Q are $k(n-1)$ for $k = 0, 1, 2, \dots$. This is a case of Morse's original application of his theory to infinite dimensional manifolds: the energy functional

$$E : \Omega_v M \longrightarrow \mathbb{R}, \quad E(\gamma) := \int_0^1 |\gamma'(t)|^2 dt$$

is a Morse function with critical points the geodesics and indices given by the number of conjugate points (counted with multiplicity) along the given geodesic. Morse shows that infinite dimensionality is not an issue, because the space of paths with bounded energy has the homotopy type of a finite dimensional space, namely the piecewise geodesics (where the number of corners is related to the energy bound and the injectivity radius of M). As a consequence of our example above,

$$\Omega_v S^n \simeq S^{n-1} \cup e^{2(n-1)} \cup e^{3(n-1)} \cup \dots$$

If P and Q are antipodal points on S^n, then the energy is not a Morse function, for example the minimal geodesics form an $(n-1)$-sphere, parametrized by the equator in S^n. However, Bott developed a theory for such cases, now known as Morse-Bott theory, where the critical points form a submanifold whose tangent space equals the null space of the Hessian of the given function, the *Morse-Bott condition*. Applied to the case at hand, we can derive the same cell decomposition of $\Omega_v S^n$ as above but this time the bottom cell consists of the minimal geodesics:

$$S^{n-1} = M^v \xrightarrow{2(n-1)} M = S^n$$

So Freudenthal's suspension theorem is a direct consequence of this result, without looking at higher dimensional indices or cells. More generally, for any symmetric space, this is Bott's proof of Theorem 4 above.

His approach to study the homotopy types of the classical groups was to apply these method to compact Lie groups which are symmetric spaces in their bi-invariant metric. For example, consider $M = O(2m)$ and $P =$

$\mathbb{1}$, $Q = -\mathbb{1}$. Then every geodesic γ from P to Q is of the form

$$\gamma(t) = \exp(\pi t \cdot A), \quad t \in [0, 1]$$

where A is skew-adjoint. Thus we can 'diagonalize' A by an orthogonal matrix T, i.e. TAT^{-1} is a block sum B of matrices

$$\begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$$

with $a_1, \dots, a_m \geq 0$ after normalization. Since $\gamma(1) = -\mathbb{1}$, we see that the a_i are odd integers. It is not hard to see that the length of γ is given by the formula

$$\sqrt{2(a_1^2 + \cdots + a_m^2)}$$

so that minimal length means that all $a_i = 1$. We note that the length determines the homotopy class h of such a path so that we don't need to mention it for minimal length paths (this stays true in all considerations below as well). We conclude that

$$A^2 = T^{-1}B^2T = T^{-1}(-\mathbb{1})T = -\mathbb{1},$$

so that A is a complex structure on $\mathbb{R}^{2m}$. Just like for the standard complex structure we have

$$A = \exp((\pi/2) \cdot A) = \gamma(1/2)$$

and we obtain the following result.

Proposition 5 (Bott)**.** *The space $\mathcal{B}_2(2m)$ of minimal geodesics in $O(2m)$ with basepoint v as above is isometric to the space $\mathcal{M}_2(2m)$ of unitary structures on $\mathbb{R}^{2m}$ (consisting of $J \in O(2m)$ with $J^2 = -\mathbb{1}$). Moreover, this is a totally geodesic submanifold of $O(2m)$ and just like for antipodal points on S^n, the homeomorphism is given by sending a geodesic γ to its midpoint $\gamma(1/2)$.*

We are introducing a notation that is consistent with

$$\mathcal{B}_1(m) = O(m) = \mathcal{M}_1(m)$$

and will lead to the Bott and Milnor spaces in the limit when $m \mapsto \infty$. Now recall that $\mathcal{B}_2(2m)$ is again a symmetric space by Bott's theorem so that we can iterate the construction: Pick a complex structure J_1 and study the space $\mathcal{B}_3(4m)$ of minimal geodesics in $\mathcal{B}_2(4m)$ from J_1 to $-J_1$ (they automatically lie in a fixed homotopy class).

By a very similar discussion as above, it turns out that the midpoint map gives an isometry

$$\mathcal{B}_3(4m) \cong \mathcal{M}_3(4m) := \{J \in O(4m) \mid J^2 = -\mathbb{1},\ JJ_1 = -J_1J\}.$$

Note that the right hand side is the space of (orthogonal) quaternion structures on $\mathbb{R}^{4m}$ that are compatible with the given unitary (or orthogonal complex) structure J_1. Such structure form a totally geodesic submanifold of $O(4m)$. More generally, we make the following

Definition 6. Assume that $\mathbb{R}^m$ is a C_n-module (or $*$-representation) for some $n \geq 1$, with an action that sends e_i to J_i for $i = 1, \dots, n$. Then we define *Milnor spaces*

$$\mathcal{M}_{n+1}(m) := \{J \in O(m) \mid J^2 = -\mathbb{1}, JJ_i = -J_iJ \quad \forall i = 1, \dots, n-1\}$$

to be 'the space of all C_n-structures' on $\mathbb{R}^m$, compatible with the given C_{n-1}-structure. If d_n is the minimal dimension of a C_n-module, then this definition applies if d_n divides m. Otherwise we define these spaces to be empty.

Proposition 7 (Bott)**.** *All $\mathcal{M}_{n+1}(m)$ are totally geodesic submanifolds of $O(m)$. The space of minimal geodesics in $\mathcal{M}_{n+1}(m)$ from J_n to $-J_n$ is isometric to $\mathcal{M}_{n+2}(m)$ via the midpoint map.*

Proof. Let's assume the first sentence and show the second assertion. Any geodesic γ from J_n to $-J_n$ is of the form

$$\gamma(t) = J_n \cdot \exp(\pi t \cdot A), \quad t \in [0, 1]$$

for some skew-adjoint matrix A. One checks that $\gamma(1/2) = J_n \cdot A$ has square $-\mathbb{1}$ and anticommutes with $J_1, \dots J_n$ if and only if γ lies in the submanifold $\mathcal{M}_{n+1}(m)$. □

Definition 8. Inductively, let the *Bott spaces* $\mathcal{B}_{n+2}(m)$ be the space of minimal geodesics in $\mathcal{B}_{n+1}(m)$ from J_n to $-J_n$ (for those m where such a path exists). Then the previous discussion shows that the midpoint map gives an isometry

$$\mathcal{B}_{n+2}(m) \cong \mathcal{M}_{n+2}(m)$$

Theorem 9 (Bott)**.** *Let d_n be as in Definition 6 and recall Bott's notation for the cell decomposition of the space of loops. Then*

$$\mathcal{B}_{n+1}(m) \xrightarrow{\frac{m}{d_n}-1} \mathcal{B}_n(m)$$

This implies in particular that for $m \mapsto \infty$, the smallest dimension of a cell needed to get the loop space from the space of minimal geodesics also goes to infinity. Thus in the limit, one gets a homotopy equivalence

$$\Omega_v \mathcal{B}_n \simeq \mathcal{B}_{n+1}.$$

To make this precise, we define the (finite rank) Milnor spaces $\mathcal{M}_n = \mathcal{M}_n(\infty)$ as the union of all $\mathcal{M}_n(m)$ inside $O(\infty)$, which is the union of all $O(m)$. In fact, we only take the union over those m that are divisible by d_n.

With a similar definition of $\mathcal{B}_n = \mathcal{B}_n(\infty)$, the midpoint maps give homeomorphisms

$$\mathcal{B}_n(m) \approx \mathcal{M}_n(m) \quad \forall\, m = 1, 2, \dots \quad \text{(including } m = \infty\text{)}.$$

between these Bott and Milnor spaces. Now by Morita equivalence $\mathcal{M}_n \approx \mathcal{M}_{n+8}$ because C_{n+8} is a real matrix ring over C_n. As a consequence,

Corollary 10 (Bott)**.** *There are homeomorphisms and homotopy equivalences*

$$\mathcal{B}_n \approx \mathcal{B}_{n+8} \simeq \Omega^8 \mathcal{B}_n$$

and the homotopy groups of $O(\infty)$ are 8-periodic.

These groups are known as the 'stable' homotopy groups of the orthogonal group because

$$\pi_i O(m) \cong \pi_i O(\infty) \quad \forall i < m - 1$$

2. Milnor spaces of Clifford module structures

We recall the following model for our graded Hilbert space $\mathcal{H}_n$. For each $n \geq 1$, let H_n be a separable Hilbert space that is a C_{n-1}-module such that each irreducible representation of C_{n-1} appears with infinite multiplicity. Tensoring with C_n we obtain a graded C_n-module

$$\mathcal{H}_n := H_n \otimes_{C_{n-1}} C_n.$$

Here C_n acts on $\mathcal{H}_n$ by multiplication from the right. The tensor product is formed using the embedding $C_{n-1} \hookrightarrow C_n$ defined by the identification

$$C_{n-1} \xrightarrow{\cong} C_n^{ev},\ e_i \mapsto e_n e_i,\ \text{for } i = 1, ..., n-1.$$

It will be useful to think of $\mathcal{M}_n$ as a subspace of the group of orthogonal operators $O(H_n)$. To do so, we interpret H_n as a module over C_{n-2} via

$$C_{n-2} \xrightarrow{\cong} C_{n-1}^{ev},\ \tilde{e}_i \mapsto e_{n-1} e_i,$$

where we denote by $\tilde{e}_i$, $i = 1, ..., n-2$, the standard generators of C_{n-2}.[1] For $n \geq 2$ the space $\mathcal{M}_n$ is the same as the space of orthogonal operators J on H_n satisfying

- $J^2 = -\mathbb{1}$, or, equivalently, J is skew-adjoint.
- J anti-commutes with $\tilde{e}_1, ..., \tilde{e}_{n-2}$.
- $J = e_{n-1}$ on a subspace of finite codimension

[1] The point of the tildes is that in the proof of Proposition 22 it will be useful to have diffferent notions for elements in C_{n-2} and C_{n-1}.

This follows from from embedding $\mathbb{R}^\infty$ into H_n, for example by picking an orthonormal basis and looking at finite linear combinations of basis vectors. In the case $n = 1$ we have, by definition,

$$\mathcal{M}_1 = \{A \in O(H_1) \,|\, A \equiv \mathbf{1} \text{ modulo finite rank operators } \}.$$

From our point of view, the main result from the previous section is the following.

Theorem 11. *There are homeomorphisms $\mathcal{B}_n \approx \mathcal{M}_n$, for all $n \geq 1$.*

3. Infinitesimal Generators

In this section we will review some basic facts about self-adjoint (unbounded) operators, reminding the reader of a nice topology on this space. Let H be a separable complex Hilbert space. Denote by Inf the set of all self-adjoint operators on H with compact resolvent. Note that we do not require an element $\mathcal{D} \in \text{Inf}$ to be densely defined. By 'self-adjoint' we mean that $\mathcal{D}$ defines a self-adjoint operator on the closure of its domain. The compact resolvent condition means that the spectrum of $\mathcal{D}$ consists of eigenvalues of finite multiplicity that do not have an accumulation point in $\mathbb{R}$. Hence, if the domain of $\mathcal{D}$ is infinite dimensional, the operator $\mathcal{D}$ on $\overline{\text{dom}(\mathcal{D})}$ is necessarily unbounded. Because of this, we will think of $\text{dom}(\mathcal{D})^\perp$ as the eigenspace of $\mathcal{D}$ associated with the 'eigenvalue' ∞.

Functional calculus gives a bijection, see e. g. [HG]

$$\text{Inf} \longleftrightarrow \text{Hom}(C_0(\mathbb{R}), \mathcal{K})$$

where the right hand side is the space of all C^*-homomorphisms from (complex valued) continuous functions on $\mathbb{R}$ that vanish at ∞ to the compact operators $\mathcal{K}$ on H. Note that both of these C^*-algebras do not have a unit. Below, we will also deal with C^*-algebras that do have a unit and in this case Hom will denote those C^*-homomorphisms that preserve the unit.

We define the space $\widetilde{\text{Inf}}$ to be just as above, except that we do *not* require the spectrum to be discrete (and the eigenspaces can be infinite dimensional). The "Cayley transform" is defined for such operators by functional calculus using the Möbius transformation

$$c(x) := \frac{x+i}{x-i}$$

which takes $\mathbb{R} \cup \{\infty\}$ to the unit circle S^1. It defines the mapping from the very left to the very right in the following theorem.

Theorem 12. *There are bijections*

$$\widetilde{\text{Inf}} \overset{a}{\longleftrightarrow} \text{Hom}(C_0(\mathbb{R}), \mathcal{B}) \overset{c}{\cong} \text{Hom}(C(S^1), \mathcal{B}) \overset{b}{\longleftrightarrow} \mathcal{U}$$

where $\mathcal{B}$ and $\mathcal{U}$ are the bounded respectively unitary operators on H. Moreover, the bijection b on the right, given by functional calculus, is a homeomorphism from the pointwise norm topology on $\mathrm{Hom}(C(S^1), \mathcal{B})$ *to the operator norm topology on $\mathcal{U}$.*

Definition 13. We give $\widetilde{\mathrm{Inf}}$ the topology coming from the above bijections. This is sometimes referred to as *generalized norm topology* because of Lemma 15 below.

Remark 14. Just like Inf has an interpretation in terms of configuration spaces, by using the pattern of eigenvalues and their eigenspaces, the space $\widetilde{\mathrm{Inf}}$ can be interpreted as the space of all projection valued measures on $\mathbb{R}$, see [RS, Thm.VIII.6]. The fact that the operators must not be densely defined is reflected in the fact that the projection corresponding to all of $\mathbb{R}$ is not necessarily the identity but projects onto the domain. Thus the result becomes cleaner than in [RS] where the map b is not onto.

Theorem 12 is well known, we just need to collect various bits and pieces of the argument, for example from Rudin [R] or Reed-Simon [RS]. These authors only define the adjoint of a *densely defined* operator D because otherwise the adjoint is not determined by the formula

$$\langle Dv, w \rangle = \langle v, D^* w \rangle$$

In particular, self-adjoint operators are assumed to be have dense domain. As a consequence, [R, Thm.13.19] proves that the Cayley transform gives an inclusion of all *densely defined* self-adjoint operators onto the space of unitary operators without eigenvalue 1. If one allows nondense domains, i.e. eigenvalues ∞ (defining the adjoint also to be ∞ on that subspace), then the Cayley transform takes the eigenspace at ∞ to the eigenspace at 1 and therefore becomes onto all unitary operators, i.e. gives the desired bijection $\widetilde{\mathrm{Inf}} \leftrightarrow \mathcal{U}$.

Proof of Theorem 12. The map a is given by functional calculus which is well defined on self-adjoint operators that are densely defined. Since the functions f vanish at ∞ one can extend this for all $D \in \widetilde{\mathrm{Inf}}$ by defining $f(D)$ to be zero on the orthogonal complement of the domain of D. For the second map, note that $C(S^1)$ is obtained from $C_0(\mathbb{R})$ by adding a unit $\mathbb{1}$ (and using the above Möbius transformation c). We get an isomorphism between the two spaces of C^*-homorphisms since we require that $\mathbb{1}$ maps to $\mathbb{1}$ (if the algebras have units). Finally, the map b is given by evaluating a homomorphism at the identity map $z : S^1 \to S^1$. It is clear that the composition from left to right is therefore the Cayley transform and hence a bijection. Recall that by Fourier decomposition, there is an isomorphism

of complex C^*-algebras
$$C(S^1) \cong C^*(\mathbb{Z})$$
where $\mathbb{Z}$ is the infinite cyclic group, freely generated by an element z (which corresponds to the above identity z on S^1). It follows that $C^*(\mathbb{Z})$ is free as a C^*-algebra on one unitary element z and hence C^*-homomorphisms out of it are just unitary elements in the target. Moreover, the bijection is given by evaluating functions on this unitary z which is our map b above.

To show that b is a homeomorphism, we need to show that a sequence φ_n of C^*-homomorphism converges if and only if $\varphi_n(z)$ converges (in norm). By definition, the φ_n converge if $\varphi_n(f)$ converge (in norm) for all f, so one direction is obvious. For the other, assume that $u_n := \varphi_n(z)$ converges to $u \in \mathcal{U}$ and note that $\varphi_n(f) = f(\varphi_n(z)) = f(u_n)$. We want to show that $f(u_n)$ converges to $f(u)$ and we claim that this is easy to check in the case when f is a Laurent polynomials. For the general case, pick $\epsilon > 0$ and choose a Laurent polynomial $p = p(z)$ such that $\|f - p\|_{sup} < \epsilon/3$ and an $N >> 0$ such that $\|p(u_n) - p(u)\| < \epsilon/3$ for $n > N$. Then for $n > N$ we have
$$\|f(u_n) - f(u)\| = \|(f(u_n) - p(u_n)) + (p(u_n) - p(u)) + (p(u) - f(u))\| < \epsilon$$
and hence $f(u_n)$ converges to $f(u)$. This argument is very similar to the one in [RS, Thm.VIII.20(a)]. □

Lemma 15. *The Cayley transform on bounded operators*
$$\mathcal{B}^{sa} \subset \widetilde{\mathrm{Inf}} \longleftrightarrow \mathcal{U}$$
is an open embedding, i.e. the generalized norm topology on $\widetilde{\mathrm{Inf}}$ *extends the operator norm topology on* $\mathcal{B}^{sa}$*, the bounded self-adjoint operators on* H*.*

Again this result is well known, see for example [RS, Thm.VIII.18]. Reed and Simon use the resolvent instead of the Cayley transform but this is just a different choice of Möbius transformation, using $x \mapsto (x + i)^{-1}$ instead of c. This has the effect that the image of $\mathbb{R}$ is not the unit circle but a circle of radius $1/2$ inside the unit circle. Therefore, one does not get unitary operators but there is certainly no difference for the induced topology. Unfortunately, in the above Theorem VIII.18, Reed and Simon assume an additional property on the sequence considered, namely that it is uniformly bounded. It turns out, however, that this assumption is unnecessary which is an easy consequence of Theorem VIII.23(b) in [RS].

Remark 16. It is interesting to recall from [R, Thm.13.19] that the Cayley transform can also be applied to *symmetric* operators, i.e. those that are formally self-adjoint and with $\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D^*)$. The result is an isometry U with
$$\mathrm{Dom}(U) = \mathrm{Range}(D + i \cdot \mathbb{1}) \quad \text{and} \quad \mathrm{Range}(U) = \mathrm{Range}(D - i \cdot \mathbb{1})$$

D is closed if and only if U is closed and D is self-adjoint if and only if U is unitary. Using the Cayley transform and its inverse one sees that the self-adjoint extensions of D are in 1-1 correspondence with unitary isomorphisms between the orthogonal complements of $\mathrm{Dom}(U)$ and $\mathrm{Range}(U)$. In particular, self-adjoint extensions exist if and only if these complements have the same dimensions, usually referred to as the *deficiency indices*.

An example to keep in mind is the infamous right shift which is an isometry with deficiency indices 0 and 1. Thus its inverse Cayley transform has no self-adjoint extension.

Let $\mathrm{Inf}^{\mathrm{fin}}$ be the space of all unbounded, self-adjoint operators on H, with finite spectrum and multiplicity (possibly non-densely defined).

Proposition 17. *The Cayley transform induces the following bijections*

$$\mathrm{Inf} \overset{a}{\longleftrightarrow} \mathrm{Hom}(C_0(\mathbb{R}), \mathcal{K}) \overset{c}{\cong} \mathrm{Hom}(C(S^1), \mathcal{K} + \mathbb{C}\cdot \mathbb{1}) \overset{b}{\longleftrightarrow} \mathcal{U} \cap (\mathcal{K} + \mathbb{C}\cdot \mathbb{1})$$

$$\mathrm{Inf}^{fin} \overset{a}{\longleftrightarrow} \mathrm{Hom}(C_0(\mathbb{R}), \mathcal{FR}) \overset{c}{\cong} \mathrm{Hom}(C(S^1), \mathcal{FR}+\mathbb{C}\cdot \mathbb{1}) \overset{b}{\longleftrightarrow} \mathcal{U}\cap(\mathcal{FR}+\mathbb{C}\cdot \mathbb{1})$$

where $\mathcal{K}$ and $\mathcal{FR}$ are the compact respectively finite rank operators on H. Moreover, the bijections b on the right, given by functional calculus, are homeomorphisms from the pointwise norm topology on the spaces of C^*-homomorphisms to the operator norm topology on $\mathcal{U}$.

Proof. The Cayley transforms give the bijections from the very left to the very right because one can read off the conditions of being compact respectively finite rank from the spectrum and multiplicities of the operators. These conditions are mapped into each other by definition of the spaces. The fact that the maps b are homeomorphisms is proved exactly as in Theorem 12 □

We now have complete control over the topology on our various spaces of configurations. Note that the largest space $\widetilde{\mathrm{Inf}}$ is homeomorphic to $\mathcal{U}$ and hence contractible by Kuiper's theorem, whereas the subspaces $\mathrm{Inf}^{\mathrm{fin}}$ and Inf are homotopy equivalent and have a very interesting topology.

We shall now add some bells and whistles, like grading, real structure and Clifford action to make these spaces even more interesting. In a first step, assume that our complex Hilbert space H has a real structure, i.e. we have

$$H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

for some real Hilbert space $H_{\mathbb{R}}$. If we think of the the real structure (aka complex conjugation) on H as a grading involution α (which has the property that the even and odd parts are isomorphic) then the above Proposition 17 leads to the following result.

Proposition 18. *The Cayley transform induces homeomorphisms*

$$\mathrm{Inf}_{odd}(H) \overset{c}{\approx} O(H_{\mathbb{R}}) \cap (\mathcal{K} + \mathbb{C} \cdot \mathbb{1}) \quad \textit{and} \quad \mathrm{Inf}^{fin}_{odd}(H) \overset{c}{\approx} O(H_{\mathbb{R}}) \cap (\mathcal{FR} + \mathbb{C} \cdot \mathbb{1})$$

Here $\mathrm{Inf}_{odd}(H)$ *denotes the subspace of* odd *operators in* $\mathrm{Inf}(H)$ *(which are still* $\mathbb{C}$*-linear) and* $O(H_{\mathbb{R}})$ *is the usual orthogonal group, thought of as the subgroup of real operators in the unitary group* $\mathcal{U}(H)$*.*

Proof. Since both sides have the subspace topology, it suffices to show that the Cayley transform is a bijection between the spaces given in the Proposition. An operator D in $\mathrm{Inf}(H)$ is odd if and only if $D^\alpha := \alpha D \alpha = -D$. Since our grading involution α is $\mathbb{C}$-antilinear, we also have $i^\alpha = -i$ and therefore

$$\overline{c(D)} = c(D)^\alpha = \left(\frac{D+i}{D-i}\right)^\alpha = \frac{-D-i}{-D+i} = \frac{D+i}{D-i} = c(D)$$

Since the operators in $\mathcal{U}(H)$ that commute with complex conjugation are clearly those in the real orthogonal group $O(H_{\mathbb{R}})$, we get $c(D) \in O(H_{\mathbb{R}})$ and as before, $c(D) - \mathbb{1}$ is compact (respectively finite rank). Conversely, a similar calculation shows that if $c(D)$ is real then $D^\alpha = -D$. □

Let $\mathcal{H}_n$ be as in the introduction, a $\mathbb{Z}/2$-graded real Hilbert space with a $*$-action of the real Clifford algebra C_n. For example, the above discussion is the case $n = 1$ if we define $\mathcal{H}_1 = H$ with the grading and $C_1 = \mathbb{C}$-action as above. Note that in this case one can think of $\mathbb{C}$-linear operators, say in $\mathcal{K}(H)$, as $\mathbb{R}$-linear operators that commute with the C_1-action. This motivates the following definition.

Definition 19. We denote by $\mathcal{K}_n$ (resp. $\mathcal{FR}_n$) the space of all C_n-linear self-adjoint compact (resp. finite rank) operators on $\mathcal{H}_n$, and by Inf_n (resp. $\mathrm{Inf}^{\mathrm{fin}}_n$) the subspace of Inf (resp. $\mathrm{Inf}^{\mathrm{fin}}$) that consist of all C_n-linear and odd operators.

Furthermore, the real graded C^*-algebra $\mathcal{S}$ is given by real valued functions in $\mathbb{C}_0(\mathbb{R})$ with trival $*$ and grading involutions induced by $x \mapsto -x$, i.e. the usual decomposition into even and odd functions.

To motivate the use of self-adjoint operators, we make the following easy observation that comes from the above case $n = 1$.

Lemma 20. *The restriction to self-adjoint elements give bijections*

$$\mathrm{Hom}_{gr}(C_0(\mathbb{R}), \mathcal{K}(H)) \longleftrightarrow \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{K}^{sa}(H))$$

where H is a complex Hilbert space with grading involution as above and Hom_{gr} *denotes grading preserving $*$-homomorphisms. The analogous statement holds for* $\mathcal{FR}$ *in place of* $\mathcal{K}$*.*

Proof. Recall that $\mathcal{S}$ are just the real valued functions in $C_0(\mathbb{R})$ (aka the self-adjoint elements in this complex C^*-algebra) and that the grading involutions agree. Moreover, there is an isomorphism

$$\mathcal{S} \otimes_{\mathbb{R}} \mathbb{C} \cong C_0(\mathbb{R})$$

The same statements apply to $\mathcal{K}$ (respectively $\mathcal{FR}$) and therefore the complexification map gives an inverse to the restriction map in the lemma. □

The same argument as above then leads to the following result.

Proposition 21. *Functional calculus induces the homeomorphisms*

$$\mathrm{Inf}_n \approx \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{K}_n) \quad \textit{and} \quad \mathrm{Inf}_n^{fin} \approx \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{FR}_n)$$

It remains to identify the image of these spaces under the Cayley transform in $O(H_n)$, where we now assume the same setting as in Section 2: For $n \geq 1$, let H_n be a real Hilbert space that is a C_n^{ev}-module and consider the graded C_n-module

$$\mathcal{H}_n := H_n \otimes_{C_n^{ev}} C_n \cong H_n \otimes_{\mathbb{R}} \mathbb{C}.$$

The last isomorphism should be interpreted as saying the the complex structure on $\mathcal{H}_n$ is given by the last basis element $e_n \in C_n$ and that the grading can be thought of as corresponding to complex conjugation, just like in our previous discussion.

Proposition 22. *For all $n \geq 1$ the Cayley transform (composed with left multiplication by e_{n-1}) gives homeomorphisms with the Milnor spaces*

$$\mathrm{Inf}_n^{fin} \approx \mathcal{M}_n \quad \textit{and} \quad \mathrm{Inf}_n \approx \bar{\mathcal{M}}_n.$$

Here $\bar{\mathcal{M}}_n$ is the closure of the Milnor space $\mathcal{M}_n$ in $O(H_n)$ with respect to the operator norm. This space differs from $\mathcal{M}_n$ in that the last condition in Section 2 is replaced by

- $J \equiv e_{n-1}$ modulo compact operators.

Proof. We will show that there is a homeomorphism $\mathrm{Inf}_n \approx \bar{\mathcal{M}}_n$ which restricts to the desired homeomorphism on finite configurations. The case $n = 1$ was discussed above because in this case we have by definition

$$O(H_1) \cap (\mathbb{1} + \mathcal{K}) = \bar{\mathcal{M}}_1 \quad \text{and} \quad O(H_1) \cap (\mathbb{1} + \mathcal{FR}) = \mathcal{M}_1.$$

Now, let $n \geq 2$. Recall that the complex structure on $\mathcal{H}_n$ is given by e_n, hence the relation $De_n = e_n D$ gives the $\mathbb{C}$-linearity of $c(D)$. We claim that the relations $De_i = e_i D$ for the remaining $n-1$ generators e_i of C_n imply that the generators e_i of C_{n-1} satisfy

$$e_i c(D) = c(D)^{-1} e_i.$$

To see this, note that we have the relations[2]

$$e_i(D \pm i) = (D \mp i)e_i \text{ and } e_i(D \pm i)^{-1} = (D \mp i)^{-1}e_i$$

which together yield

$$e_i c(D) = e_i(D - i)(D + i)^{-1} = (D + i)(D - i)^{-1}e_i = c(D)^{-1}e_i.$$

Note that all these are operators on $\mathcal{H}_n$ but that our *odd* operator D gives an action of $c(D)$ on H_n. We assert that the same relation holds for this operator $c(D)$ on H_n. First, note that since $c(D)$ is $\mathbb{C}$-linear, i.e. it commutes with e_n, we have $e_ne_ic(D) = c(D)^{-1}e_ne_i$. Next, one checks that under the isomorphism $C_n^{ev} \cong C_{n-1}$ the action of $e_ne_i \in C_n$, $i = 1, ..., n-1$, corresponds to the automorphism $e_i \otimes \mathrm{id}$ of $H_n \otimes \mathbb{C}$. This together with the relation we computed for $c(D) \otimes \mathrm{id}$ implies $e_ic(D) = c(D)^{-1}e_i$ for $i = 1, \dots, n-1$.

Hence we see that the Cayley transform c gives a homeomorphism

$$\mathrm{Inf}_n \approx \{\, A \in O(H_n) \mid A \equiv \mathbf{1} \bmod \mathcal{K}(H_n) \text{ and } e_iA = A^{-1}e_i \text{ for } i = 1, ..., n-1\}$$

The space on the right-hand side is not quite $\bar{\mathcal{M}}_n$ yet. However, we claim that it can be identified with $\bar{\mathcal{M}}_n$ by associating to an operator A the complex structure

$$J := e_{n-1}A \in \bar{\mathcal{M}}_n.$$

It is clear that $J \equiv e_{n-1} \bmod \mathcal{K}(H_n)$. Furthermore, J is indeed a complex structure:

$$J^2 = e_{n-1}Ae_{n-1}A = e_{n-1}AA^{-1}e_{n-1} = -\mathbf{1}.$$

It remains to check that J anti-commutes with the generators $\tilde{e}_1, ..., \tilde{e}_{n-2}$ of C_{n-2}. The following computation shows this claim using $\tilde{e}_i = (e_{n-1}e_i)$:

$$(e_{n-1}e_i)(e_{n-1}A) = (e_{n-1}e_i)(A^{-1}e_{n-1}) = e_{n-1}Ae_ie_{n-1} = -(e_{n-1}A)(e_{n-1}e_i)$$

This completes the proof. □

4. Configuration Spaces

The unbounded operators of the previous section can be visualized as configurations on the real line: an operator $D \in \mathrm{Inf}$ is completely determined by its eigenvalues and eigenspaces and hence by the map V that associates to $\lambda \in \mathbb{R}$ the subspace $V(\lambda)$ on which $D = \lambda$. We call V a 'configuration on $\mathbb{R}$', since $V(\lambda)$ may be thought of as a label attached at $\lambda \in \mathbb{R}$. Since slightly different spaces of configurations will appear in Section 9, we give a general definition that also covers the case considered there.

Let Λ be a topological space equipped with an involution s and $\mathcal{H}$ a separable $\mathbb{Z}/2$-graded Hilbert space with grading involution α. A *configuration*

[2]The notation $(D \pm i)^{-1}$ best interpreted on each pair of eigenspaces $V_\lambda \oplus V_{-\lambda}$ of D separately; there it definitely makes sense and that is all we care about here.

on Λ *indexed by orthogonal subspaces of* $\mathcal{H}$ is a map V from Λ to the set of closed (ungraded) subspaces of $\mathcal{H}$ such that

- the subspaces $V(\lambda)$ are pairwise orthogonal
- $\mathcal{H}$ is the Hilbert sum of the $V(\lambda)$'s
- V is compatible with s and α, i.e. $V(s(\lambda)) = \alpha(V(\lambda))$ for all $\lambda \in \Lambda$.

Recall that closed subspaces of $\mathcal{H}$ correspond precisely to continuous projection operators on $\mathcal{H}$. Hence we may interpret V as a map

$$V : X \longrightarrow \mathrm{Proj}(\mathcal{H}) \subset B(\mathcal{H}).$$

To save space, we write $V_\lambda := V(\lambda)$. Define $\mathrm{supp}(V) := \overline{\{ \lambda \in \Lambda \mid V_\lambda \neq 0 \}}$.

Definition 23. The *space of configurations* $\mathrm{Conf}(\Lambda; \mathcal{H})$ *on* Λ *indexed by orthogonal subspaces of* $\mathcal{H}$ is the set of all configurations $V : \Lambda \to \mathrm{Proj}(\mathcal{H})$ equipped with the topology generated by the subbasis consisting of the sets

$$\mathcal{B}(U, L) := \{ V \in \mathrm{Conf}(\Lambda; \mathcal{H}) \mid V_U := \sum_{\lambda \in U} V_\lambda \in L, \ \mathrm{supp}(V) \cap \partial U = \emptyset \},$$

where U and L range over all open subsets $U \subset X$ and $L \subset \mathrm{Proj}(\mathcal{H})$.

We will need the following variations. Let $\Theta \subset \Lambda$ be a subspace that is preserved under the involution s. Define $\mathrm{Conf}(\Lambda, \Theta; \mathcal{H}) \subset \mathrm{Conf}(\Lambda; \mathcal{H})$ to be the subspace of configurations V such that V_λ has *finite rank* for all $\lambda \in \Theta^c := \Lambda \setminus \Theta$ and such that the subset of all $\lambda \in \Theta^c$ with $V(\lambda) \neq 0$ is *discrete* in Θ^c. Replacing the discreteness condition by requiring that there should be only *finitely many* $\lambda \in \Theta^c$ with $V_\lambda \neq 0$ we obtain the space $\mathrm{Conf}^{\mathrm{fin}}(\Lambda, \Theta; \mathcal{H})$ of configurations that are 'finite away from Θ'. Finally, if C is an $\mathbb{R}$-algebra and $\mathcal{H}$ is a C-module, we can replace subspaces of $\mathcal{H}$ by C-submodules in order to obtain spaces $\mathrm{Conf}_C(\Lambda, \Theta; \mathcal{H})$. If C is $\mathbb{Z}/2$-graded then we assume that $\mathcal{H}$ is a graded C-module but the subspaces $V(\lambda)$ are still ungraded; only those for which $s(\lambda) = \lambda$ are *graded* modules over C. Our main examples will be the Clifford algebras $C = C_n$.

Examples 24. Consider the one-point compactification $\bar{\mathbb{R}}$ of $\mathbb{R}$ equipped with the involution $s(x) := -x$. Define

$$\mathrm{Conf}_n := \mathrm{Conf}_{C_n}(\bar{\mathbb{R}}, \{\infty\}; \mathcal{H}_n),$$

where $\mathcal{H}_n$ is the graded C_n-module from the previous section. We will see in Lemma 25 that Conf_n gives a different model for the space Inf_n of unbounded operators introduced above. The homeomorphism $\mathrm{Inf}_n \to \mathrm{Conf}_n$ is given by mapping $\mathcal{D} \in \mathrm{Inf}_n$ to the configuration defined by associating to $\lambda \in \mathbb{R}$ the λ-eigenspace V_λ of $\mathcal{D}$. Here we let $V_\infty := \mathrm{dom}(\mathcal{D})^\perp$. Since $\mathcal{D}$ has compact resolvent, the set of $\lambda \in \mathbb{R}$ with $V_\lambda \neq 0$ is indeed discret in $\mathbb{R}$ and each eigenspace V_λ, $\lambda \in \mathbb{R}$, is finite-dimensional. The relation $V(s(\lambda)) = \alpha(V(\lambda))$ corresponds to $\mathcal{D}$ being odd.

In order to get a better feeling for the topology on Conf_n, let us describe a neighborhood basis for each configuration in Conf_n. This will also be useful for the proof that the map $\mathrm{Inf}_n \to \mathrm{Conf}_n$ is a homeomorphism.

We begin by pointing out that the topology on Conf_n is generated by the sets $\mathcal{B}(U, L)$, where $U \subset \mathbb{R}$ is bounded. To see this, note that, by definition of Conf_n, $\infty \in \mathrm{supp}(V)$ for all $V \in \mathrm{Conf}_n$. Hence $\mathcal{B}(U, L) = \emptyset$ whenever $\infty \in \partial U$ so that the case of unbounded $U \subset \mathbb{R}$ is irrelevant. Furthermore, if $\infty \in U$ we can use $\mathcal{B}(U, L) = \mathcal{B}(U^c, 1 - L)$ to describe $\mathcal{B}(U, L)$ in terms of $U^c := \bar{\mathbb{R}} \setminus U$. Thus it is sufficient to consider $\mathcal{B}(U, L)$ for $U \subset \mathbb{R}$ bounded.

Now, let $V \in \mathrm{Conf}_n$ and let K be a (large) positive real number such that $V_K = 0$. Let $B_K(0)$ be the ball of radius K around 0 and denote by $\lambda_1, ..., \lambda_{l_K}$ the numbers in $B_K(0)$ such that $V_{\lambda_i} \neq 0$. Let $\delta > 0$ and $\varepsilon > 0$ be (small) real numbers; we may choose δ so small that $B_\delta(\lambda_i) \cap B_\delta(\lambda_j) = \emptyset$ for $i \neq j$. Denote by $V_{K,\delta,\varepsilon}$ the set of all configurations W such that $\|V_{B_\delta(\lambda_i)} - W_{B_\delta(\lambda_i)}\| < \varepsilon$ for all i and such that $W_\lambda = 0$ for all $\lambda \in B_K(0)$ that do not lie in one of the balls $B_\delta(\lambda_i)$. Thus, an element $W \in V_{K,\delta,\varepsilon}$ almost looks like V on $B_K(0)$: the only thing that can happen is that a label V_λ 'splits' into labels W_{λ_j} with $|\lambda - \lambda_j|$ small ($< \delta$) and $\sum_j W_{\lambda_j}$ close to V_λ ($< \varepsilon$). The $V_{K,\delta,\varepsilon}$ form indeed a neighborhood basis of V: assume $V \in \bigcap_{k=1}^n \mathcal{B}(U_k, L_k)$, with $U_k \subset \mathbb{R}$ bounded. Choose K as above with $\bigcup_{k=1}^n U_k \subset B_K(0)$. Picking $\delta > 0$ so small that $B_\delta(\lambda_i) \subset \bigcap_{k=1}^n U_k$ for all i it follows easily using the triangle inequality that for $\varepsilon > 0$ sufficiently small $V_{K,\delta,\varepsilon} \subset \bigcap_{i=1}^n \mathcal{B}(U_i, L_i)$.

In particular, we see that the topology on Conf_n controls configurations well on compact subsets of $\mathbb{R}$ but not near infinity. The discussion also shows that Conf_n is first countable since we can choose K, δ, and ε in $\mathbb{Q}$.

Given any $V \in \mathrm{Conf}_n = \mathrm{Conf}_{C_n}(\bar{\mathbb{R}}, \{\infty\}; \mathcal{H}_n)$ and any function $f \in \mathcal{S}$, we can define a C_n-linear operator $f(V)$ on $\mathcal{H}_n$ by requiring that $f(V)$ has eigenvalue $f(\lambda)$ exactly on V_λ. This operator is always compact and it is of finite rank if and only if $V \in \mathrm{Conf}_n^{\mathrm{fin}} := \mathrm{Conf}_{C_n}^{\mathrm{fin}}(\bar{\mathbb{R}}, \{\infty\}; \mathcal{H}_n) \subset \mathrm{Conf}_n$, the subspace of configuration that are finite away from $\{\infty\}$.

Lemma 25. *Functional calculus $F(V)(f) := f(V)$ gives homeomorphisms*

$$F : \mathrm{Conf}_n \xrightarrow{\approx} \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{K}_n) \quad \textit{and} \quad \mathrm{Conf}_n^{fin} \xrightarrow{\approx} \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{FR}_n)$$

and so do the maps that take an operator to the configuration of its Eigenspaces and eigenvalues:

$$\mathrm{Inf}_n \approx \mathrm{Conf}_n \quad \textit{and} \quad \mathrm{Inf}_n^{fin} \approx \mathrm{Conf}_n^{fin}$$

Proof. It is clear that F is a bijection because the map that identifies operators with the eigenspaces and eigenvalues is obviously a bijection and it is the composition of F with a homeomorphism. Since Conf_n is first countable, we can check the continuity of F on sequences. To do so, assume $V_n \to V$ and fix $f \in \mathcal{S}$. We have to prove $f(V_n) \to f(V)$. Given $\varepsilon > 0$, choose $K > 0$ such that $|f(x)| < \varepsilon$ if $|x| > K$. Since the continuous map f is automatically uniformly continuous on compact sets, we can find a $\delta > 0$ such that for all $x \in B_K(0)$ we have $|f(x) - f(y)| < \varepsilon$ provided $|x - y| < \delta$. The assumption $V_n \to V$ tells us that $V_n \in V_{K,\delta,\varepsilon}$ for large n. The claim now follows from the following estimate that holds for all $W \in V_{K,\delta,\varepsilon}$:

$$\begin{aligned}
\|f(V) - f(W)\| &= \| \sum_{\lambda\in\mathbb{R}} f(\lambda)V_\lambda - \sum_{\mu\in\mathbb{R}} f(\mu)W_\mu \| \\
&\le \| \sum_{\lambda\in B_K(0),\, V_\lambda\neq 0} \left(f(\lambda)V_\lambda - \sum_{\mu\in B_\delta(\lambda)} f(\mu)W_\mu \right) \| + 2\varepsilon \\
&\le \#\{\lambda \in B_K(0) \mid V_\lambda \neq 0\,\} \cdot \left(\max_{\lambda\in B_K(0)} f(\lambda)\cdot\varepsilon + \varepsilon \right) + 2\varepsilon \\
&\le C\cdot\varepsilon,
\end{aligned}$$

where the constant C only depends on f and V. The first inequality follows by re-arranging the terms and using the triangle inequality together with $|f(x)| < \varepsilon$ for $|x| > K$. The second inequality follows from

$$\begin{aligned}
\|f(\lambda)V_\lambda - \sum_{\mu\in B_\delta(\lambda)} f(\mu)W_\mu\| &\le \|f(\lambda)(V_\lambda - W_{B_\delta(\lambda)})\| + \| \sum_{\mu\in B_\delta(\lambda)} (f(\lambda) - f(\mu))W_\mu\| \\
&\le \max_{\lambda\in B_K(0)} f(\lambda)\cdot\varepsilon + \varepsilon.
\end{aligned}$$

Note that the continuity of F implies that $\mathrm{Hom}_{gr}(S, \mathcal{K}_n)$ is also first countable so that the continuity of F^{-1} can be checked on sequences as well. Assume $f(V_n) \to f(V)$ for all $f \in \mathcal{S}$ and $V \in \mathcal{B}(U, L)$. We have to show $V_n \in \mathcal{B}(U, L)$ for n sufficiently large. More explicitly: $\mathrm{supp}(V_n) \cap \partial U = \emptyset$ for n large and $\lim_{n\to\infty} \|(V_n)_U - V_U\| = 0$.

Note that for an accumulation point $\gamma \in \mathbb{R}$ of the set $\bigcup_n \mathrm{supp}(V_n)$ we must have $V_\gamma \neq 0$, because otherwise we would also have $\|f(V_n) - f(V)\| \ge \frac{1}{2}$ for infinitely many n if we choose f to be a bump function with $f(\gamma) = 1$ that is concentrated near γ. This together with $\mathrm{supp}(V) \cap \partial U = \emptyset$ implies that there is a neighborhood $\nu(\partial U)$ of ∂U such that $(V_n)_\lambda \neq 0$ for $\lambda \in \nu(\partial U)$ occurs only for finitely many n. In particular, $\mathrm{supp}(V_n) \cap \partial U = \emptyset$ for n large. Now, choose $f \in \mathcal{S}$ such that $f|_{\bar{\mathbb{R}}\setminus U} = 0$ and $f|_{U\setminus\nu(\partial U)} = 1$. By construction, $f(V_n) = \chi_U(V_n)$ for n large, where χ_U denotes the indicator function for U. The same identity holds for V and hence we can conclude

$$\lim_{n\to\infty} \|(V_n)_U - V_U\| = \lim_{n\to\infty} \|\chi_U(V_n) - \chi_U(V)\| = \lim_{n\to\infty} \|f(V_n) - f(V)\| = 0.$$

This completes the proof. □

We close this section with the following remark concerning the functoriality of the configuration spaces that we will need in Section 9.

Remark 26. A continuous map $f : \Lambda \to \Lambda'$ that commutes with the involutions on Λ and Λ' induces

$$f_* : \mathrm{Conf}(\Lambda; \mathcal{H}) \longrightarrow \mathrm{Conf}(\Lambda'; \mathcal{H}),\ (f_*(V))_{\lambda'} := \sum_{\lambda \in f^{-1}(\lambda')} V_\lambda.$$

We show that the map f_* is continuous under the assumption that the space Λ' is normal. Let $V \in f_*^{-1}(\mathcal{B}(U, L))$. From the definition of $\mathcal{B}(U, L)$ we find $\mathrm{supp}(f_* V) \cap \partial U = \emptyset$. Since Λ' is normal, there is an open neighborhood N_V of $\mathrm{supp}(f_* V)$ such that $\overline{N_V} \cap \partial U = \emptyset$. Unravelling the definitions one finds

$$V \in \mathcal{B}(f^{-1}(U), L) \cap \mathcal{B}(f^{-1}(N_V), \{\mathrm{id}_{\mathcal{H}}\}) \subset f_*^{-1}(\mathcal{B}(U, L))$$

so that $f_*^{-1}(\mathcal{B}(U, L))$ is a neighborhood of V. Thus $f_*^{-1}(\mathcal{B}(U, L))$ is open.

The additional properties that we required in the definition of the spaces $\mathrm{Conf}(\Lambda, \Theta; \mathcal{H})$ are not stable under the pushforward of labels. In order to get an induced map we have to require f to be 'nice'. For example, if $f : (\Lambda, \Theta) \to (\Lambda', \Theta')$ is a proper map between locally compact Hausdorff spaces, we get an induced map $f_* : \mathrm{Conf}(\Lambda, \Theta; \mathcal{H}) \to \mathrm{Conf}(\Lambda', \Theta'; \mathcal{H})$.

5. Quillen categories and their classifying spaces

Fix a $\mathbb{Z}/2$-graded real C_n-module $\mathcal{H}_n$ as in the introduction (and Section 3). Then we can define the following category object $\mathfrak{C}_n$ in TOP, the category of (compactly generated) topological spaces. Libman [L] calls such category objects *internal space categories* and we follow his example. All this means is that the object set and the morphism set of $\mathfrak{C}_n$ are equipped with a (compactly generated) topology and the structure maps are continuous. The objects of $\mathfrak{C}_n$ are finite dimensional graded subspaces of $\mathcal{H}_n$, with the topology given by thinking of such subspaces as projection operators in $B(\mathcal{H}_n)$, just like in Section 4. A morphism from W_1 to W_2 exists if and only if $W_1 \subseteq W_2$ and is in this case given by

$$\mathrm{Mor}_{\mathfrak{C}_n}(W_1, W_2) := \{R \in O(W_2 - W_1) \mid R^* = R = R^{-1}, R^\alpha = -R\}.$$

Here and in the following we use the notation $W_2 - W_1$ for the orthogonal complement of W_1 in W_2 and the above operators R are odd, orthogonal involutions on this complement. These morphism spaces have the subspace topology of the orthogonal group and form the fibres of a locally trivial bundle

$$\mathrm{Mor}_{\mathfrak{C}_n} \longrightarrow \mathrm{Obj}_{\mathfrak{C}_n} \times \mathrm{Obj}_{\mathfrak{C}_n}$$

The usual construction gives a simplicial space, the nerve $N\mathfrak{C}_n$ of the internal space category $\mathfrak{C}_n$, whose geometric realization $|N\mathfrak{C}_n| =: B\mathfrak{C}_n$ is called the classifying space of $\mathfrak{C}_n$. This classifying space is a topological version of Quillen's $S^{-1}S$-construction, see [G] and it is directly related to configuration spaces.

Theorem 27. *There is a homeomorphism* $B\mathfrak{C}_n \approx \mathrm{Conf}_n^{fin}$.

Proof. Let's start with the case $n = 0$, where we suppress the index n all together. Recall that the k-simplices $x \in N_k\mathfrak{C}$ of the nerve of our internal space category are chains of graded finite dimensional subspaces

$$W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k \subset \mathcal{H}$$

together with odd, orthogonal involutions R_i on $W_i - W_{i-1}$ for $i = 1, \ldots, k$. We abbreviate this to $x = (W_i, R_i)$. The classifying space $B\mathfrak{C}$ is the quotient space

$$\Big(\coprod_{k\geq 0} N_k\mathfrak{C} \times \Delta^k\Big) \;/\; (\beta^*(x), t) \sim (x, \beta_*(t)) \quad \forall\, \beta : [m] \to [n]$$

In our context, it is convenient to replace the usual standard simplex with the following definition

$$\Delta^k := \{t = (t_1, \ldots, t_k) \in \bar{\mathbb{R}}^k \;\mid\; 0 \leq t_1 \leq \cdots \leq t_k \leq \infty\}$$

The face map $d_i : [k-1] \to [k]$ induces the map $(d_i)_* : \Delta^{k-1} \hookrightarrow \Delta^k$, given by repeating t_i, for $i = 1, \ldots, k-1$. Moreover, $(d_0)_*$ adds a first coordinate equal to 0 and $(d_k)_*$ adds a last coordinate equal to ∞. The degeneracy maps $s_i : [k] \to [k-1]$ induce $(s_i)_* : \Delta^k \twoheadrightarrow \Delta^{k-1}$, given by skipping t_{i+1} for all $i = 0, \ldots, k-1$.

We can now define a map $F : B\mathfrak{C} \to \mathrm{Inf}_0^{\mathrm{fin}}$ as follows:

$$F[x, t] := \sum_{i=1}^{k} t_i \cdot R_i \quad \forall\, x = (W_i, R_i) \in N_k C, t \in \Delta^k$$

The right hand side denotes by definition the odd, self-adjoint operator with (finite dimensional) domain W_k, kernel W_0 and equal to $t_i \cdot R_i$ on $W_i - W_{i-1}$. Recall from the previous section that such operators are best understood by the homeomorphism $\mathrm{Inf}_n^{\mathrm{fin}} \approx \mathrm{Conf}_n^{\mathrm{fin}}$ that assigns the configuration of eigenspaces and eigenvalues to an operator. We'll use this identification repeatedly in proving the following steps. Note for example that R_i has exactly the eigenvalues ± 1 because $R^2 = \mathrm{id}$ and R is odd. Therefore, the eigenvalues of the above right hand side are exactly $\pm t_i$. We need to show that

1. F is well defined, i.e. $F(\beta^*(x), t) = F(x, \beta_*(t))$ for all $\beta : [m] \to [n]$,
2. F is a bijection,
3. F and its inverse are continuous.

To show (1), we study the face and deneracy operators seperately. Let's start with $\beta = d_0 : [k-1] \to [k]$. Then $\beta^*(x)$ is the chain of subspaces where W_0 and R_1 have been removed (and the indices of the other W_i and R_i are shifted to the left). This means that the kernel of $F(\beta^*(x), t = (t_1, \dots, t_{k-1}))$ is W_1 and it equals $t_i \cdot R_i$ on $W_{i+1} - W_i$. But since $\beta_*(t) = (0, t_1, \dots, t_{k-1})$, one easily checks that this is the same operator as $F(x, \beta_*(t))$.

If $\beta = d_k : [k-1] \to [k]$ then $\beta^*(x)$ is the chain of subspaces where W_k and R_k have been removed. This means that the domain of $F(\beta^*(x), (t_1, \dots, t_{k-1}))$ is W_{k-1} (and R_k becomes irrelevant). But since $\beta_*(t) = (t_1, \dots, t_{k-1}, \infty)$, one easily checks that this is the same operator as $F(x, \beta_*(t))$.

For $i = 1, \dots, k-1$ and $\beta = d_i$, then chain $\beta^*(x)$ is obtained by composing the morphisms R_i and R_{i+1}. This means that on $W_{i+1} - W_{i-1}$ we get an orthogonal sum of these two operators. But since $\beta_*(t)$ just repeats t_i, it is clear that $F(\beta^*(x), t) = F(x, \beta_*(t))$ in this case.

For a deneracy map $\beta = s_i : [k] \to [k-1]$, where $i = 0, \dots, k-1$, the argument is even easier. Then for a $(k-1)$-simplex x, we get a chain $\beta^*(x)$ of length k by inserting the identity at the i-th subspace. This operation does not alter the operators R_j (the identity corresponds to $R = 0$ on a 0-space), it only shifts the indices $> i$ to the right. Similarly, $\beta_*(t_1, \dots, t_k) = (t_1, \dots, \hat{t}_i, \dots, t_k)$, so that again a shifting of indices $> i$ to the right occurs and $F(\beta^*(x), t) = F(x, \beta_*(t))$ follows.

For (2), we first show injectivity. Since the set $B\mathfrak{C}$ is the disjoint union of *open* k-cells e^k_x, one for each k-simplex x, it suffices to show that points in these open k-cells are mapped to distinct operators. Being in the open k-cell means that $(t_1, \dots, t_k) \in \Delta^k$ satisfies $0 < t_1 < \cdots < t_k < \infty$. Since $\pm t_i$ are exactly the eigenvalues of $F(x, t)$ it follows that the value of k and the values of t_i can be read off from knowing $F(x, t)$. Morever, the $\pm t_i$ eigenspaces together form a graded subspace of $\mathcal{H}$ which means that we can also read off $W_i - W_{i+1}$. Together with knowing the kernel W_0 and domain W_k this reconstructs also the whole chain x and proves that F is injective.

To show that F is surjective, take any operator $D \in \mathrm{Inf}_0^{\mathrm{fin}}$. Then D corresponds to a finite configuration V_λ of eigenspaces and eigenvalues $\lambda \in \mathbb{R}$. Moreover, $V_{-\lambda} = V_\lambda^\alpha$ because D is odd. Order the *positive* eigenvalues as $t_1, \dots, t_k$ and define

$$W_0 := V_0,\ W_1 := V_0 + V_1, \dots,\ W_k := V_0 + V_1 + \cdots + V_k = \mathrm{dom}(D)$$

We see that $W_i - W_{i-1} = V_i$ and hence we can define $R_i := D/t_i$ on this eigenspace. It is clear that this defines a k-simplex x such that $F(x, t) = D$.

The fact that F is a homeomorphism, statement (3) above, follows from the following two lemmas and the cases $n > 0$ are left to the reader. $\square$

Lemma 28. *Denote by* $\mathrm{Conf}_n^{(k)}$ *the subspace of configurations* $V \in \mathrm{Conf}_n^{fin}$ *such that* $V_\lambda \neq 0$ *for at most k eigenvalues* $\lambda \in \mathbb{R}_{\geq 0}$. *Then*

$$\mathrm{Conf}_n^{fin} = \operatorname*{colim}_{k\to\infty} \mathrm{Conf}_n^{(k)}.$$

Proof. We have to prove that $\mathrm{Conf}_n^{\mathrm{fin}}$ carries the colimit topology, i.e. that $U \subset \mathrm{Conf}_n^{\mathrm{fin}}$ is open if and only if $U \cap \mathrm{Conf}_n^{(k)}$ is open in $\mathrm{Conf}_n^{(k)}$ for all k. 'Only if' is obvious. Conversely, assume $U \cap \mathrm{Conf}_n^{(k)}$ is open for all k. Let $V \in U$ and $\kappa := \mathrm{rank}(V_{\mathbb{R}}) = \sum_{\lambda\in\mathbb{R}} \mathrm{rank}(V_\lambda)$. By assumption, there exists a neighborhood $V_{K,\delta,\epsilon}$ of V such that $U \cap V_{K,\delta,\epsilon} \subset U \cap \mathrm{Conf}_n^{(\kappa)}$. We may assume $\varepsilon < \frac{1}{2}$ so that $\mathrm{rank}(W_{[-K,K]}) = \mathrm{rank}(V_{[-K,K]}) \leq \kappa$ for all $W \in V_{K,\delta,\epsilon}$.

We claim that $U \cap V_{K,\delta,\epsilon} \subset U \cap \mathrm{Conf}_n^{(k)}$ for all k. Namely, if there was a $W \in U \cap V_{K,\delta,\epsilon}$ not in $\mathrm{Conf}_n^{(k)}$ for some $k > \kappa$, then there would also be a $W'_\lambda \in \mathrm{Conf}_n^{(\kappa)}$ that does not lie in U: simply define $W'_\lambda = W_\lambda$ on $[-K, K]$ and $W'_\infty = 1 - W_{[-K,K]}$.

We conclude $U \cap V_{K,\delta,\epsilon} \subset U$ and hence $U \subset \mathrm{Conf}_n^{\mathrm{fin}}$ is open. □

Lemma 29. *Let* $B\mathfrak{C}^{(k)}$ *be the image of* $\coprod_{i=0}^k N_i\mathfrak{C} \times \Delta^i$ *in* $B\mathfrak{C}$. *Then for all k, F restricts to a homeomorphism*

$$B\mathfrak{C}^{(k)} \xrightarrow{\approx} \mathrm{Conf}_n^{(k)}.$$

6. Super semigroups of operators

In this section we will define super semigroups of operators (SGOs) using as little super mathematics as possible. We will only need basic definitions and results from the theory of supermanifolds, as can be found in Chapter 2 of [DM]. Super manifolds are particular ringed spaces, i.e. topological spaces together with a sheaf of rings, and morphisms are maps of ringed spaces. The local model for a supermanifold of dimension $(p|q)$ is Euclidian space $\mathbb{R}^p$ equipped with the sheaf of commutative super $\mathbb{R}$-algebras

$$U \mapsto C^\infty(U) \otimes \Lambda^*(\mathbb{R}^q).$$

This ringed space is the supermanifold $\mathbb{R}^{p|q}$.

Definition 30. A *supermanifold M* of dimension $(p|q)$ is a pair $(|M|, O_M)$ consisting of a (Hausdorff and second countable) topological space $|M|$ together with a sheaf of commutative super $\mathbb{R}$-algebras O_M that is locally isomorphic to $\mathbb{R}^{p|q}$.

To every supermanifold M there is an associated *reduced manifold*

$$M^{red} := (|M|, \mathcal{O}_M/\text{nil})$$

obtained by dividing out nilpotent functions. By construction, this gives a smooth manifold structure on the underlying topological space $|M|$ and there is an inclusion of supermanifolds $M^{red} \hookrightarrow M$. For example $(\mathbb{R}^{p|q})^{red} = \mathbb{R}^p$.

The main invariant of a supermanifold M is its ring of functions $C^\infty(M)$, defined as the global sections of the sheaf $\mathcal{O}_M$. For example, $C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \Lambda^*(\mathbb{R}^q)$. It turns out that the maps between supermanifolds M and N are just given by grading preserving algebra homomorphisms between the rings of functions:

$$\text{Hom}(M, N) \cong \text{Hom}_{Alg}(C^\infty(N), C^\infty(M))$$

Example 31. Let $E \to M$ be a real vector bundle of fiber dimension q over the smooth manifold M^p. Then $(M, \Gamma(\Lambda^* E))$ is a supermanifold of dimension $(p|q)$. Bachelor's theorem says that every supermanifold is isomorphic (but not canonically) to one of this type. This result does not hold in analytic categories, and it shows that in the smooth category, supermanifolds are only interesting when one takes their morphisms seriously and doesn't just consider isomorphism classes.

The twisted super group $\mathbb{R}^{1|1}$. Define the 'twisted' super Lie group structure on $\mathbb{R}^{1|1}$ by

$$\mu : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \longrightarrow \mathbb{R}^{1|1}, \ (t, \theta), (s, \eta) \mapsto (t + s + \theta\eta, \theta + \eta).$$

This super Lie group plays a special role in super geometry, the reason being the particular structure of its super Lie algebra: $\text{Lie}(\mathbb{R}^{1|1}) \cong \mathbb{R}[D]$ is *the* super Lie algebra generated freely by one *odd* generator D. *Thus,* $\mathbb{R}^{1|1}$ *may be considered the odd analogue of the Lie group* $\mathbb{R}$. For example, integrating an odd vector field on a supermanifold M leads to a flow $M \times \mathbb{R}^{1|1} \to M$, and formulating the flow property involves the 'twisted' group structure.

From the definition of μ it is clear that the open sub supermanifold $\mathbb{R}^{1|1}_{>0}$ defined by the inclusion $\mathbb{R}_{>0} \subset \mathbb{R}$ inherits the structure of a super semigroup.[3] Now we can already guess what a SGO should be: just as an ordinary semigroup of operators is a homomorphism from $\mathbb{R}_{>0}$ to an algebra of operators, a super semigroup of operators will be a homomorphism from the super semigroup $\mathbb{R}^{1|1}_{>0}$ to a ($\mathbb{Z}_2$-graded) operator algebra. In order to make sense of such a homomorphism, we will consider the latter to be a generalized super semigroup using the 'functor of points' formalism (see

[3] A *super (Lie) semigroup* is a supermanifold M together with an associative multiplication $M \times M \to M$. In terms of the functor of points language: the morphism sets $\text{Hom}(S, M)$ carry semigroup structures, functorially in S.

[DM], §§2.8-2.9). Note that we, implicitly, already used the 'functor of points' language when writing down the group law μ. The formula above tells us what the product of two elements in the group $\mathrm{Hom}(S, \mathbb{R}^{1|1})$ is. Since the rule holds functorially for all supermanifolds S, this defines the map μ by the Yoneda lemma.

Finally, we would like to remark that the structure of $\mathrm{Lie}(\mathbb{R}^{1|1})$ and the existence of an *odd* infinitesimal generator $\mathcal{D}$ for a SGO Φ, are closely related: $\mathcal{D}$ is nothing but the image of D under the derivative of Φ. However, making this precise requires some work (note that Φ maps to an infinite-dimensional space!). We will avoid such problems altogether: the super Lie algebras do not appear in our argument.

Generalized supermanifolds and super Lie groups. We will use the following, somewhat primitive, extension of the notion of supermanifolds:

Definition 32. A *generalized supermanifold* M is a contravariant functor from supermanifolds to sets.[4] Similarly, if M takes values in the category of (semi)groups, we call it a *generalized super (semi)group*. Morphisms in all these categories are natural transformations.

Examples 33. (1) The Yoneda lemma implies that supermanifolds are embedded as a full subcategory in generalized supermanifolds by associating to a supermanifold M the functor

$$S \mapsto M(S) := \mathrm{Hom}(S, M).$$

The analogous statement holds for super (semi)groups. For example, we will consider $\mathbb{R}^{1|1}_{>0}$ as a generalized super semigroup by identifying it with the contravariant functor

$$S \mapsto \mathrm{Hom}(S, \mathbb{R}^{1|1}_{>0})$$

from supermanifolds to semigroups.
(2) Every $\mathbb{Z}_2$-graded real Banach space $B = B_0 \oplus B_1$ may be considered as a generalized supermanifold as follows. We define the value of the functor B on a super domain $U = (|U|, C^\infty(\_)[\theta_1, ..., \theta_q]) \subset \mathbb{R}^{p|q}$ to be

$$B(U) := (C^\infty(|U|, B)[\theta_1, ..., \theta_q])^{ev}.$$

The superscript *ev* indicates that we pick out the even elements, so that an element $f \in B(U)$ is of the form

$$f = \sum_I f_I \theta^I$$

[4] We use this simple notion here in order to avoid dealing with infinite-dimensional supermanifolds.

where $I \subset \{1, ..., q\}$ and $\theta^I := \prod_{j \in I} \theta_j$ and each f_I is a smooth map $|U| \to B_{|I|}$. For a map $\varphi : U' \to U$ between super domains, the map $B(\varphi)$ is defined using the formal Taylor expansion, just as in the case of usual supermanifolds. This functor on super domains may be extended to the whole category of supermanifolds by gluing.
(3) If B is a $\mathbb{Z}_2$-graded Banach algebra, $B(U)$ is an algebra and thus B is a generalized super semigroup. Again, B may be extended to all supermanifolds by gluing.

Remark 34. Giving a morphism from an ordinary supermanifold T to a generalized supermanifold B amounts to prescribing the image of the universal element id $\in \mathrm{Hom}(T, T)$ in $B(T)$. Hence $B(T)$ is exactly the set of morphisms from T to B.

Now assume that, in addition, T and B carry super (semi)group structures. A map $\Phi : T \to B$ is a homomorphism if

$$\begin{tikzcd} \mathrm{Hom}(S,T) \times \mathrm{Hom}(S,T) \arrow[r] \arrow[d, "\Phi\times\Phi"] & \mathrm{Hom}(S,T) \arrow[d, "\Phi"] \\ B(S) \times B(S) \arrow[r] & B(S). \end{tikzcd}$$

commutes for all supermanifolds S. Again, it suffices to check the commutativity for the universal element

$$pr_1 \times pr_2 \in \mathrm{Hom}(T \times T, T) \times \mathrm{Hom}(T \times T, T).$$

Definition 35. Let $\mathcal{H}$ be a $\mathbb{Z}_2$-graded Hilbert space, and denote by $B(\mathcal{H})$ the Banach algebra of bounded operators on $\mathcal{H}$ equipped with the $\mathbb{Z}_2$-grading inherited from $\mathcal{H}$.
(1) A *super semigroup of operators* on $\mathcal{H}$ is a morphism of generalized super semigroups

$$\Phi : \mathbb{R}^{1|1}_{>0} \longrightarrow B(\mathcal{H}).$$

As explained in the previous remark, Φ is of the form $A + \theta B$, where

$$A : \mathbb{R}_{>0} \to B^{ev}(\mathcal{H}) \text{ and } B : \mathbb{R}_{>0} \to B^{odd}(\mathcal{H})$$

are smooth maps. The homomorphism property amounts to certain relations between A and B (cf. the proof of Proposition 37).
(2) If $K \subset B(\mathcal{H})$ is a subset, we say Φ is a *super semigroup of operators with values in* K if the images of A and B are contained in K.
(3) If $\mathcal{H}$ is a module over the Clifford algebra C_n, we say Φ is C_n*-linear* if it takes values in C_n-linear operators.

Examples 36. SGOs arise in a natural way from Dirac operators. We give two examples of that type and then extract their characteristic properties to describe a more general class of examples. The verification of the SGO properties for these more general examples also includes the case of Dirac operators.
(1) Let $\mathcal{D}$ be the Dirac operator on a closed spin manifold X. There is a corresponding SGO on the Hilbert space of L^2-sections of the spinor bundle S over X. It is given by the super semigroup of operators

$$\mathbb{R}^{1|1}_{>0} \longrightarrow B(L^2(S)),\ (t,\theta) \mapsto e^{-t\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}\ (= e^{-t\mathcal{D}^2+\theta\mathcal{D}})$$

and takes values in the compact, self-adjoint operators $K^{sa}(L^2(S)) \subset B(L^2(S))$.
(2) If $\dim X = n$, one can consider the C_n-linear spinor bundle and the associated C_n-linear Dirac operator (see [LM], chapter 2, §7). Using the same formula as in the previous example one obtains a C_n-linear SGO.
(3) Now, let $\mathcal{H}$ be any $\mathbb{Z}_2$-graded Hilbert space. For any closed subspace $V_\infty \subset \mathcal{H}$ invariant under the grading involution and any odd, self-adjoint operator $\mathcal{D}$ on $V_\infty^\perp$ with compact resolvent, there is a unique super semigroup of self-adjoint, compact operators $\Phi = A + \theta B$ defined (using functional calculus) by

$$A(t) = e^{-t\mathcal{D}^2} \text{ and } B(t) = \mathcal{D}e^{-t\mathcal{D}^2} \text{ on } V_\infty^\perp$$

and $A(t) = B(t) = 0$ on V_∞. The first thing to check is that the maps A and B are indeed smooth; this follows easily using the fact that the map $\mathbb{R}_{>0} \to C_0(\mathbb{R})$, $t \mapsto e^{-tx^2}$, is smooth. Since $\mathcal{D}$ is self-adjoint, the same holds for A and B. Finally, we have to show that Φ is a homomorphism. Let t, θ, s, η be the usual coordinates on $\mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}_{>0}$. It suffices to consider the universal element $pr_1 \times pr_2 = (t,\theta) \times (s,\eta)$. The computation, which, of course, heavily uses that odd coordinates θ and η square to zero, goes as follows (cf. [ST], page 38):

$$\begin{aligned}
& \Phi(t,\theta)\Phi(s,\eta) \\
= \ & (e^{-t\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2})(e^{-s\mathcal{D}^2} + \eta\mathcal{D}e^{-s\mathcal{D}^2}) \\
= \ & e^{-t\mathcal{D}^2}e^{-s\mathcal{D}^2} + e^{-t\mathcal{D}^2}\eta\mathcal{D}e^{-s\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}e^{-s\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}\eta\mathcal{D}e^{-s\mathcal{D}^2} \\
= \ & e^{-(t+s)\mathcal{D}^2} + (\theta+\eta)\mathcal{D}e^{-(t+s)\mathcal{D}^2} + \theta\mathcal{D}\eta\mathcal{D}e^{-(t+s)\mathcal{D}^2} \\
= \ & (1 - \theta\eta\mathcal{D}^2)e^{-(t+s)\mathcal{D}^2} + (\theta+\eta)\mathcal{D}e^{-(t+s)\mathcal{D}^2} \\
= \ & e^{-(t+s+\theta\eta)\mathcal{D}^2} + (\theta+\eta)\mathcal{D}e^{-(t+s+\theta\eta)\mathcal{D}^2} \\
= \ & \Phi(t+s+\theta\eta, \theta+\eta)
\end{aligned}$$

The second to last equality uses the typical Taylor expansion in super geometry.

We call $\mathcal{D}$ the *infinitesimal generator* of Φ. We will see presently that every super semigroup of self-adjoint, compact operators has a unique infinitesimal generator and is hence one of our examples. Note that if V_∞ is a C_n-submodule and if $\mathcal{D}$ is C_n-linear, then A and B will also be C_n-linear.

Next, we will construct infinitesimal generators for super semigroups of operators. We restrict ourselves to the compact, self-adjoint case, which makes the proof an easy application of the spectral theorem for compact, self-adjoint operators. However, invoking the usual theory of semigroups of operators it should not be too difficult to prove the result for more general SGOs.

Proposition 37. *Every super semigroup Φ of compact, self-adjoint operators on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ has a unique infinitesimal generator $\mathcal{D}$ as in Example 3 above and is hence of the form*

$$\Phi(t,\theta) = e^{-t\mathcal{D}^2} + \theta\mathcal{D}e^{-t\mathcal{D}^2}.$$

If Φ is C_n-linear, so is $\mathcal{D}$.

We need the following technical lemma:

Lemma 38. *Let $A, B : \mathbb{R}_{>0} \to K^{sa}(H)$ be smooth families of self-adjoint, compact operators on the Hilbert space H, and assume that the following relations hold for all $s, t > 0$:*

$$A(s+t) = A(s)A(t) \tag{1}$$

$$B(s+t) = A(s)B(t) = B(s)A(t) \tag{2}$$

$$A'(s+t) = -B(s)B(t). \tag{3}$$

Then H decomposes uniquely into orthogonal subspaces H_λ, $\lambda \in \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, such that on H_λ

$$A(t) = e^{-t\lambda^2} \text{ and } B(t) = \lambda e^{-t\lambda^2}$$

(where we set $e^{-\infty} = 0$, $\infty \cdot e^{-\infty} = 0$). For $\lambda \in \mathbb{R}$, the dimension of H_λ is finite. Furthermore, the subset of λ in $\mathbb{R}$ with $H_\lambda \neq 0$ is discrete.

Proof. The identities (1) – (3) above show that all operators $A(s)$, $B(t)$ commute. We apply the spectral theorem for self-adjoint, compact operators to obtain a decomposition of H into simultaneous eigenspaces H_λ of the $A(s)$ and $B(t)$; the label λ takes values in $\bar{\mathbb{R}}$ and will be explained presently. We define functions $A_\lambda, B_\lambda : \mathbb{R}_{>0} \to \mathbb{R}$ by

$$A(t)x = A_\lambda(t)x \text{ and } B(t)x = B_\lambda(t)x \text{ for all } x \in H_\lambda.$$

Clearly, A_λ and B_λ are smooth and satisfy the same relations as A and B.

From (1) we see that A_λ is non-negative, and (3) shows $A'_\lambda \leq 0$, hence A_λ is decreasing. On the other hand, (1) implies $A_\lambda(\frac{1}{n}) = \sqrt[n]{A_\lambda(1)}$, so that

$$A_\lambda(0) := \lim_{t\to 0} A_\lambda(t) \text{ exists and equals 0 or 1.}$$

In the first case we conclude $A_\lambda \equiv 0$ and thus also $B_\lambda \equiv 0$; the label of the corresponding subspace is $\lambda = \infty$. In the second case, we have $A_\lambda(1) \neq 0$ and using (1) again we compute

$$A'_\lambda(s) = \frac{A_\lambda(1)}{A_\lambda(1)} \lim_{t\to 0} \frac{A_\lambda(s+t) - A_\lambda(s)}{t} = \frac{A_\lambda(s)}{A_\lambda(1)} \lim_{t\to 0} \frac{A_\lambda(1+t) - A_\lambda(1)}{t} = -\lambda^2 A_\lambda(s),$$

where $\lambda^2 := -A'_\lambda(1)/A_\lambda(1)$ defines the label λ up to choice of a sign. By uniqueness of solutions of ODEs, we must have

$$A_\lambda(t) = e^{-t\lambda^2}.$$

Finally, (3) gives

$$B_\lambda(t) = \lambda e^{-t\lambda^2},$$

picking the appropriate sign for λ. □

Proof of Propsition 37. Let $\Phi = A + \theta B$ be a super semigroup of compact, self-adjoint operators. As before, we consider $U = \mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}_{>0}$ with coordinates t, θ, s, η. For the universal element $pr_1 \times pr_2 = (t,\theta) \times (s,\eta)$ the homomorphism property of Φ gives that

$$\begin{aligned}
\Phi(t+s+\theta\eta, \theta+\eta) &= A(t+s+\theta\eta) + (\theta+\eta)B(t+s+\theta\eta) \\
&= A(t+s) + A'(t+s)\theta\eta + (\theta+\eta)(B(t+s) + B'(t+s)\theta\eta) \\
&= A(t+s) + \theta B(t+s) + \eta B(t+s) + \theta\eta A'(t+s)
\end{aligned}$$

equals

$$\begin{aligned}
\Phi(t,\theta)\Phi(s,\eta) &= (A(t) + \theta B(t))(A(s) + \eta B(s)) \\
&= A(t)A(s) + \theta B(t)A(s) + \eta A(t)B(s) - \theta\eta B(t)B(s).
\end{aligned}$$

Comparing the coefficients[5] yields exactly the relations in Lemma 38. Using the corresponding decompostion of $\mathcal{H}$ into subspaces $\mathcal{H}_\lambda$ we define the operator $\mathcal{D}$ by letting $\mathcal{D} = \lambda$ on $\mathcal{H}_\lambda$. From the construction it is clear that $\mathcal{D}$ is the desired infinitesimal generator. Since A is even and B is odd, it follows that $\mathcal{D}$ is an odd operator. If Φ is C_n-linear, so is $\mathcal{D}$. □

[5]Just to make the formal aspect of this computation clearer, we would like to point out that the considered identity is an equation in the algebra $K^{sa}(\mathcal{H})(\mathbb{R}^{1|1}_{>0} \times \mathbb{R}^{1|1}_{>0}) = C^\infty(\mathbb{R}_{>0} \times \mathbb{R}_{>0}, K^{sa}(\mathcal{H}))[\theta,\eta]^{ev}$.

7. Completions by compact operators

Let $\mathcal{H}$ be a $\mathbb{Z}_2$-graded Hilbert space and $C \subset B(\mathcal{H})$ a subspace of the algebra of bounded operators on $\mathcal{H}$. We denote by $\mathcal{SGO}(C)$ the set of super semigroups of operators with values in C and in particular

$$\mathcal{SGO}_n := \mathcal{SGO}(\mathcal{K}_n) \quad \text{and} \quad \mathcal{SGO}_n^{\text{fin}} := \mathcal{SGO}(\mathcal{FR}_n)$$

We endow $\mathcal{SGO}(C)$ with the topology of uniform convergence on compact subsets, i.e.

$$\Phi_n = A_n + \theta B_n \longrightarrow \Phi = A + \theta B$$

if and only if for all compact $K \subset \mathbb{R}_{>0}$ we have

$$A_n(t) \longrightarrow A(t) \text{ and } B_n(t) \longrightarrow B(t) \text{ uniformly on } K$$

with respect to the operator norm on $B(\mathcal{H})$.

We have a triangle (and an analogous one for finite rank operators)

$$\begin{tikzcd}
\mathcal{SGO}_n \arrow[dr, "I"'] & & \mathrm{Hom}_{gr}(\mathcal{S}, \mathcal{K}_n) \arrow[ll, "R"'] \\
& \mathrm{Conf}_n \arrow[ur, "F"'] &
\end{tikzcd}$$

Here I maps a super semigroup of operators to its infinitesimal generator, F is given by functional calculus,

$$F(\mathcal{D})(f) := f(\mathcal{D}),$$

and R is given by

$$R(\varphi) := \varphi(e^{-tx^2}) + \theta\varphi(xe^{-tx^2}).$$

Lemma 39. *The maps I, F, and R are homeomorphims, and similarly for finite rank operators.*

Proof. From the previous discussion it is clear that the composition of the three arrows is the identity no matter where in the triangle we start. We already know from Lemma 25 that F is a homeomorphism. We complete the proof by showing that R is a homeomorphism.

The continuity of R^{-1} follows from the following assertion. We claim that we have convergences of operators

$$f(\mathcal{D}_n) \longrightarrow f(\mathcal{D}) \text{ for all } f \in C_0(\mathbb{R})$$

if and only if the two sequences converge:

$$e^{-\mathcal{D}_n^2} \longrightarrow e^{-\mathcal{D}^2} \text{ and } \mathcal{D}_n e^{-\mathcal{D}_n^2} \longrightarrow \mathcal{D}e^{-\mathcal{D}^2}.$$

The first obviously implies the second condition. To see the converse, note that the assumption implies that $f(\mathcal{D}_n) \to f(D)$ for all f that can be written as a polynomial in the functions e^{-x^2} and xe^{-x^2}. Furthermore, since e^{-x^2} and

xe^{-x^2} generate $C_0(\mathbb{R})$ as a C^*-algebra, cf. [HG], Remark 1.4, the set of such f is dense in $C_0(\mathbb{R})$. Using that $\|f(\mathcal{D})\| \leq \|f\|$ for all $\mathcal{D}$ and the triangle inequality we can deduce that $f(\mathcal{D}_n) \to f(D)$ holds for all $f \in C_0(\mathbb{R})$.

The continuity of R amounts to showing that if $f(\mathcal{D}_n) \to f(\mathcal{D})$ for all f, then $e^{-t\mathcal{D}_n^2} \to e^{-t\mathcal{D}^2}$ and $\mathcal{D}_n e^{-t\mathcal{D}_n^2} \to \mathcal{D}e^{-t\mathcal{D}^2}$ *uniformly* for all t in a compact subset $K \subset \mathbb{R}_{>0}$. As before, we can use $\|f(\mathcal{D})\| \leq \|f\|$ and the triangle inequality to see that for a given $\varepsilon > 0$ we can find N such that we do not only have $\|f(\mathcal{D}_n) - f(\mathcal{D})\| \leq \varepsilon$ for all $n \geq N$, but that this estimate also holds for all g in a small neighborhood of f. This together with the compactness of K and the continuity of the maps $t \mapsto e^{-tx^2}$ and $t \mapsto xe^{-tx^2}$ implies the claim. □

Remark 40. The arguments in the last parts of the proof can be used to show that we could also have equipped $\mathcal{SGO}_n$ with the topology that controls *all* derivatives of a super semigroup map Φ and still would have obtained the same topological space. We find this interesting, because this is the topology that one usually considers on spaces of smooth maps.

8. Euclidean field theories

The following preliminary definition is motivated by [ST].

Definition 41. Let $\mathcal{H}$ be a $\mathbb{Z}_2$-graded Hilbert space.

(1) A *supersymmetric Euclidian field theory E of dimension* $(1|1)$ *based on* $\mathcal{H}$ is a super semigroup of operators on $\mathcal{H}$ that takes values in self-adjoint Hilbert-Schmidt operators $\mathrm{HS}^{sa}(\mathcal{H})$.
(2) If $\mathcal{H}$ is a graded C_n-module and E is C_n-linear, we say that E has *degree n*.
(3) For all $n \in \mathbb{Z}$, we denote by $\mathcal{EFT}_n \subset \mathcal{SGO}_n$ the subspace of EFTs of degree n.

Examples 42. The examples of SGOs arising from infinitesimal generators $\mathcal{D}$ define susy EFTs if the eigenvalues of $\mathcal{D}$ converge to infinity sufficiently fast. This is, for example, the case for Dirac operators on closed spin manifolds, cf. [LM], Chapter 3, §5.

Proposition 43. *For all $n \geq 1$, we have homotopy equivalences*

$$\mathcal{EFT}_n \simeq \mathcal{SGO}_n^{fin} \simeq \mathcal{M}_n$$

Proof. We obtain a diagram

$$\begin{tikzcd} \mathcal{SGO}_n^{\mathrm{fin}} \arrow[r] & \mathcal{EFT}_n \arrow[r] & \mathcal{SGO}_n \\ \mathrm{Conf}_n^{\mathrm{fin}} \arrow[u, "\approx"] \arrow[d, "\approx"] \arrow[rr] & & \mathrm{Conf}_n \arrow[u, "\approx"] \arrow[d, "\approx"] \\ \mathcal{M}_n \arrow[rr] & & \bar{\mathcal{M}}_n. \end{tikzcd}$$

We will see that the horizontal arrow in the middle is a homotopy equivalence. Thus the same is true for the top and bottom rows. In fact, the homotopy involved preserves the subspace $\mathcal{EFT}_n \subset \mathcal{SGO}_n$ which implies that $\mathcal{SGO}_n^{\mathrm{fin}} \hookrightarrow \mathcal{EFT}_n$ is a homotopy equivalence and hence the result. Now, consider the family of maps $h_t : \bar{\mathbb{R}} \longrightarrow \bar{\mathbb{R}}$ defined by

$$h_t(x) := \begin{cases} \frac{x}{1-t|x|} & \text{if } x \in (-\frac{1}{t}, \frac{1}{t}) \\ \infty & \text{else.} \end{cases}$$

These induce a homotopy

$$H_t := (h_t)_* : \mathrm{Conf}_n \longrightarrow \mathrm{Conf}_n$$

from the identity on Conf_n to H_1 whose image is $\mathrm{Conf}_n^{\mathrm{fin}}$. Thus, we see that the inclusion $\iota : \mathrm{Conf}_n^{\mathrm{fin}} \hookrightarrow \mathrm{Conf}_n$ is a homotopy equivalence with homotopy inverse H_1. From the construction it is clear that the H_t define a homotopy on $\mathcal{SGO}_n$ that preserves the subspace $\mathcal{EFT}_n$. Hence $H_1|_{\mathcal{EFT}_n}$ is a homotopy inverse to the inclusion $\mathcal{SGO}_n^{\mathrm{fin}} \hookrightarrow \mathcal{EFT}_n$. In fact, this argument works for all spaces that lie between $\mathcal{SGO}_n^{\mathrm{fin}}$ and $\mathcal{SGO}_n$. □

9. Spaces of Fredholm operators

In this chapter we relate the spaces Conf_n to the spaces of skew-adjoint Fredholm operators considered by Atiyah and Singer in [AS].

Fredholm operators. Recall that a *Fredholm operator* is a bounded operator whose kernel and cokernel are finite dimensional. Let $\mathrm{Fred}(H) \subset B(H)$ be the subspace of Fredholm operators on the infinite dimensional separable real Hilbert space H. Denote by $C(H) := B(H)/K(H)$ the C^*-algebra of bounded operators modulo compact operators (a.k.a. *Calkin algebra*) and by $\pi : B(H) \to C(H)$ the projection. Then $\mathrm{Fred}(H)$ is precisely the preimage of the units in $C(H)$ under π, i.e. we have

$$T \in B(H) \text{ is Fredholm} \iff \pi(T) \in C(H) \text{ is invertible.}$$

We will need the following facts about the spectrum $\sigma(T)$ of a self-adjoint bounded operator T. Let $\sigma_{\mathrm{ess}}(T) := \sigma(\pi(T))$ be the *essential spectrum* of T, i.e. the spectrum of $\pi(T)$ in $C(H)$. Then there is a decomposition

$$\sigma(T) = \sigma_{\mathrm{ess}}(T) \amalg \sigma_{\mathrm{discrete}}(T),$$

where $\sigma_{\mathrm{discrete}}(T)$ consists precisely of the isolated points in $\sigma(T)$ such that the corresponding eigenspace has finite dimension. From the definition of the essential spectrum it is clear that

$$\sigma_{\mathrm{ess}}(T) \cap (-\varepsilon(T), \varepsilon(T)) = \emptyset \quad \text{for} \quad \varepsilon(T) := \|\pi(T)^{-1}\|_{C(H)}^{-1},$$

where $\|.\|_{C(H)}$ is the C^*-norm on the Calkin algebra. In other words: the essential spectrum of T has a gap of size $\varepsilon(T)$ around 0. Note that the map $\varepsilon : \mathrm{Fred}(H) \to \mathbb{R}_{>0}$ is continuous.

K-theory and Fredholm operators. The most important invariant of a Fredholm operator T is its *index*

$$\mathrm{index}(T) := \dim(\ker T) - \dim(\mathrm{coker}\, T).$$

It turns out that the index is invariant under deformations, i.e. it is a locally constant function on $\mathrm{Fred}(H)$. More precisely, it defines an isomorphism

$$\pi_0 \,\mathrm{Fred}(H) \xrightarrow{\cong} \mathbb{Z}, \;\; [T] \mapsto \mathrm{index}(T).$$

This is a special case of the well-known result that $\mathrm{Fred}(H)$ is a classifying space for the real K-theory functor KO^0. More explicitly, for all compact spaces X there are natural isomorphisms

$$KO^0(X) \cong [X, \mathrm{Fred}(H)].$$

The isomorphism is defined as follows. Consider $[f] \in [X, \mathrm{Fred}(H)]$. Changing f by a homotopy one can achieve that the dimensions of the kernel and the cokernel of $f(x)$ are locally constant. This implies that they define vector bundles $\ker f$ and $\mathrm{coker}\, f$ over X. The image of $[f]$ is defined to be

$$[\ker f] - [\mathrm{coker}\, f] \in KO^0(X).$$

For $X = pt$ this reduces to the above isomorphism

$$\pi_0 \,\mathrm{Fred}(H) \cong [pt, \mathrm{Fred}(H)] \xrightarrow{\cong} KO^0(pt) \cong \mathbb{Z}.$$

Atiyah and Singer showed that the other spaces in the Ω-spectrum representing real K-theory can also be realized as spaces of Fredholm operators.

The Atiyah-Singer spaces $\mathcal{F}_n$. Let $n \geq 1$ and let H_n be a real Hilbert space with an action of C_{n-1}, just as before. Define

$$\widetilde{\mathcal{F}}_n := \{ T_0 \in \mathrm{Fred}(H_n) \mid T_0^* = -T_0 \text{ and } T_0 e_i = -e_i T_0 \text{ for } i = 1 \ldots, n-1 \}.$$

Furthermore, let $\mathcal{F}_n := \widetilde{\mathcal{F}}_n$ if $n \not\equiv 3$ (4). In the case $n \equiv 3$ (4) define $\mathcal{F}_n \subset \widetilde{\mathcal{F}}_n$ to be the subspace of operators T_0 satisfying the following additional condition (AS): the essential spectrum of the self-adjoint operator $e_1 \cdots e_{n-1} T_0$ contains positive *and* negative values ("$e_1 \cdots e_{n-1} T_0$ is neither essentially positive nor negative"). Atiyah and Singer introduce this condition, because it turns out that for $n \equiv 3$ (4) the space $\widetilde{\mathcal{F}}_n$ has three connected components two of which are contractible. However, for the relation with K-theory only the third component, whose elements are characterized by the above requirement on the essential spectrum of $e_1 ... e_{n-1} T_0$, is interesting. In fact, the main result of [AS] is that for all $n \geq 1$ the space $\mathcal{F}_n$ represents the functor KO^{-n}. We shall reprove this result in terms of our configuration spaces.

The elements in $\widetilde{\mathcal{F}}_n$ can also be interpreted as odd operators on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}_n = H_n \otimes_{C_{n-1}} C_n$. If we define

$$\widetilde{\mathcal{F}}_n^{gr} := \{ T \in \mathrm{Fred}(\mathcal{H}_n) \mid T \text{ is odd, } C_n\text{-linear, and self-adjoint} \}$$

we can identify $\widetilde{\mathcal{F}}_n$ and $\widetilde{\mathcal{F}}_n^{gr}$ using the homeomorphism

$$\psi_{\otimes e_n} : \widetilde{\mathcal{F}}_n \xrightarrow{\approx} \widetilde{\mathcal{F}}_n^{gr},\ T_0 \mapsto T := T_0 \otimes e_n.$$

In terms of the decomposition $\mathcal{H}_n \cong H_n \oplus H_n$ one can express T as

$$T = T_0 \otimes e_n \qquad \text{or equivalently} \qquad T = \begin{pmatrix} 0 & T_0^* \\ T_0 & 0 \end{pmatrix}$$

It is important to note that the skew-symmetry of T_0 is equivalent to the relation $Te_n = e_n T$. This correspondence actually extends to the well known case $n = 0$ where one starts with all Fredholm operators on H_0 and gets all odd, self-adjoint Fredholm operators on $H_0 \oplus H_0$.

Lemma 44. *Let $n \equiv 3$ (4) and recall from* [LM, Prop.5.9] *that in this case the action of the 'volume element' $e := e_1 \cdots e_n \in C_n$ distinguishes the two distinct (ungraded) irreducible C_n-modules. (Since e is a central orthogonal involution, it acts as* $\pm$ id *on these modules.)*

1. *The Atiyah-Singer condition (AS) that the operator $e_1 \cdots e_{n-1} T_0 : H_n \to H_n$ is neither essentially positive nor negative is equivalent to the same condition on $eT : \mathcal{H}_n \to \mathcal{H}_n$.*
2. *Let $W_\pm$ denote the Hilbert sum of the essential eigenspaces of T with positive (respectively negative) eigenvalues. Then eT is essentially positive (respectively negative) if and only if the volume element*

e has a finite dimensional (-1)*-eigenspace on* W_+ *(respectively on* W_-*).*

(3) *The (AS) condition is equivalent to the* (± 1)*-eigenspaces of e, restricted to* W_+*, both being infinite dimensional.*

(4) W_+ *is a* C_n*-module and the (AS) condition is equivalent to* W_+ *containing both irreducible* C_n*-modules infinitely often.*

Proof. For part (1), just note that eT is an even operator whose diagonal entries are given by $e_1 \cdots e_{n-1} T_0 : H_n \to H_n$.
For part (2) we observe that $eT = Te$ and hence we can find simultanous eigenspace decompositions for these self-adjoint operators. Then a vector $v \in \mathcal{H}_n$ is in an essentially positive eigenspace of eT if and only if either $v \in W_+$ and $e(v) = +1$, or $v \in W_-$ and $e(v) = -1$. Since T is odd, its spectrum is symmetric and in particular, the grading involution α takes W_+ to W_-. α also anti-commutes with e and hence

$$e_{|W_+} = e \circ \alpha_{|W_-} = -\alpha \circ e_{|W_-}$$

so that α takes the $(+1)$-eigenspace of $e_{|W_+}$ to the (-1)-eigenspace of $e_{|W_-}$. In particular, these vector spaces have the same dimension. This finishes the prove of the part (2) as well as part (3).
To prove part (4) notice that T is C_n-linear and therefore W_+ is a C_n-module (which is not graded since α takes it to W_-). The claim follows from the well known algebraic fact stated at the beginning of the lemma. □

We will now show that $\widetilde{\mathcal{F}}_n^{gr}$ is homotopy equivalent to a configuration space. Let $\widetilde{\mathbb{R}} := [-\infty, \infty]$ be the two-point compactification of $\mathbb{R}$ equipped with the involution $s(x) := -x$.

Lemma 45. *The subspace* $\mathcal{A} \subset \widetilde{\mathcal{F}}_n^{gr}$ *of all operators* T *with* $\|T\| = 1$ *and* $\varepsilon(T) = 1$ *is a strong deformation retract of* $\widetilde{\mathcal{F}}_n^{gr}$*. Furthermore,* $\mathcal{A}$ *is homeomorphic to the configuration space* $\mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$ *as defined in Chapter 4.*

Proof. Define a homotopy $H : \widetilde{\mathcal{F}}_n^{gr} \times [0,1] \to \widetilde{\mathcal{F}}_n^{gr}$ by

$$(T,t) \mapsto H_t(T) := (t + (1-t)\|T\|) \cdot \phi\Big((t\varepsilon(T)^{-1} + (1-t)\|T\|^{-1}) \cdot T\Big).$$

Here $\phi : \mathbb{R} \to [-1,1]$ is defined by $\phi|_{[-1,1]} = \mathrm{id}$, $\phi|_{[1,\infty)} \equiv 1$ and $\phi|_{(\infty,-1]} \equiv -1$, and $\phi(\_)$ denotes the functional calculus with ϕ. The continuity of H follows from the continuity of $\|.\|$ and ε and from the usual continuity properties of functional calculus, see [RS], Theorem VIII.20. Also, C_n-linearity and parity of T are preserved under functional calculus. Furthermore,

$$H_0 = \mathrm{id}_{\widetilde{\mathcal{F}}_n^{gr}},\ H_t = \mathrm{id}_{\mathcal{A}} \text{ for all } t, \text{ and } H_1(\widetilde{\mathcal{F}}_n^{gr}) \subset \mathcal{A}.$$

Hence $\mathcal{A}$ is a strong deformation retract of $\widetilde{\mathcal{F}}_n^{gr}$.

Now, for all $T \in \mathcal{A}$ we have $\sigma(T) \subset [-1,1]$ and all $\lambda \in \sigma(T) \cap (-1,1)$ are eigenvalues of finite multiplicity. The spectral theorem for self-adjoint operators implies that the eigenspaces $V(T)_\lambda$ of T are pairwise orthogonal and span all of $\mathcal{H}_n$. Since T is odd, $V(T)_{-\lambda} = \alpha(V(T)_\lambda)$, where α is the grading involution on $\mathcal{H}_n$. We thus obtain a map

$$\mathcal{A} \to \mathrm{Conf}_{C_n}([-1,1], \{\pm 1\}; \mathcal{H}_n),\ T \mapsto V(T)$$

by associating to T the configuration $\lambda \mapsto V(T)_\lambda$ on $[-1,1]$. Here the involution on $[-1,1]$ is $x \mapsto -x$. It is easy to see that this map is a homeomorphism. Finally, we can use the obvious homeomorphism $[-1,1] \approx \widetilde{\mathbb{R}}$ to identify $\mathrm{Conf}_{C_n}([-1,1], \{\pm 1\}; \mathcal{H}_n)$ with $\mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$. □

We can now relate $\mathcal{F}_n$ and Conf_n and we will rediscover the Atiyah-Singer condition (AS), defining the spaces $\mathcal{F}_n \subset \widetilde{\mathcal{F}}_n$ for $n \equiv 3\ (4)$. By part (3) of Lemma 44, the (AS) condition can be expressed in terms of the action of the volume element $e = e_1 \cdots e_n$ on the essentially positive eigenspaces W_+ of T. Under the deformation retraction of the above lemma, these essential eigenspaces become the $(+1)$-eigenspace of the resulting operator in $\mathcal{A}$ and then turn into the $(+\infty)$-eigenspaces of the configuration $W \in \mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$ that corresponds to T. By part (4) of Lemma 44, the Atiyah-Singer condition (AS) then is the equivalent to the following condition on the label $W_{+\infty} = W_+$ of the configuration W: The ungraded C_n-module $W_{+\infty}$ contains both irreducible C_n-modules infinitely often.

We will now see why this is a very natural condition in terms of our configuration spaces. Consider the map $p : \widetilde{\mathbb{R}} \to \bar{\mathbb{R}}$ that is the identity on $\mathbb{R}$ and that maps $\pm\infty$ to ∞. It induces a continuous map

$$p_* : \mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n) \longrightarrow \mathrm{Conf}_{C_n}(\bar{\mathbb{R}}, \{\infty\}; \mathcal{H}_n) = \mathrm{Conf}_n$$

which can be composed with the homotopy equivalence

$$H : \widetilde{\mathcal{F}}_n^{gr} \xrightarrow{\simeq} \mathrm{Conf}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$$

from the previous lemma. After restricting to $\mathcal{F}_n$ we get the desired result:

Theorem 46. *For all $n \geq 1$, p_*H restricts to a homotopy equivalence*

$$p_*H|_{\mathcal{F}_n} : \mathcal{F}_n \longrightarrow \mathrm{Conf}_n\,.$$

Proof. Define

$$\mathrm{Conf}'_n := H(\mathcal{F}_n) \cap \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n).$$

Since the vertical arrows in the commutative diagram

$$\begin{array}{ccc} H(\mathcal{F}_n) & \xrightarrow{p_*} & \mathrm{Conf}_n \\ \simeq\uparrow & & \uparrow\simeq \\ \mathrm{Conf}'_n & \xrightarrow{p_*} & \mathrm{Conf}^{\mathrm{fin}}_n \end{array}$$

are homotopy equivalences (for the right arrow this was done in the proof of Proposition 43; the same argument works for the arrow on the left), it follows that $p_*|_{H\mathcal{F}_n}$ is a weak homotopy equivalence exaclty if this is the case for $p_*|_{\mathrm{Conf}'_n}$. This, in turn, follows from Theorem 48 and Lemma 52 below. Thus we can conclude that $p_*H|_{\mathcal{F}_n}$ is a weak homotopy equivalence. Since $\mathcal{F}_n$ and Conf_n both have the homotopy type of a CW-complex, the map $p_*H|_{\mathcal{F}_n}$ is a homotopy equivalence, cf. [Mi2]. □

Before we proceed, we need to collect some representation theoretic facts about graded Clifford modules. Recall from [LM] that for $n \not\equiv 3\ (4)$ the Clifford algebra C_n is simple, where it is the product of two simple algebras in the other cases. Therefore, there is a unique irreducible C_n-module (and a unique graded irreducible C_{n+1}-module) for $n \not\equiv 3\ (4)$ otherwise there are exactly two such modules.

Lemma 47. *Let M be a graded C_n-module.*

(1) *If M contains all (one or two) irreducible C_n-modules infinitely often then the C_n-action on M extends to a graded C_{n+1}-action.*
(2) *Let M_0 be a graded irreducible C_n-module. Then there is a graded vector space R, the multiplicity space, such that M is isomorphic to the graded tensor product $M_0 \otimes R$.*
(3) *With this notation, the grading preserving Clifford linear orthogonal group $O_{C_n}(M)$ is isomorphic to*

$$O_{C_n}(M) \cong O(R) \cong O(R^{ev}) \times O(R^{odd})$$

In particular, this group is contractible (by Kuiper's theorem) if and only if the multiplicity spaces R^{ev} and R^{odd} are either zero or both infinite dimensional. This is equivalent to M containing either only one type of graded irreducible C_n-module, or containing both infinitely often.

Proof. There are two cases to consider for proving (1): If only one graded irreducible C_n-module exists, then take any irreducible C_{n+1}-module M_0 and restrict it to C_n. It is clear that over C_n, M must be given by infinitely many copies of M_0.

Let's say there are two graded irreducible C_n-modules M_0, M_1 and hence n is divisible by 4. By assumption, M is the sum of infinitely many copies of $M_0 \perp M_1$. It then suffices to show that $M_0 \perp M_1$ has an C_{n+1}-action. We first claim that $M_1 \cong M_0^{op}$, i.e. M_1 is obtained from M_0 by flipping the grading. Using the characterization of ungraded C_{n-1}-modules given in Lemma 44, it suffices to show that the volume element $\tilde{e}_1 \cdots \tilde{e}_{n-1}$ acts with different sign on M_0^{ev} and M_0^{odd}. Here $\tilde{e}_i = e_i e_n$ are the usual generators of $C_{n-1} \cong C_n^{ev}$. Since n is divisible by 4, it follows that

$$\tilde{e}_1 \cdots \tilde{e}_{n-1} = (e_1 e_n) \cdots (e_{n-1} e_n) = e_1 \cdots e_n =: e$$

is also the volume element in C_n. Writing $M_0^{odd} = e_i \cdot M_0^{ev}$ for some i, our claim follow from $ee_i = -e_i e$. Finally, the module $M_0 \perp M_0^{op}$ has a C_{n+1}-action given by the element $e_{n+1} = f\alpha$, where f flips the two summands and α is the grading involution.

For part (2) one again needs to know that in the case that there are two graded irreducible C_n-modules M_0, M_1, they differ from each other by flipping the grading. This was proven above. Part (3) is obvious. □

Theorem 48. *The restriction*

$$p := p_*|_{\mathrm{Conf}'_n} : \mathrm{Conf}'_n \longrightarrow \mathrm{Conf}^{fin}_n$$

is a quasi-fibration with contractible fibers (see Definition 50).

Remark 49. The map H is surjective. Hence $\mathrm{Conf}'_n = \mathrm{Conf}^{\mathrm{fin}}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$ for $n \not\equiv 3\,(4)$. In the case $n \equiv 3\,(4)$ the space Conf'_n is the unique connected component of $\mathrm{Conf}^{\mathrm{fin}}_{C_n}(\widetilde{\mathbb{R}}, \{\pm\infty\}; \mathcal{H}_n)$ that is not contractible. On the two remaining components the map p_* is *not* a quasi-fibration as well shall see below (the fibres have distinct homotopy groups).

Proof of Theorem 48. We begin by proving that the fibers of p are contractible. Fix $V \in \mathrm{Conf}^{\mathrm{fin}}_n$. The fiber $p^{-1}(V)$ consists of all $W \in \mathrm{Conf}'_n$ such that $V_\lambda = W_\lambda$ for $\lambda \in \mathbb{R}$ and V_∞ is the orthogonal sum of W_∞ and $W_{-\infty}$. Since $W_{-\infty} = \alpha(W_\infty)$, where α is the grading involution on $\mathcal{H}_n$, we may identify $p^{-1}(V)$ with the space of decompositions of the graded C_n-module V_∞ of the form $V_\infty = W_\infty \perp \alpha(W_\infty)$, where W_∞ is an ungraded C_n-submodule of V_∞ that for $n \equiv 3$ (4) satisfies the (AS) condition: both irreducible C_n-modules appear infinitely often in W_∞.

Without the (AS) condition, it is straight forward to show that this space of decompositions of V_∞ is homeomorphic to the following space of C_{n+1}-structures on V_∞:

$$\widetilde{\mathfrak{C}}_{n+1}(V_\infty) := \{e_{n+1} \in O(V_\infty) \mid e_{n+1}^2 = -1\!\!1, e_{n+1}e_i = -e_i e_{n+1},\ i = 1, \dots, n\}$$

Namely, given e_{n+1}, one can define $W_{\pm\infty}$ to be the (± 1)-eigenspaces of $e_{n+1}\alpha$ (and vice versa). Under this correspondence, the (AS) condition translates into the requirement that e_{n+1} defines a C_{n+1} module structure on V_∞ which contains both graded irreducibles infinitely often. We denote this subspace of $\widetilde{\mathfrak{C}}_{n+1}(V_\infty)$ simply by $\mathfrak{C}_{n+1}(V_\infty)$ and observe that all these module structures e_{n+1} on V_∞ are isomorphic.

We show in the following 4 steps that the fibre $p^{-1}(V) \approx \mathfrak{C}_{n+1}(V_\infty)$ is contractible under our assumptions.

Step 1: By our basic assumption, the ambient Hilbert space $\mathcal{H}_n$ contains all graded irreducible C_n-modules infinitely often. Since V was a finite configuration to start with, it follows that V_∞ has the same property and by part (1) of Lemma 47 it follows that $\mathfrak{C}_{n+1}(V_\infty)$ is not empty.

Step 2: Since any two points in $\mathfrak{C}_{n+1}(V_\infty)$ lead to C_{n+1}-module structures on V_∞ that are isomorphic, the orthogonal group $O_{C_n}(V_\infty)$ acts transitively (by conjugation) on $\mathfrak{C}_{n+1}(V_\infty)$. The stabilizer of a particular C_{n+1}-structure is $O_{C_{n+1}}(V_\infty)$ and hence

$$\mathfrak{C}_{n+1}(V_\infty) \approx O_{C_n}(V_\infty)/O_{C_{n+1}}(V_\infty)$$

We shall show that this space is contractible, as a quotient of two contractible groups.

Step 3: As a C_n-module, V_∞ contains both graded irreducible C_n-modules infinitely often, that's what we need by part (3) of Lemma 47 for the contractibility of the larger group $O_{C_n}(V_\infty)$.

Step 4: For the smaller group $O_{C_{n+1}}(V_\infty)$, the (AS) condition tells us again that the assumptions of part (3) of Lemma 47 are satisfied.

To finish the proof of Theorem 48, it remains to show that p is indeed a quasi-fibration. We will use the criterion in Lemma 52 but first we give the relevant definitions. □

Definition 50. A map $p : E \twoheadrightarrow B$ is a *quasi-fibration* if for all $b \in B$, $i \in \mathbb{N}$, and $e \in p^{-1}(b)$ p induces an isomorphism

$$\pi_i(E, p^{-1}(b), e) \xrightarrow{\cong} \pi_i(B, b).$$

From the long exact sequence of homotopy groups for a pair it follows that p is a quasi-fibration exactly if there is a long exact homotopy sequence connecting fibre, total space and base space of p, just like for a fibration. However, p does not need to have any (path) lifting properties as the following example shows.

Example 51. The prototypical example of a quasi-fibration that's not a fibration is the projection of a 'step'

$$(-\infty, 0] \times \{0\} \cup \{0\} \times [0, 1] \cup [0, \infty) \times \{1\} \subset \mathbb{R}^2$$

onto the x-axis. Even though all fibers have the same homotopy type (they are contractible), the map doesn't have the lifting property of a fibration, since it is impossible to lift a path that passes through the origin.

The following sufficient condition for a map to be a quasi-fibration is proved in [DT]:

Theorem 52. *The map* $p : E \twoheadrightarrow B$ *is a* quasi-fibration *if there exists a filtration*

$$F_0 \subset F_1 \subset F_2 \subset \ldots \text{ of } B \text{ such that}$$

(1) *For all i the restriction* $p|_{F_i \setminus F_{i-1}}$ *is a fibration.*
(2) *For all i there exists a neighborhood* N_i *of* F_i *in* F_{i+1} *and a homotopy* h *on* N_i *such that* $h_0 = \mathrm{id}$ *and* $h_1(N_i) \subset F_i$*.*
(3) h *is covered by* $H : p^{-1}(N_i) \times I \to p^{-1}(N_i)$ *with* $H_0 = \mathrm{id}$ *and for all*

$$x \in N_i \text{ we have } H_1(p^{-1}(x)) \subset p^{-1}(h_1(x))$$

To complete the proof of Theorem 48 we have to show that p is indeed a quasi-fibration. This follows from theorem 52; we only outline the argument. The filtration F_i is defined by

$$F_i := \{\, c \in \mathrm{Conf}_n \mid \dim(\oplus_{x\in\mathbb{R}} c(x)) \le 2i \,\}.$$

The neighborhoods N_i consist of configurations $c \in F_{i+1}$ such that $c(x) \neq 0$ for exactly one $x \in \mathbb{R}_{>1}$ with $\dim c(x) = 1$. The map H_1 is the inclusion of a smaller unitary group into a bigger one. This completes the proof of Theorem 46.

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