Name:
Take Home Exam, Feb. 26, 2008

You can use any resources for this test (e.g., books, notes, internet) with the exception of fellow mathematicians: talking to anybody except me about this exam before you turn your exam in constitutes a violation of the honor code. Please return the exam to me in class on Wednesday, Feb. 27. Don’t forget part (b) of problem 4 on the back page. I’ll be around most of the day today, except 2-3:30 PM. Good Luck!

1. (10 points) Let $A$ be a subspace of a topological space $X$ and denote by $i: A \to X$ the inclusion map. Show that if $A$ is a retract of $X$, then the induced map $i_*: H_q(A) \to H_q(X)$ is injective.

Proof. The assumption that $A$ is a retract of $X$ means that there is a map $r: X \to A$ making the following diagram commutative:

$$
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow{=} & & \downarrow{r} \\
A & \xrightarrow{r} & X
\end{array}
$$

commutative, where $\mathbf{1}$ denotes the identity map. Applying the homology functor, it follows that the diagram

$$
\begin{array}{ccc}
H_q(A) & \xrightarrow{i_*} & H_q(X) \\
\downarrow{=} & & \downarrow{r_*} \\
H_q(A) & \xrightarrow{r_*} & H_q(X)
\end{array}
$$

is commutative. Hence $i_*$ is injective, since if $i_*(a) = i_*(a')$, then $a = r(i_*(a)) = r(i_*(a')) = a'$.

2. (15 points) Suppose the following diagram of abelian groups and group homomorphisms is commutative with exact rows:

$$
\begin{array}{ccccccc}
\ldots & \xrightarrow{c_{q+1}} & C_{q+1} & \xrightarrow{\partial_{q+1}} & A_q & \xrightarrow{f_q} & B_q & \xrightarrow{g_q} & C_q & \xrightarrow{\partial_q} & A_{q-1} & \xrightarrow{} & \ldots \\
& & c_q & \downarrow{a_q} & b_q & \downarrow{c_q} & \downarrow{a_{q-1}} & & & & & & \\
& & C'_q & \xrightarrow{\partial'_q} & A'_q & \xrightarrow{f'_q} & B'_q & \xrightarrow{g'_q} & C'_q & \xrightarrow{\partial'_q} & A'_{q-1} & \xrightarrow{} & \ldots
\end{array}
$$
Assuming in addition that the maps \( c_q \) are isomorphisms show that there is a long exact sequence of the form

\[
\longrightarrow A_q \longrightarrow A'_q \oplus B_q \longrightarrow B'_q \longrightarrow A_{q-1} \longrightarrow A'_{q-1} \oplus B_{q-1} \longrightarrow B'_{q-1} \longrightarrow
\]

First define carefully the homomorphisms in the above sequence. Then prove exactness at each location.

**Proof.** We define the maps in the above sequence as follows:

\[
\begin{align*}
\alpha_q &: A_q \rightarrow A'_q \oplus B_q & a &\mapsto (a_q(a), f_q(a)) \\
\beta_q &: A'_q \oplus B_q \rightarrow B'_q & (a', b) &\mapsto f'_q(a') - b_q(b) \\
\gamma_q &: B'_q \rightarrow A_{q-1} & b' &\mapsto \partial_q c_q^{-1} g'_q(b')
\end{align*}
\]

**Exactness at** \( B'_q \). First we show \( \gamma_q \beta_q = 0 \). For \( (a', b) \in A'_q \oplus B_q \) we have

\[
\gamma_q \beta_q(a', b) = \partial_q c_q^{-1} g'_q(f'_q(a') - b_q(b)) = -\partial_q c_q^{-1} g'_q b_q b = \partial_q g_q b = 0
\]

Here the second equality holds due to \( g'_q f'_q = 0 \), the third follows from the commutativity of the third square, and the last is due to \( \partial_q g_q = 0 \).

To show \( \ker \gamma_q \subseteq \im \beta_q \) let \( b' \in B'_q \) with \( \gamma_q b' = \partial_q c_q^{-1} g'_q b' = 0 \). By exactness at \( C_q \) there is an element \( b \in B_q \) such that \( g_q b = c_q^{-1} g'_q b' \) or equivalently

\[
g'_q b' = c_q g_q b = g'_q b_q b,
\]

where the second equality follows from commutativity of the third square. It follows that \( b' - b_q b \) is in the kernel of \( g'_q \) and hence by exactness at \( B'_q \), there is an element \( a' \in A'_q \) with \( f'_q a' = b' - b_q b \). This implies

\[
\beta_q(a', -b) = f'_q a' + b_q b = b'
\]

which shows that \( b' \) is in the image of \( \beta_q \).

**Exactness at** \( A'_q \oplus B_q \). First we show \( \beta_q \alpha_q = 0 \). For \( a \in A_q \) we have

\[
\beta_q \alpha_q a = \beta_q(a_q(a), f_q(a)) = f'_q a_q a - b_q f_q a = 0
\]

due to the commutativity of the second square.
To show ker $\beta_q \subset \text{im} \alpha_q$, let $(a', b) \in A'_q \oplus B_q$ with 
$$\beta_q(a', b) = f'_q a' - b_q b = 0.$$ 
Then we have 
$$c_q g_q b = g'_q b_q b = g'_q f'_q a' = 0,$$
where the first equality is due to the commutativity of the third square, and the last is due to exactness at $B'_q$. Since $c_q$ is an isomorphism, this implies $g_q b = 0$ and hence by exactness at $B_q$, there is an element $a \in A_q$ with $f_q a = b$. If we could show $a_q a = a'$, we would be done. However we can only say the following:
$$f'_q (a_q a - a') = f'_q a_q a - f'_q a' = b_q f_q a - b_q b = 0$$
where the second equality follows from the commutativity of the second square and our assumption $f'_q a' = b_q b$. Since $f'_q$ is not necessarily injective, we can’t conclude that $a_q a = a'$, but thanks to exactness at $A'_q$, it implies that there is an element $c' \in C'_{q+1}$ with $\partial_{q+1} c' = a_q a - a'$. Moreover, since $c_{q+1}$ is an isomorphism, there is a $c \in C_{q+1}$ with $c_{q+1} c = c'$. Now we modify the element $a \in A_q$ by defining $\bar{a} \overset{\text{def}}{=} a - \partial_{q+1} c$. We calculate
$$f_q \bar{a} = f_q (a - \partial_{q+1} c) = f_q a = b$$
$$a_q \bar{a} = a_q (a - \partial_{q+1} c) = a_q a - \partial'_{q+1} c_{q+1} c = a_q a - (a_q a - a') = a'$$
This shows that $\alpha_q (\bar{a}) = (a', b)$ as desired.

**Exactness at $A_q$.** First let us show $\alpha_q \circ \gamma_{q+1} = 0$. For $b' \in B'_{q+1}$ we have
$$\alpha_q \gamma_{q+1} b' = \alpha_q (\partial_{q+1} c_{q+1}^{-1} g'_{q+1} b')$$
$$= (a_q \partial_{q+1} c_{q+1}^{-1} g'_{q+1} b', f_q \partial_{q+1} c_{q+1}^{-1} g'_{q+1} b')$$
$$= (\partial'_{q+1} g'_{q+1} b', 0) = (0, 0)$$
since the compositions $f_q \partial_{q+1}$ and $\partial'_{q+1} g'_{q+1}$ are zero due to the exactness at $A_q$ resp. $A'_q$.
To show ker $\alpha_q \subset \text{im} \gamma_{q+1}$, let $a \in A_q$ with $\alpha_q a = (a_q a, f_q a) = (0, 0)$. By exactness at $A_q$ there is an element $c \in C_{q+1}$ with $\partial_{q+1} c = a$. Then
$$\partial'_{q+1} c_{q+1} c = a_q \partial_{q+1} c = a_q a = 0$$
and hence by exactness at $C'_{q+1}$, there is an element $b' \in B'_{q+1}$ with $g'_{q+1}b' = c_{q+1}$. This implies

$$\gamma_{q+1}b' = \partial_{q+1}c_{q+1}^{-1}g'_{q+1}b' = \partial_{q+1}c = a,$$

which shows that $a$ is in the image of $\gamma_{q+1}$.

3. (10 points) Let $x_1, \ldots, x_l$ be distinct points of $\mathbb{R}^n$. Calculate the reduced homology groups of the space $\mathbb{R}^n \setminus \{x_1, \ldots, x_l\}$. Hint: Compare the homology groups of $\mathbb{R}^n \setminus \{x_1, \ldots, x_l\}$ with those of $\mathbb{R}^n$ by analyzing the long exact homology sequence of this pair of spaces.

**Proof.** Consider the following portion of the long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus X)$, $X \overset{\text{def}}{=} \{x_1, \ldots, x_l\}$:

$$\tilde{H}_{q+1}(\mathbb{R}^n) \xrightarrow{\partial} \tilde{H}_q(\mathbb{R}^n \setminus X) \xrightarrow{\partial} \tilde{H}_q(\mathbb{R}^n)$$

The reduced homology groups of $\mathbb{R}^n$ vanish since $\mathbb{R}^n$ is contractible, and hence the map $\partial$ in the sequence above is an isomorphism.

Let $B_i \subset \mathbb{R}^n$ be a collection of mutually disjoint open balls with center $x_i \in B_i$, and let $B \overset{\text{def}}{=} \bigcup_{i=1}^l B_i$ be the union of these balls. We note that the pair $(B, B \setminus X)$ is obtained from the larger pair $(\mathbb{R}^n, \mathbb{R}^n \setminus X)$ by excising $U = \mathbb{R}^n \setminus B$. We note that the closure of $U$ is contained in the open set $\mathbb{R}^n \setminus X$ and hence we obtain the excision isomorphism

$$H_q(\mathbb{R}^n, \mathbb{R}^n \setminus X) \cong H_q(B, B \setminus X).$$

To calculate this homology group, we note that the pair $(B, B \setminus X)$ is the disjoint union of the pairs $(B_i, B_i \setminus \{x_i\})$, and hence

$$H_q(B, B \setminus X) \cong \bigoplus_{i=1}^l H_q(B_i, B_i \setminus \{x_i\}).$$

Finally we note that $B_i$ is a manifold of dimension $n$ and hence the local homology group $H_q(B_i, B_i \setminus \{x_i\})$ is isomorphic to $\mathbb{Z}$ for $q = n$ and trivial for $q \neq n$. Putting the various isomorphisms together, we obtain:

$$\tilde{H}_q(\mathbb{R}^n \setminus X) = \begin{cases} 
\mathbb{Z} & q = n - 1 \\
0 & q \neq n - 1
\end{cases}$$

\[ \square \]
4. a) (10 points) Show that if \( f: S^n \to S^n \) is a continuous map of degree 0, then there are points \( x, y \in S^n \) such that \( f(x) = x \) and \( f(y) = -y \). Hint: Show that if \( f(y) \neq -y \) for all \( y \in S^n \), you could construct a homotopy between \( f \) and the identity map. Use a similar argument to show that \( f(x) \neq x \) for all \( x \in S^n \) leads to a contradiction.

Proof. Let us assume that \( f(y) \neq -y \) for all \( y \in S^n \). This implies that for all \( y \in S^n \) and \( t \in [0, 1] \) the vector \((1 - t)f(y) + ty\) is non-zero and hence

\[
H: S^n \times [0, 1] \to S^n \quad H(y, t) = \frac{(1 - t)f(y) + ty}{\| (1 - t)f(y) + ty \|}
\]

is a homotopy between \( f \) and the identity map \( \mathbb{I} \). This leads to the contradiction \( 0 = \text{deg}(f) = \text{deg}(\mathbb{I}) = 1 \).

Similarly, if we assume that \( f(x) \neq x \) for all \( x \in S^n \), then for all \( x \in S^n \) and \( t \in [0, 1] \) the vector \((1 - t)f(x) - tx\) is non-zero and hence

\[
H: S^n \times [0, 1] \to S^n \quad H(x, t) = \frac{(1 - t)f(x) - tx}{\| (1 - t)f(x) - tx \|}
\]

is a homotopy between \( f \) and \(-\mathbb{I}\). This leads to the contradiction \( 0 = \text{deg}(f) = \text{deg}(-\mathbb{I}) = \pm 1 \). \( \square \)

b) (5 points) Let \( F \) be a vector field on the disk \( D^n \subset \mathbb{R}^n \); i.e., \( F \) is a continuous map \( F: D^n \to \mathbb{R}^n \) which we picture by drawing the vector \( F(x) \) with its tail at the point \( x \). Show that if \( F(x) \neq 0 \) for all \( x \in D^n \), then there must be some point \( x \) on \( \partial D^n \) where \( F \) points radially outward, and another point \( y \) on \( \partial D^n \) where \( F \) points radially inward (i.e., \( F(x) = ax \) and \( F(y) = -by \) for some positive real numbers \( a, b \)).

Proof. Due to the assumption \( F(x) \neq 0 \) for all \( x \in D^n \), we can construct a homotopy

\[
H: S^n \times [0, 1] \to S^n \quad H(x, t) = \frac{F(tx)}{\|F(tx)\|}
\]

between the map \( f: S^n \to S^n \) given by \( f(x) = \frac{F(x)}{\|F(x)\|} \) and the constant map \( F_0: S^n \to S^n \), \( x \mapsto \frac{F(x)}{\|F(x)\|} \). Since \( F_0 \) is not surjective, its degree is zero and hence \( \text{deg}(f) = \text{deg}(F_0) = 0 \). This allows us to apply part (a) to the map \( f \) and we conclude that there exists points \( x, y \in S^n \) such that \( f(x) = x \), \( f(y) = -y \). In terms of the vector field \( F \) this means that \( F(x) = \|F(x)\|f(x) = \|F(x)\|x \) and \( F(y) = \|F(y)\|f(y) = -\|F(y)\|y \) as desired. \( \square \)