Lecture 1: Introduction

This seminar is about zero-dimensional subschemes of $\mathbb{P}^n(k), (k = \overline{k}, \text{char} k = 0)$, alternatively, about saturated homogeneous ideals $I \subset k[x_0, \ldots, x_n] = R$ for which the Krull dimension of $R/I = 1$.

The simplest examples of such ideals correspond to points in $\mathbb{P}^n$. So, let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of s distinct points in $\mathbb{P}^n$. Then P_i corresponds to the prime ideal $\wp_i$ in R of height n, and $\wp_i = (L_{i1}, \ldots, L_{in})$ where the $L_{ij}, j = 1, \ldots n$ are linearly independent linear forms. Hence $I = \wp_1 \cap \ldots \cap \wp_s$ is the saturated ideal corresponding to $\mathbb{X}$. These examples are all the <u>*reduced*</u> ideals corresponding to (reduced) zero-dimensional subschemes of $\mathbb{P}^n$.

We can write $R = \oplus_{i=0}^{\infty} R_i$ (R_i the vector space of forms in R of degree i) where $\dim_k R_i = \binom{i+n}{n}$ and $I = \oplus_{i \geq 0} I_i$. The *Hilbert Function* of I, or of $A = R/I = \oplus A_i$, or of $\mathbb{X}$, is the numerical function

$$H(\mathbb{X}, t) := H(A, t) = \dim_k A_t \ .$$

Example: Consider three general points in $\mathbb{P}^2$. After a change of variables we can assume the points are

$$P_1 = [1:0:0],\ P_2 = [0:1:0],\ P_3 = [0:0:1] \ .$$

We have that I, the ideal of these three points, is

$$I = \wp_1 \cap \wp_2 \cap \wp_3 = (y, z) \cap (x, z) \cap (x, y) = (xy, xz, yz) \ .$$

One verifies that $(R/I)_n = < \overline{x}^n, \overline{y}^n, \overline{z}^n >$ for all $n \geq 1$ and so the Hilbert function of these three points is: 1 3 3 $3 \cdots$.

It is easy to check that if we had, instead, chosen our three points less generally (i.e. if all were on a line of $\mathbb{P}^2$), then the Hilbert function would have been: 1 2 3 $3 \cdots$.

If we let $M_1, \ldots, M_{\binom{d+n}{n}}$ be the monomial basis for R_d then an arbitrary element of R_d looks like

$$c_1 M_1 + \cdots c_{\binom{d+n}{n}} M_{\binom{d+n}{n}} = F$$

where the $c_i \in k$ are arbitrary.

In order that F vanish at the point P, i.e. $F(P) = 0$, we must have

$$M_1(P)c_1 + \cdots + M_{\binom{d+n}{n}}(P)c_{\binom{d+n}{n}} = 0$$

i.e. we must have a certain *linear* expression in the c_i's vanish.

So, if we consider s points, $P_1, \ldots, P_s$ in $\mathbb{P}^n$, then the forms of degree d which vanish at these points are precisely the solutions to the system of linear equations

$$\begin{array}{ccccccccc} M_1(P_1)c_1 & + & \cdots & \cdots & + & M_{\binom{d+n}{n}}(P_1)c_{\binom{d+n}{n}} & = & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ M_1(P_s)c_1 & + & \cdots & \cdots & + & M_{\binom{d+n}{n}}(P_s)c_{\binom{d+n}{n}} & = & 0 \end{array}$$

which we write

$$\mathcal{M}_d \begin{pmatrix} c_1 \\ . \\ . \\ c_{\binom{d+n}{n}} \end{pmatrix} = 0$$

where $\mathcal{M}_d$ is the $s \times \binom{d+n}{n}$ coefficient matrix of the system of equations.

Since the set of solutions to this system of linear equations is precisely the vector space I_d, the dimension of the space of solutions is,

$$\dim_k I_d = \binom{d+n}{n} - rk\mathcal{M}_d \ .$$

Thus

$$H(R/I, d) = \binom{d+n}{n} - \dim_k I_d = rk\mathcal{M}_d \ .$$

It is well known, and not hard to prove, that for any integer s, we can pick points $P_1, \ldots, P_s$ so that the matrices $\mathcal{M}_d$ *<u>all</u>* have the maximum rank possible, i.e.

$$rk\mathcal{M}_d = \min\{s, \binom{d+n}{n}\} \ .$$

This tells us then:

A general set $\mathbb{X}$ of s points in $\mathbb{P}^n$ has Hilbert function

$$H(\mathbb{X}, t) = \min\{s, \binom{t+n}{n}\} \ .$$

What can we say about the Hilbert functions of non-reduced zero-dimensional subschemes of $\mathbb{P}^n$?

To make life simple, we shall begin by assuming that our subscheme is supported at a single point, P (which we might as well assume is the point $P = [1 : 0 : \cdots : 0]$) i.e. from an algebraic point of view we are looking at a *primary* ideal q with radical $\sqrt{q} = \wp = (x_1, \cdots, x_n)$.

There are many interesting classes of primary ideals for $\wp = (x_1, \ldots, x_n)$ which we could consider, but for the moment, one class stands out, thanks to a theorem of Macaulay (see [Z-S, Vol. II, Appendix]).

Theorem: Let $I = (F_1, \ldots, F_s)$ be an ideal of R of height s (i.e. I is a *complete intersection ideal*). Then I^r is unmixed with respect to height, i.e. all the primary components of I^r have the same height s.

In particular, if $I = \wp$ is prime then I^r is a $\wp$-primary ideal.

We can apply this theorem, for example, to the prime ideal $\wp = (x_1, \ldots, x_n)$ above and so we obtain that all the ideals of the form $\wp^r$ are $\wp$-primary.

Our interest in this class of ideals does not only come from the fact that they are (unexpectedly!) $\wp$-primary, but also because these ideals were much studied classically. I will now explain the source of the classical interest in these ideals.

Let $F \in \wp$ be a homogeneous polynomial of degree d. If we dehomogenize F with respect to x_0, we obtain $f \in S = k[x_1, \ldots, x_n]$ (I'll abuse the notation here and use the same variables.) The point P above then becomes $P = (0, \ldots, 0) = \mathbf{0} \in \mathbb{A}^n(k)$. We can write

$$f = f_0 + f_1 + \cdots + f_d \text{ where } \deg f_i = i \ .$$

Moreover, since $F \in \wp$ we have that $f(P) = 0$ i.e. $f_0 = 0$.

Recall that if $f_1 = a_1x_1 + \cdots + a_nx_n$ then we can rewrite f_1 (at least if the characteristic of k is 0) as

$$f_1 = ((\partial f/\partial x_1)|_{\mathbf{0}})\, x_1 + \cdots + ((\partial f/\partial x_n)|_{\mathbf{0}})\, x_n$$

and, if all the first partials of f do not vanish at $P = \mathbf{0}$, then P is a smooth point of $V(f)$ and $f_1 = 0$ is the equation of the tangent hyperplane to $V(f)$ at P.

In fact, $F \in \wp \backslash \wp^2 \Leftrightarrow$ at least one of these first partials does not vanish at $\mathbf{0}$. Put another way,

$$F \in \wp^2 \Leftrightarrow (\partial F/\partial x_i)|_{\mathbf{0}} = 0 \text{ for all } i = 1, \ldots, n \ .$$

(recall Euler's Theorem). Moreover, this happens,

$$\Leftrightarrow P \text{ is a singular point of } V(F) \ .$$

So, if $I = \wp^2$ then I_d consists of all the forms of degree d which have a singularity at P. This vector space is a classic example of a *linear system* of hypersurfaces of $\mathbb{P}^n$, i.e. a linear subspace of R_d. Moreover, it is a subspace for which it is easy to see the linear equations that describe it (namely certain coefficients of the dehomogenized F's from R_d have to vanish.)

We can continue in this way by considering the Taylor expansion of f around $\mathbf{0}$ and thus reinterpret the coefficients of f_2 as giving us the various second partial derivatives of f (evaluated at $\mathbf{0}$). More precisely, if $a_{\alpha,\beta} x_\alpha x_\beta$ is a term of f_2 then

$$a_{\alpha,\beta} = \begin{cases} (\partial f/\partial x_\alpha \partial x_\beta)|_{\mathbf{0}} & \text{if } \alpha \neq \beta \\ \\ (1/2!)(\partial f/\partial x_\alpha^2)|_{\mathbf{0}} & \text{if } \alpha = \beta \ . \end{cases}$$

Notice further that <u>*all*</u> the second partial derivatives of f vanish at $\mathbf{0} \Leftrightarrow F \in \wp^3 \Leftrightarrow P$ is a singular point of $V(F)$ having multiplicity ≥ 3.

More generally:

$$\underline{\textit{all}} \text{ the partial derivatives of } f \text{, of order } \leq t \text{, vanish at } P \Leftrightarrow F \in \wp^{t+1}$$

$$\Leftrightarrow P \text{ is a singular point of } V(F) \text{ having multiplicity } \geq t+1 \ .$$

Notice also that if $F \in \wp^t$ and $\deg F = d$ ($t \leq d$ obviously) then

$$f = f_t + \cdots + f_d$$

and clearly any such $f \in S$ gives an $F \in \wp^t$ by homogenization. It is a simple consequence of this fact that

$$H(R/\wp^t, s) = \begin{cases} \binom{s+n}{n} & \text{if } s < t \\ \\ \binom{t-1+n}{n} & \text{if } s \geq t. \end{cases}$$

Definition: Let $P \in \mathbb{P}^n$ and let P correspond to $\wp \subset R = k[x_0, x_1, \ldots, x_n]$. If t is any positive integer then the subscheme of $\mathbb{P}^n$ defined by the $\wp$-primary ideal $\wp^t$ is called a *fat point in* $\mathbb{P}^n$ *supported on* P and is denoted $(P; t)$.

Observe that a single fat point $(P; t)$ in $\mathbb{P}^n$ behaves like $\binom{t-1+n}{n}$ distinct general points of $\mathbb{P}^n$ (at least from the point of view of the Hilbert function).

Examples:

In $\mathbb{P}^2$ **:** $\wp = (x_1, x_2) \subset k[x_0, x_1, x_2] = R$. Then,

$$H(R/\wp^2, -) : 1 \quad 3 \quad 3 \quad \cdots$$

$$H(R/\wp^3, -) : 1 \quad 3 \quad 6 \quad 6 \quad \cdots .$$

In $\mathbb{P}^3$**:** $\wp = (x_1, x_2, x_3) \subset k[x_0, x_1, x_2, x_3] = R$. Then

$$H(R/\wp^2, -) : 1 \quad 4 \quad 4 \quad \cdots$$

$$H(R/\wp^3, -) : 1 \quad 4 \quad 10 \quad 10 \quad \cdots .$$

There is nothing to stop us from extending our earlier definition to include more than one point at a time.

Definition: Let $P_1, \ldots, P_s$ be distinct points in $\mathbb{P}^n(k)$ with corresponding prime ideals $\wp_1, \ldots, \wp_s$. Let $\alpha_1, \ldots, \alpha_s$ be any set of positive integers. The subscheme of $\mathbb{P}^n$ defined by the ideal $I = \wp_1^{\alpha_1} \cap \cdots \cap \wp_s^{\alpha_s}$ is called a *scheme of fat points in* $\mathbb{P}^n$ and is denoted $(P_1, \ldots, P_s; \alpha_1, \ldots, \alpha_s)$.

Remarks:

1) I is a saturated homogeneous ideal. This is clear since the way we wrote I gives its primary decomposition and there is no primary component for the irrelevant ideal.

2) Since I is a saturated ideal there is no ambiguity in referring to it as **THE** ideal of the fat points $(P_1, \ldots, P_s; \alpha_1, \ldots, \alpha_s)$.

3) **WARNING:**

In general $(\wp_1 \cap \ldots \cap \wp_s)^\alpha$ is *not* the ideal of the fat points $(P_1, \ldots, P_s; \alpha, \ldots, \alpha)$.

> **e.g.** If $s = 3$ and, as earlier, we let P_1, P_2, P_3 be the coordinate points of $\mathbb{P}^2$ so that $I = \wp_1 \cap \wp_2 \cap \wp_3 = (xy, xz, yz)$ then I^2 only begins in degree 4. But $\wp_1^2 \cap \wp_2^2 \cap \wp_3^2$ contains a cubic equation, namely xyz. (Draw the picture!).

In general $(\wp_1 \cap \ldots \cap \wp_s)^\alpha \subseteq \wp_1^\alpha \cap \ldots \cap \wp_s^\alpha$ but we need not have equality because the primary decomposition of $(\wp_1 \cap \ldots \cap \wp_s)^\alpha$ is:

$$(\wp_1 \cap \ldots \cap \wp_s)^\alpha = \wp_1^\alpha \cap \ldots \cap \wp_s^\alpha \cap q$$

(thanks to Macaulay's theorem again) where $\sqrt{q} = (x_0, x_1 \ldots, x_n)$ i.e. $(\wp_1 \cap \ldots \cap \wp_s)^\alpha$ need not be saturated.

Problem: If $P_1, \ldots, P_s$ are sufficiently general points of $\mathbb{P}^n$ with corresponding prime ideals $\wp_1, \ldots, \wp_s \subset R = k[x_0, \ldots, x_n]$ and $\alpha_1, \ldots, \alpha_s$ are a given set of non-negative integers, set $I = \wp_1^{\alpha_1} \cap \ldots \cap \wp_s^{\alpha_s}$. What is the Hilbert function of R/I? (Recall that, from very general considerations about Hilbert functions we know that eventually $H(R/I, -)$ takes on the constant value $\sum_{i=1}^{s} \binom{\alpha_i - 1 + n}{n}$.)

Notice that this is a sort of differential interpolation problem. We are asking the dimension of the space of "hypersurfaces" of a given degree which pass through a given set of points and have, at those points, a singularity of multiplicity at least α_i.

We have seen, also, that if $s = 1$ then $R/\wp^t$ has the Hilbert function of $\binom{t-1+n}{n}$ distinct *general* points of $\mathbb{P}^n$. So, it is natural to ask:

Question 1: For sufficiently general sets of points (as above) does I have the Hilbert function of $\sum_{i=1}^{s} \binom{\alpha_i - 1 + n}{n}$ distinct general points of $\mathbb{P}^n$?

There is a first simple answer to this question. **NO!**

Examples:

Let P_1, P_2 be any two points of $\mathbb{P}^2$, P_i corresponding to $\wp_i$, and let $\alpha_1 = \alpha_2 = 2$ so that $I = \wp_1^2 \cap \wp_2^2$.

Then Question 1 asks if I has the Hilbert function of 6 general points of $\mathbb{P}^2$. Since 6 general points of $\mathbb{P}^2$ have Hilbert function $1 \quad 3 \quad 6 \quad 6 \quad \cdots$ there should be no conic in I. But, if L is the equation of the line connecting P_1 and P_2 then $L^2 \in I$.

Another example comes as follows: let $P_1, \ldots, P_5$ be 5 general points in $\mathbb{P}^2$ with corresponding prime ideals $\wp_i$. Consider $I = \wp_1^2 \cap \ldots \cap \wp_5^2$. We want to know if I has the Hilbert function of $5 \cdot 3 = 15$ general points of $\mathbb{P}^2$. Since 15 general points of $\mathbb{P}^2$ have Hilbert function $1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 15 \quad \cdots$ there should be no quartic in the ideal I. But, 5 points of $\mathbb{P}^2$ always lie on a conic and if C is the equation of that conic then C^2 is a quartic in I.

With these examples one begins to wonder if Question 1 *ever* has a positive response!

Theorem 1: (J. Alexander, A. Hirschowitz) Fix any integer n. If $s >> 0$ and if the $\alpha_i \leq 2$ then I (as above) does have the Hilbert function of $\sum_{i=1}^{s} \binom{\alpha_i - 1 + n}{n}$ distinct general points of $\mathbb{P}^n$.

Remarks:

1) My formulation of Theorem 1 is a much weaker statement than that actually proved by Alexander and Hirschowitz. I'll give the precise statement later. For my purposes, this is the easiest way to give the idea of their result.
2) To give some idea of how much better than Theorem 1 the real theorem is, it suffices to note that (when $n = 2$) the two examples I've given above are the only examples for which the answer is no! (of course, when all the $\alpha_i \leq 2$.)
3) As a small indication of how much more complicated the situation is for higher exponents, I should mention that Giuliana Fattabi has recently observed (although it can be deduced, with some effort, from earlier work of S. Giuffrida) that if $P_1, \ldots, P_6$ are any 6 general points of $\mathbb{P}^2$, with corresponding prime ideals $\wp_i$, then the ideals $\wp_1^a \cap \ldots \cap \wp_6^a$ give a negative answer to Question 1 for all $a \geq 14$.
4) There has been a great deal of recent work on this problem. In addition to the work of Alexander and Hirschowitz referred to above, there have been several very interesting things done by M.V. Catalisano, A. Gimigliano, B. Harbourne, Trung and G. Valla. In particular, there is a wonderful Survey Article by Gimigliano (Our Thin Knowledge

of Fat Points) in another of the Queen's Papers in Pure and Applied Mathematics (No. 83, The Curves Seminar at Queen's, Volume VI).

Very recently, there have been some fascinating preprints by A. Iarrobino *et. al.* on this subject. Iarrobino's approach is quite different from the one taken by all the authors above and one of the purposes of these lectures is to make the approach of Iarrobino better known to people who have worked in this area (and to understand it better myself!). There are some lovely things that come out of Iarrobino's approach which give some unexpected connections between the Problem mentioned above and some very classical questions about secant varieties of the Veronese varieties. The connection will be made via Waring's Problems for homogeneous polynomials. This classical connection was brought to people's attention recently by R. Lazarsfeld. I will explain that also in the suceeding lectures.

Lecture 2: Inverse Systems

One of the fundamental ideas in Iarrobino's approach to the study of many questions concerning 0-dimensional subschemes of $\mathbb{P}^n$ is to use Macaulay's *Inverse Systems.* My impression is that this topic is not very well known to many people working in commutative algebra and algebraic geometry, particularly young people. I think, therefore, that it will be useful to include something on this basic notion in these notes. In this early discussion I will concentrate on the case of characteristic zero, but I will remedy that in a later Lecture.

In this section we will consider <u>*two*</u> polynomial rings at the same time:

$$R = k[x_1, \ldots, x_n] \quad \text{and} \quad S = k[y_1, \ldots, y_n] \ .$$

As I mentioned earlier, in order to avoid certain difficulties I will always assume that the field k has characteristic zero. Before long I will also assume that it is algebraically closed. We will think of the polynomials of R as representing partial differential operators and the polynomials of S as the "real" polynomials on which the differential operators act. This action is sometimes called the "apolarity" action of R on S. We begin with a precise definition of this action by saying

$$x_i \circ y_j = (\partial/\partial y_i)(y_j) = \begin{cases} 0 & \text{if } i \neq j \\ \\ 1 & \text{if } i = j \end{cases} .$$

In this way, the $\{x_i\}$ of R_1 behave like the basis dual to the $\{y_i\}$ of S_1. Hence R_1 can be thought of as the dual space of S_1.

If we use the standard (and formal) properties of differentiation, we can extend this action of R_1 on S_1 to:

$$R_i \quad \times \quad S_j \longrightarrow S_{j-i} \ .$$

where $r_i \times s_j := r_i \circ s_j$.

Example: Let $F_2 = x_1^2 + x_1x_2$ and let $G_4 = y_1^4 + y_2^4$. Then $F_2 \circ G_4 \in S_2$ and

$$F_2 \circ G_4 = 12y_1^2 \ .$$

Remarks:

1) Notice that the action of R on S makes S into an R-module. I.e.

$$\begin{aligned} &i) && r \circ (s_1 + s_2) = r \circ s_1 + r \circ s_2 \ ; \\ &ii) && (r_1 r_2) \circ s = r_1 \circ (r_2 \circ s) \ ; \\ &iii) && (r_1 + r_2) \circ s = r_1 \circ s \ + \ r_2 \circ s \ ; \\ &iv) && \text{and } 1 \circ s = s \ . \end{aligned} \qquad (*)$$

In addition, if $c \in k$ then

$$v) \qquad r \circ (cs) = (cr) \circ s = c(r \circ s) \ .$$

2) Note also that the action of R on S lowers degree. Thus, S is *not* a finitely generated R-module. (Some authors, in an attempt to keep the action going in the "right" direction, actually reverse the ordering on S. In that case, S is different from 0 only in non-positive degrees).

If we write a monomial of the ring R as x^α (where $\alpha = (a_1, \ldots, a_n), a_i \in \mathbb{Z}, a_i \geq 0$) and a monomial of the ring S as y^β, with β described analogously, then we say

$$\alpha \leq \beta \Leftrightarrow a_i \leq b_i \text{ for all } i \Leftrightarrow x^\alpha | x^\beta \text{ in } R \ .$$

If x^α does not divide x^β in R we write $\alpha \not\leq \beta$.

Proposition 2.1: Let x^α, y^β be as above, then

$$x^\alpha \circ y^\beta = \begin{cases} 0 & \text{if } \alpha \not\leq \beta \\ \Pi_{i=1}^n((b_i)!/(b_i - a_i)!) y^{\beta - \alpha} & \text{if } \alpha \leq \beta \end{cases} \ .$$

(Note: $0! = 1$) One sees, from this proposition, how zero characteristic enters into the picture.

Thus, in the example above we only had to observe that y_1^2 only divided y_1^4 and that $y_1 y_2$ did not divide either y_1^4 nor y_2^4.

Notice that in view of $i), iii)$ and $iv)$ of $(*)$ above, the apolarity action induces a k-bilinear pairing

$$R_j \times S_j \longrightarrow k$$

for each $j = 0, 1, \ldots$.

Now, whenever one has a k-bilinear pairing $V \times W \to k$ given by $v \times w \to v \circ w$, one has two induced k-linear maps:

$$\phi : V \longrightarrow Hom_k(W, k) \qquad \text{and} \qquad \chi : W \longrightarrow Hom_k(V, k) \ ,$$

where

$$\phi(v) := \phi_v \qquad \text{and} \qquad \phi_v(w) = v \circ w \ .$$

Similarly,

$$\chi(w) := \chi_w \qquad \text{and} \qquad \chi_w(v) = v \circ w \ .$$

Definition: The bilinear pairing $V \times W \to k$ is called *nonsingular* (or sometimes a *perfect pairing*) if the maps ϕ and χ (above) are isomorphisms.

It is well-known, and easy to prove that:

Proposition 2.2: The bilinear pairing $V \times W \to k$ is nonsingular $\Leftrightarrow$ for any basis $\{v_1, \ldots, v_n\}$ of V and $\{w_1, \ldots, w_n\}$ of W the matrix $(b_{ij} = v_i \circ w_j)$ is an invertible matrix.

With this proposition, the following is clear.

Proposition 2.3: The bilinear pairing

$$R_j \times S_j \longrightarrow k$$

induced by the apolarity action of R on S, is nonsingular.

Proof: Order the monomials of R_j by $x^{\alpha_1}, \ldots, x^{\alpha_t}$ and those of S_j by $y^{\alpha_1}, \ldots, y^{\alpha_t}$. Then, with respect to these ordered bases of R_j and S_j, the matrix of the bilinear form is a diagonal matrix (Prop. 2.1) whose ith diagonal entry is c_i, where, if $\alpha_i = (a_{i1}, \ldots, a_{in})$ then $c_i = \Pi_{j=1}^{n}(a_{ij})! \neq 0$. Thus the matrix for the pairing is invertible.

Note: The fact that the diagonal entries of the matrix described above are not 1 (if $j > 1$) means that, for $j > 1$, the bases $\{x^\alpha\}$ of R_j and $\{y^\beta\}$ of S_j are <u>*not*</u> dual bases – but they almost are! We will come back to this point later when we consider the situation in characteristic $p \neq 0$.

Remark: If $V \times W \to k$ is a pairing and $V_1 \subseteq V$ is a subspace then $V_1^\perp \subseteq W$ is the subspace of W consisting of

$$\{w \in W | v \circ w = 0 \text{ for all } v \in V_1\} \ .$$

This subspace of W is often referred to as V_1 "perp".

Alternatively,:

$$V_1^\perp = \{w \in W \mid \chi_w(V_1) = 0\}$$

(where χ is as defined above).

Likewise, if $W_1 \subseteq W$ is a subspace, we define $W_1^\perp \subseteq V$ by

$$W_1^\perp = \{v \in V \mid v \circ w = 0, \text{ for all } w \in W_1\} \ .$$

Proposition 2.4: Let $V \times W \longrightarrow k$ be a nonsingular pairing where $n = \dim_k V = \dim_k W$.

If $V_1 \subseteq V$ and $\dim_k V_1 = t$ then $\dim_k V_1^\perp = n - t$.

Proof: Let $v_1, \ldots, v_t$ be a basis for V_1 and extend that basis to a basis for all of V, $\{v_1, \ldots, v_t, v_{t+1}, \ldots, v_n\} = \mathcal{B}$. Now let $\{w_1, \ldots, w_t, w_{t+1}, \ldots, w_n\}$ be the basis for W which is dual to $\mathcal{B}$.

Clearly $w_{t+1}, \ldots, w_n \in V_1^\perp$.

On the other hand, let $w = a_1w_1 + \ldots + a_tw_t + a_{t+1}w_{t+1} + \ldots + a_nw_n$ be an arbitrary element of $V_1^\perp$. Since $v_1 \circ w = a_1$ and $v_1 \circ w = 0$ we get that $a_1 = 0$. Similarly $a_2 = \cdots = a_t = 0$ and so w is in the subspace spanned by $w_{t+1}, \ldots, w_n$. This gives that $V_1^\perp =< w_{t+1}, \ldots, w_n >$ and so $\dim_k V_1^\perp = n - t$.

Before giving the definition of Inverse Systems I want to remind you that we have before us a very general situation, namely that of a ring (R) and a module over that ring

(S). In that context there is a very simple thing one can look at – an ideal of R and the submodule of S which it annihilates or, looking from the other side, a submodule of S and the ideal of all the elements in R which annihilate that submodule. We shall, eventually, consider both of these things for the special ring and module before us, but for now we shall consider only one.

Definition: Let I be a homogeneous ideal of the ring R. The *inverse system* of I, denoted I^{-1}, is the R-submodule of S consisting of all the elements of S annihilated by I.

Remarks:

1) Suppose that $I = (F_1, \ldots, F_t)$ and $G \in S$. Then $G \in I^{-1}$ if and only if $F_1 \circ G = \cdots = F_t \circ G = 0$. Since finding all G for which $F \circ G = 0$ is nothing more than finding all the polynomial solutions to the differential equation defined by F, one sees that finding I^{-1} is the same thing as solving (with polynomial solutions) a finite set of differential equations.
2) I^{-1} is a graded submodule of S.
3) I^{-1} is not necessarily closed under multiplication, i.e. I^{-1} is not (generally) an ideal of S.

Example: Suppose that $I = (x_1) \subseteq k[x_1, x_2]$. Then, by definition,

$$I^{-1} = \{G \in S \mid (\partial/\partial y_1)(G) = 0\} .$$

Since I^{-1} is graded, it is enough to know what I^{-1} looks like in every degree.

Let $ay_1 + by_2 \in S_1$, then $(\partial/\partial y_1)(ay_1 + by_2) = a$. Thus $(I^{-1})_1 =< y_2 >$.

Let $ay_1^2 + by_1y_2 + cy_2^2 \in S_2$. Then $(\partial/\partial y_1)(ay_1^2 + by_1y_2 + cy_2^2) = 2ay_1 + by_2$, and this $= 0 \Leftrightarrow a = 0, b = 0$. Thus, $(I^{-1})_2 =< y_2^2 >$.

Continuing in this way it is easy to see that

$$I^{-1} = k \oplus < y_2 > \oplus < y_2^2 > \oplus < y_2^3 > \oplus \cdots$$

Notice several things about this example. First of all I^{-1} is *not* a finitely generated R-submodule of S (recall the direction of the R-action!), nor is it the ideal of S generated by y_2.

How do we go about finding I^{-1} more generally?

As we stated above, I^{-1} is a graded module. Thus it is enough to know $(I^{-1})_j$ for every j.

Now, by definition

$$I_j \times I_j^{\perp} \to 0$$

i.e. I_j certainly annihilates $I_j^{\perp}$, so,

$$(I^{-1})_j \subseteq I_j^{\perp} \ .$$

Proposition 2.5:

$$(I^{-1})_j = I_j^{\perp}$$

Proof: We already have an inclusion, so we may as well suppose that $G \in I_j^{\perp}$ and try to show that $G \in (I^{-1})_j$.

Since $G \in I_j^{\perp}$ we have that $h \circ G = 0$ for all $h \in I_j$. It will be enough to prove:

> *Claim:* $F \circ G = 0$ for all $F \in I$.
>
> *Pf:* *Case 1 :* $\deg(F) > j$. In this case $F \circ G = 0$ simply because the degree of F is big with respect to the degree of G.
>
> *Case 2 :* $\deg(F) < j$. In this case let $\alpha = (a_1, \ldots, a_n)$ where
>
> $$\sum_{i=0}^{n} a_i = j - \deg(F) \ .$$
>
> Then $\deg(x^\alpha F) = j$ and $x^\alpha F \in I_j$. Thus $(x^\alpha F) \circ G = 0$, i.e. $x^\alpha \circ (F \circ G) = 0$. But this means that $F \circ G$ is annihilated by every monomial x^α. Since $\deg(x^\alpha) = j - \deg(F)$ and $F \circ G \in S_{j-\deg(F)}$ and the apolarity pairing is non-singular, this implies tht $F \circ G = 0$, as we wanted to show.

This is a very useful proposition as it implies that the inverse system of I can be constructed graded piece by graded piece. There are some interesting consequences of this proposition.

Remarks:

1) $\dim_k(I^{-1})_j = \dim_k(R_j/I_j) := H(R/I, j)$.

Pf: We've already noted that $(I^{-1})_j = I_j^{\perp}$ and that $\dim_k I_j^{\perp} = \dim_k S_j - \dim_k I_j$. Since $\dim_k S_j = \dim_k R_j$ we are done.

We will have occassion to use this remark quite often. It reduces the computation of the Hilbert function to a discussion of the size of the inverse system of the ideal. One could also use it in another way to determine the size of $(I^{-1})_j$. For example,

2) I^{-1} is a finitely generated R-module $\Leftrightarrow$ I is an artinian ideal.

To see why this is so consider the nature of the R-action on S – it is clear that I^{-1} is finitely generated $\Leftrightarrow$ $(I^{-1})_j = 0$ for all $j >> 0$. By our first Remark above, this occurs $\Leftrightarrow$ $H(R/I, j) = 0$ for all $j >> 0$. This last is true if and only if I is an artinian ideal.

3) The Proposition also gives us a very simple description of the inverse system of a monomial ideal.

Since $(I^{-1})_j = I_j^{\perp}$ and we know exactly what $I_j^{\perp}$ looks like when I_j is a vector space spanned by monomials of degree j, namely

$$I_j^{\perp} = <\text{ the monomials of } S_j \text{ not “in” } I_j > \ .$$

(I have put the word “in” in quotes because I_j is not in S at all.)

I.e. the inverse system of a monomial ideal is, what has been called in the literature, an *order ideal* of monomials.

Yet another way to say this is: I^{-1} is the R-submodule of S spanned by a set of monomials which form a k-basis for R/I. (Again, note the abuse of language as $I^{-1} \subset S$, it is not in R/I.)

There is a simple thing we can say about the inverse system of an intersection of ideals.

Proposition 2.6: Let I and J be ideals of the ring R. Then

$$(I \cap J)^{-1} = I^{-1} + J^{-1} \ .$$

Since the inverse system is constructed graded piece by graded piece (Proposition 2.5), Proposition 2.6 will be an immediate consequence of the following Lemma.

Lemma 2.7: Let $V \times W \longrightarrow k$ be a nonsingular bilinear pairing with $\dim_k V = \dim_k W = n$. Let U_1 and U_2 be subspaces of V, then

$$(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp \ .$$

Pf: I will leave, as a simple exercise, the fact that $U_1^\perp \cap U_2^\perp = (U_1 + U_2)^\perp$.

$\underline{\supseteq}$:

Now $U_1 \cap U_2 \subseteq U_i$ implies that $U_i^\perp \subseteq (U_1 \cap U_2)^\perp$ for $i = 1, 2$. Thus, $U_1^\perp + U_2^\perp \subseteq (U_1 \cap U_2)^\perp$.

$\underline{\subseteq}$:

As for this inclusion we have:

$$\dim_k(U_1^\perp + U_2^\perp) = \dim_k U_1^\perp + \dim_k U_2^\perp - \dim_k(U_1^\perp \cap U_2^\perp)$$

which by the exercise above

$$= (n - \dim_k U_1) + (n - \dim_k U_2) - \dim_k(U_1 + U_2)^\perp$$

$$= n - \dim_k U_1 + n - \dim_k U_2 - [n - \dim_k(U_1 + U_2)]$$

$$= n - \dim_k U_1 + n - \dim_k U_2 - [n - (\dim_k U_1 + \dim_k U_2 - \dim_k(U_1 \cap U_2))]$$

$$= n - \dim_k(U_1 \cap U_2) = \dim_k(U_1 \cap U_2)^\perp \ .$$

Since we already have proved one containment, this equality of dimensions means the two spaces are equal.

Aside:

I've said that the ring of polynomials S is being considered, in this context, as a module over the polynomial ring R. In some sense, S seems to have lost its ring structure in the process! One thing one might ask is the following: suppose we <u>*try*</u> to remember the ring structure on S and consider an ideal $J \neq 0, J \subset S$ and we consider all the partial differential operators which annihilate this <u>*entire*</u> ideal. It's easy to see that such an annihilator is an ideal of R. But, this is not an interesting idea, as the following lemma demonstrates. (I am grateful to Tony Iarrobino for this simple and elegant proof.)

Lemma 2.8: Let $F \in S_j$ and consider $(F)_{j+d} = FS_d = V$, the $(j+d)-th$ graded piece of the ideal generated by F. The only form of degree d in R which annihilates V is the zero form.

Proof: Consider any multiplicative ordering on the monomials of S which respects degree and write

$$F = m_1 + m_2 + \cdots m_r \text{ where } m_1 > \cdots > m_r \ .$$

Let $m_1 = cy^{\alpha}$ (deg $\alpha = j$).

Let $\mathbf{n}$ be any monomial of S_d. Then

$$\mathbf{n}F = \mathbf{n}m_1 + \cdots + \mathbf{n}m_r \ .$$

Let $\mathbf{m} = x^{\alpha}$ (where $m_1 = cy^{\alpha}$) be the analogous monomial of R_d. Then

$$\mathbf{m} \circ \mathbf{n}F = \mathbf{n} + b_2 + \cdots + b_r$$

where $b_i = 0$ if m_1 does not divide $\mathbf{n}m_i$. Notice that, if we don't think of the b_i that are $= 0$, then we have written the monomials of $\mathbf{m} \circ \mathbf{n}F$ in decreasing order. Thus, $\mathbf{n}$ is the leading monomial of $\mathbf{m} \circ \mathbf{n}F$ and we have shown that the pairing

$$R_j \times V \longrightarrow S_d$$

is onto S_d (in fact, we have shown that $\mathbf{m} \times V \to S_d$ is onto).

Now, let $\mathbf{r} \in R_d$ and suppose that $\mathbf{r}$ annihilates FS_d, i.e.

$$\mathbf{r} \circ (FS_d) = 0 \ .$$

Then $(\mathbf{r}R_j) \circ (FS_d) = 0$. But, we can rewrite this as $\mathbf{r} \circ (R_j \circ FS_d) = 0$. But, we say above that the term $R_j \circ FS_d = S_d$, so we obtain $\mathbf{r} \circ S_d = 0$. But, the pairing $R_d \times S_d \to k$ is nonsingular. So, $\mathbf{r} = 0$ as we wanted to show.

Lecture 3: Inverse Systems of Fat Points

Let's make our discussion of inverse systems more precise in the case of a single fat point. We have

$$R = k[x_0, x_1, \ldots, x_n] \quad \leftrightarrow \quad \mathbb{P}^n(k)$$

$$\wp = (L_1, \ldots, L_n) \qquad \leftrightarrow \quad P \in \mathbb{P}^n$$

Since the L_i are linearly independent linear forms, we can make a linear change of variables in $\mathbb{P}^n$ so that $P = [1 : 0 : \ldots : 0]$ and $\wp = (x_1, \ldots, x_n)$, which is a monomial ideal.

Let $I = \wp^{\ell+1}$, i.e. a $\wp$-primary ideal which defines a single fat point in $\mathbb{P}^n$. By what we saw in the last lecture, and since I is a monomial ideal, we have:

$$I^{-1} = k - \text{span of } \{y^\beta \mid x^\beta \notin I\} \ .$$

Thus, to know I^{-1} it suffices to know exactly which monomials are <u>not</u> in I.

Clearly, $I_t = (0)$ for $t \leq \ell$. Thus

$$(I^{-1})_t = S_t \text{ for } t \leq \ell.$$

(It will help, in describing the rest of I^{-1} if we write $T = k[y_1, \ldots, y_n]$).

Let's group the monomials of S_t according to the power of y_0 which divides them. If we do that we get:

$$S_t = < y_0^t > \oplus < y_0^{t-1} T_1 > \oplus \cdots \oplus < y_0^{t-\ell} T_\ell > \oplus \left[< y_0^{t-(\ell+1)} T_{\ell+1} > \oplus \cdots \oplus T_t \right]$$

Notice that everything inside the large brackets is in $\wp^{\ell+1}$ and the monomials in the first part of the expression above, are not. I.e. we have

$$\left[(\wp^{\ell+1})^{-1}\right]_t = < y_0^t > \oplus < y_0^{t-1} T_1 > \oplus \cdots \oplus < y_0^{t-\ell} T_\ell >$$

$$= y_0^{t-\ell} S_\ell \ .$$

To have this all in a formal statement, we write

Proposition 3.1: Let $\wp = (x_1, \ldots, x_n) \subset k[x_0, \ldots, x_n] = R$ and let $S = k[y_0, \ldots, y_n]$. If $\ell \geq 0$ then

$$(\wp^{\ell+1})^{-1} = S_0 \oplus S_1 \oplus \cdots \oplus S_\ell \oplus y_0 S_\ell \oplus y_0^2 S_\ell \oplus \cdots \ .$$

Now, suppose that P is an arbitrary point in $\mathbb{P}^n$, i.e. $P = [p_0 : p_1 : \ldots : p_n]$ where, with no loss of generality, we may as well assume that $p_0 \neq 0$ (the discussion would proceed in the same way for any non-zero coordinate of P). If we then set $p_0 = 1$, i.e. fix all the projective coordinates of P, we can write (abusively) $P = [1 : p_1 : \ldots : p_n]$. Then, the ideal of P in the $\{x_i\}$-coordinates is $\wp = (x_1 - p_1 x_0, \ldots, x_n - p_n x_0)$.

Thus, if we make the change of variables:

$$\begin{aligned} x_0' &= x_0 \\ x_1' &= x_1 - p_1 x_0 \\ &\vdots \\ x_n' &= x_n - p_n x_0 \end{aligned}$$

then the point P has $\{x'\}$-coordinates $[1 : 0 : \ldots : 0]$. If we let $y_0', \ldots, y_n'$ in S_1 be the dual basis to $x_0', \ldots, x_n'$ in R_1, then we know precisely how to describe the inverse system of $\wp^{\ell+1}$, in the $\{y_i'\}$-coordinates:

$$\left[(\wp^{\ell+1})^{-1} \right]_t = \begin{cases} (y_0')^{t-\ell} S_\ell & \text{if } t \geq \ell \\ \\ S_t & \text{if } t < \ell \end{cases} .$$

All that remains is a description of y_0' in the $\{y_i\}$ -coordinate system.

By general nonsense about bilinear forms (and the fact that the matrix of the pairing between R_1 and S_1, with respect to the bases $\{x_i\}$ and $\{y_i\}$ respectively, is I_{n+1}) we get that

$$\begin{pmatrix} x_0' \\ \vdots \\ x_n' \end{pmatrix} = A \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \text{ where } A = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ -p_1 & 1 & 0 & \cdots & \cdots & 0 \\ -p_2 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -p_n & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

and hence that the dual basis is

$$\begin{pmatrix} y_0' \\ \vdots \\ y_n' \end{pmatrix} = (A^{-1})^t \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} .$$

Since

$$(A^{-1})^t = \begin{pmatrix} 1 & p_1 & p_2 & \cdots & \cdots & p_n \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

we obtain

$$\begin{gathered} y_0' = y_0 + p_1 y_1 + \cdots + p_n y_n \\ y_1' = y_1 \\ \vdots \\ y_n' = y_n \end{gathered} \quad .$$

In Summary: If $P = [p_0 : p_1 : \ldots : p_n] \in \mathbb{P}^n$ and $P \leftrightarrow \wp$, then

$$(\wp^{\ell+1})^{-1} = S_0 \oplus S_1 \oplus \cdots \oplus S_\ell \oplus LS_\ell \oplus L^2 S_\ell \oplus \cdots$$

where $L = p_0 y_0 + p_1 y_1 + \cdots + p_n y_n$.

Coupling these observations with Proposition 2.6 gives the following theorem (which I first saw in a paper of Ensalem and Iarrobino):

Theorem 3.2: Let $P_1, \cdots, P_s$ be points of $\mathbb{P}^n$ and suppose that $P_i = [p_{i0} : p_{i1} : \ldots : p_{in}]$. Let

$$L_{P_i} = p_{i0} y_0 + p_{i1} y_1 + \cdots + p_{in} y_n \in S = k[y_0, \ldots, y_n] \ .$$

Then, if $I = \wp_1^{n_1+1} \cap \cdots \cap \wp_s^{n_s+1} \subset R = k[x_0, \ldots, x_n]$ we have:

$$(I^{-1})_j = \begin{cases} S_j & \text{for } j \leq \max\ \{n_i\} \\ L_{P_1}^{j-n_1} S_{n_1} + \cdots + L_{P_s}^{j-n_s} S_{n_s} & \text{for } j \geq \max\ \{n_i + 1\} \end{cases} .$$

By the first remark after Proposition 2.5 we have, as an immediate corollary,

Corollary 3.3: Let $I = \wp_1^{n_1+1} \cap \cdots \cap \wp_s^{n_s+1} \subseteq R = k[x_0, \ldots, x_n]$ be as above, where $\wp_i \leftrightarrow P_i$.

Then

$$H(R/I,j)=\dim_k (I^{-1})_j$$
$$=\begin{cases}\dim_k R_j & \text{for } j\le \max\ \{n_i\}\\ \dim_k < L_{P_1}^{j-n_1}S_{n_1},\ldots ,L_{P_s}^{j-n_s}S_{n_s}> & \text{for } j\ge \max\ \{n_i+1\}\end{cases}\quad .$$

Notice that the Theorem and the Corollary above show that there is a *very* strong relationship between the Hilbert function of a set of fat points and ideals generated by powers of linear forms.

More precisely, the last expression in Theorem 3.2 says that

$$(I^{-1})_j \text{ is the } j\text{th graded piece of the ideal } (L_{P_1}^{j-n_1},\ldots ,L_{P_s}^{j-n_s}) \text{ for } j\ge \max\{n_i+1\}\ .$$

Thus, associated to the ideal of fat points

$$I=\wp_1^{n_1+1}\cap \wp_2^{n_2+1}\cap \ldots \cap \wp_s^{n_s+1}$$

is an infinite family of ideals, generated by powers of linear forms, each of which has a graded piece which interests us. These ideals will be denoted (for $j\ge \max\{n_i+1\}$);

$${}_jJ=(L_{P_1}^{j-n_1},\ldots ,L_{P_s}^{j-n_s})\ .$$

Thus,

$$(I^{-1})_j=({}_jJ)_j \qquad (\dagger)$$

Notice also that the ideals ${}_jJ$ all have radical

$$\sqrt{{}_jJ}=(L_{P_1},\ldots ,L_{P_s})\ .$$

But, what is this last ideal?

Proposition 3.4: Let $P_1,\ldots ,P_s\in \mathbb{P}^n$ and let $L_{P_1},\ldots ,L_{P_s}\in S=k[y_0,\ldots ,y_n]$ be the linear forms associated to these points. The following are equivalent:

1) $P_1,\ldots ,P_s$ span a $\mathbb{P}^t\subseteq \mathbb{P}^n$;

2) $\dim_k \langle L_{P_1}, \ldots, L_{P_s} \rangle = t+1$;

3) $\mathrm{ht}(L_{P_1}, \ldots, L_{P_s}) = t+1$.

Proof: 2) $\Leftrightarrow$ 3) is obvious.

1) $\Leftrightarrow$ 2) : Now, $P_1, \ldots, P_s$ span a $\mathbb{P}^t \subseteq \mathbb{P}^n \Leftrightarrow$ if $I = \wp_1 \cap \ldots \cap \wp_s$ then $\dim_k I_1 = n-t$. This last occurs, $\Leftrightarrow H(R/I, 1) = t+1$ and this occurs $\Leftrightarrow$ (Proposition 2.5, Remark 1) $\dim_k (I^{-1})_1 = t+1$.

But, $(I^{-1})_1 = \langle L_{P_1}, \ldots, L_{P_s} \rangle$ and so we are done.

Let's consider these ideas, in some detail, in the following example.

Example 3.5: Let $P_1, \ldots, P_6$ be 6 points of $\mathbb{P}^2$ with no 3 on a line and the 6 not on a conic. Write

$$I = \wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap \wp_4 \cap \wp_5 \cap \wp_6 \subseteq R = k[x_0, x_1, x_2] \ .$$

(This is an ideal of multiplicity $3+3+3+1+1+1 = 12$.)

Let $L_1, \ldots, L_6 \in S = k[y_0, y_1, y_2]$ be the linear forms associated to $P_1, \ldots, P_6$ respectively. Then, for $j \geq 2$ we have the ideals

$$\begin{gathered} {}_2J = (L_1, L_2, L_3, L_4^2, L_5^2, L_6^2) \\ {}_3J = (L_1^2, L_2^2, L_3^2, L_4^3, L_5^3, L_6^3) \\ {}_4J = (L_1^3, L_2^3, L_3^3, L_4^4, L_5^4, L_6^4) \\ {}_5J = (L_1^4, L_2^4, L_3^4, L_4^5, L_5^5, L_6^5) \\ \vdots \\ \text{etc.} \end{gathered}$$

From our comments above, the ideals ${}_jJ$ always have $\sqrt{{}_jJ} = (y_0, y_1, y_2)$.

Let's see what we can say about the Hilbert functions of all the ideals above. We know that $H(R/I, -)$ is eventually 12 and (since $\sqrt{{}_jJ} = (y_0, y_1, y_2)$ for all j) $H(S/{}_jJ, -)$ is eventually 0.

Now ${}_2J = (L_1, L_2, L_3) = (y_0, y_1, y_2)$, and that's about all there is to say! It follows from this that there is no conic in I, but that is something it was equally easy to deduce from a knowledge of I.

Notice also that there is no cubic in I, i.e. $H(R/I,3) = 10$. This gives that $\dim_k({}_3J)_3 = 10$. Thus, $H(S/{}_3J,-): 1\ \ 3\ \ 3\ \ 0\ \ \cdots$.

I claim that $H(R/I,4) = 12$. To see this, start with the ideal (of multiplicity 9), $\wp_1^2 \cap \wp_2^2 \cap \wp_3^2$. This contains a unique (up to scalar) cubic (draw the curve!), and no conics. Thus, the Hilbert function of this ideal is $1\ \ 3\ \ 6\ \ 9\ \ 9\ \cdots$. Since P_4 is on none of the lines which form the cubic in the ideal, $\wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap \wp_4$ has Hilbert function $1\ \ 3\ \ 6\ \ 10\ \ 10\ \cdots$.

It is equally easy to find a quartic in $\wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap \wp_4$ which is not in $\wp_5$ and so $\wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap \wp_4 \cap \wp_5$ has Hilbert function $1\ \ 3\ \ 6\ \ 10\ \ 11\ \ 11\ \cdots$. Finally the quartic consisting of: the cubic above and the line through P_4 and P_5 doesn't contain P_6, so $\wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap \wp_4 \cap \wp_5 \cap \wp_6$ has Hilbert function $1\ \ 3\ \ 6\ \ 10\ \ 12\ \ 12\ \ \cdots$, as we wanted to show.

This gives that $\dim_k({}_4J)_4 = 12$. Thus, the Hilbert function of ${}_4J$ begins $1\ \ 3\ \ 6\ \ 7\ \ 3\ \ ?$. We need to know $\dim_k({}_4J)_5$. But, from Theorem 3.2, since

$$({}_4J)_5 = \langle L_1^3S_2, L_2^3S_2, L_3^3S_2, L_4^4S_1, L_5^4S_1, L_6^4S_1 \rangle$$

we have

$$\dim_k({}_4J)_5 = H(R/\wp_1^3 \cap \wp_2^3 \cap \wp_3^3 \cap \wp_4^2 \cap \wp_5^2 \cap \wp_6^2, 5)\ .$$

Now it is easy to se that any quintic in the ideal $\wp_1^3 \cap \wp_2^3 \cap \wp_3^3 \cap \wp_4^2 \cap \wp_5^2 \cap \wp_6^2$ is a conic times the cubic (draw picture!), where the conic is in $\wp_1 \cap \wp_2 \cap \wp_3 \cap \wp_4^2 \cap \wp_5^2 \cap \wp_6^2$. But, there are no conics in this last ideal. Thus, $\dim_k({}_4J)_5 = 21$ and we know the complete Hilbert function of ${}_4J$, namely

$$H(S/{}_4J,-) = 1\ \ 3\ \ 6\ \ 7\ \ 3\ \ 0\cdots$$

(Notice that in this last calculation we observed another interesting fact - which is quite general): each of the ideals ${}_jJ$ is associated to a *finite* number of ideals of fat points. In the case above we had:

$$({}_4J)_4 \quad \leftrightarrow \quad (\wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap \wp_4 \cap \wp_5 \cap \wp_6)_4$$

$$({}_4J)_5 \quad \leftrightarrow \quad (\wp_1^3 \cap \wp_2^3 \cap \wp_3^3 \cap \wp_4^2 \cap \wp_5^2 \cap \wp_6^2)_5$$

the fact that the righthand space had dimension 0 was enough to finish this list.)

Similarly,

$$\begin{array}{lll} a)\ ({}_5J)_5 & \leftrightarrow & (\wp_1^2 \cap \wp_2^2 \cap \wp_3^2 \cap \wp_4 \cap \wp_5 \cap \wp_6)_5 \\ b)\ ({}_5J)_6 & \leftrightarrow & (\wp_1^3 \cap \wp_2^3 \cap \wp_3^3 \cap \wp_4^2 \cap \wp_5^2 \cap \wp_6^2)_6 \\ c)\ ({}_5J)_7 & \leftrightarrow & (\wp_1^4 \cap \wp_2^4 \cap \wp_3^4 \cap \wp_4^3 \cap \wp_5^3 \cap \wp_6^3)_7 \end{array}$$

etc.

until the ideal of fat points on the right hand side has nothing in it. (In this case, the list ends, as the piece $(**)_7 = (0)$). Notice also that since we know the Hilbert function of I in degree 5, then $\dim_k({}_5J)_5 = 12$, and in the last set of objects above, only the dimension of $b)$ has yet to be calculated. I'll leave that calculation (you should only find one thing in the ideal!) as an **Exercise**.

So, (in a way that I have not made precise) the two families of ideals

$$\{\ \wp_1^{2+t} \cap \wp_2^{2+t} \cap \wp_3^{2+t} \cap \wp_4^t \cap \wp_5^t \cap \wp_6^t \ \mid\ t \geq 0, t \in \mathbb{Z}\} \subseteq R$$

and

$$\{\ (L_1^s, L_2^s, L_3^s, L_4^{s+1}, L_5^{s+1}, L_6^{s+1})\ \mid\ s \in \mathbb{Z}, s \geq 1\} \subseteq S$$

are intricately related to each other. Of course, this example can be made quite general, and clearly there are general statements here waiting to be made. The relationship between such infinite families of ideals has not been studied very extensively, although Iarrobino has some results in this direction of study.

Remarks: 1) Inasmuch as there is an algorithm for calculating the Hilbert function of any ideal of the form $I = \wp_1^{n_1} \cap \ldots \cap \wp_6^{n_6}$ (when the $\wp_i$ correspond to points with the property that no three are on a line and the six are not on a conic) there is, consequently, an algorithm for finding the Hilbert function of any ideal of the form $(L_1^{m_1}, \ldots, L_6^{m_6})$ for the 6 corresponding linear forms. This has been studied very little.

2) I'm not aware of any results about the Hilbert function of ideals of the form

$$(L_1^{n_1}, \ldots, L_t^{n_t})$$

which *don't* come from a knowledge of the Hilbert function of ideals of fat points. I.e. the "balance of trade" between these two studies is that theorems about fat points are the major "export" item!

Corollary 3.3 takes on a particularly nice form when all the n_i in it are equal.

Corollary 3.6: Let $P_1, \ldots, P_s$ be points in $\mathbb{P}^n$, where $P_i \leftrightarrow \wp_i \subseteq R = k[x_0, \ldots, x_n]$. Let

$$I = \wp_1^{a+1} \cap \ldots \cap \wp_s^{a+1} .$$

Then, for $j \geq a+1$, we have:

$$H(R/I, j) = \dim_k \left[(L_1^{j-a}, \ldots, L_s^{j-a}) \right]_j .$$

As another simple corollary we can deduce a very nice (and undoubtedly well known) fact about the forms of degree d in a polynomial ring.

Corollary 3.7: Let $L_1, \ldots, L_s$ be a general set of linear forms in $S = k[y_0, \ldots, y_n]$. Then, for any integer j, the vector space

$$V = \langle L_1^j, \ldots, L_s^j \rangle$$

is as big as it can be, i.e.

$$\dim_k V = \min\{s, \dim_k S_j\} .$$

Proof: Let $P_1, \ldots, P_s$ be s general points in $\mathbb{P}^n$. Let $L_1, \ldots, L_s$ be the linear forms in $S = k[y_0, \ldots, y_n]$ corresponding to these points. Then, from Corollary 3.6 we have:

$$H(R/\wp_1 \cap \ldots \cap \wp_s, j) = \dim_k (L_1^j, \ldots, L_s^j)_j .$$

But, it is well known that a general set of s points in $\mathbb{P}^n$ has Hilbert function

$$\min\{s, \dim_k S_j\} = \min\{s, \binom{j+n}{n}\} \text{ for each } j ,$$

which finishes the proof.

Lecture 4: Waring's Problem

It follows from the corollary we proved at the end of the last lecture (Corollary 3.7), that if $t = \dim_k S_j$ and we choose t general linear forms in S_1, then every form in S_j is a linear combination of the jth powers of these fixed linear forms.

This last remark is very reminiscent of the so-called Waring Problems for integers, and I cannot pass up the opportunity to make a small side-trip to talk about these problems.

In 1770 E. Waring (in his paper *Meditationes Algebricae*) stated, without proof, the following:

1) Every natural number is a sum of (at most) 9 positive cubes;
2) Every natural number is a sum of (at most) 19 biquadratics;

and so on .....

It is believed that Waring believed (!) that for every natural number $j \geq 2$, there is a number $N(j)$ such that every positive integer n can be written:

$$n = a_1^j + \cdots + a_{N(j)}^j \text{ where } a_i \geq 0 \ .$$

Definition: If such an $N(j)$ exists, we call the least such $g(j)$.

So, Waring was asserting:

$$g(3) = 9$$

$$g(4) = 19$$

and $g(j)$ exists.

(Note, of course, Lagrange's famous theorem which says that $g(2) = 4$).

In fact, Waring's belief was justified by Hilbert.

Theorem: (Hilbert - 1909) $g(j)$ exists for every $j \geq 2$.

In fact, it is now known that:

a) $g(3) = 9$ and $g(4) = 19$, as Waring stated (although this last equality was only proved in the last few years).

b) When $j > 4$, there are at most three possibilities for $g(j)$ (they are too complicated to write down here). But, to give you some idea of the "state of the art" on this problem, there is the following theorem:

Theorem: If, for a given $j > 4$, we have

$$2^j \left\{ (3/2)^j \right\} + \left[(3/2)^j \right] \leq 2^j \tag{$*$}$$

(where $[x]$ = the greatest integer $\leq x$ and $\{x\}$ = the fractional part of x)
then

$$g(j) = 2^j + \left[(3/2)^j \right] - 2 \ .$$

E.g. $(*)$ is true for $j = 5$, so $g(5) = 2^5 - 7 - 2 = 37$.

Moreover, it is believed that $(*)$ holds for every j and it is known that $(*)$ does not hold, for at most, a finite number of j. It appears as if the problem of determining $g(j)$ is close to a final resolution.

However, the problem above is only one of the Waring problems – the so-called "Little" Waring Problem! The "Big" Waring problem starts with the observation that although $g(3) = 9$, only the numbers 23 and 239 actually **require** 9 cubes for their representation and only 15 other numbers (the largest being 8042) actually require 8 cubes. So, one is naturally lead to the following:

Definition: Let $G(j)$ be the least integer such that *all sufficiently large integers* are the sum of $\leq G(j)$, j^{th} powers of integers.

So, $G(j) \leq g(j)$ and, by the remarks above, $G(3) \leq 7$. In fact, it is not known if $G(3) < 7$, although it is known that $G(4) = 16$. In general, little is known about the numbers $G(j)$ (although e.g. $G(6) \leq 27$, $G(7) \leq 36$ etc). I just want to remark that this is an area of very active research (see e.g. T.D. Wooley, *Large Improvements in Waring's Problem*: Annals Of Math., 135 (131-164) 1992).

It can sometimes be the case that $G(j) = g(j)$. E.g. it follows from Gauss's observation that every number congruent to 7 mod(8) is a sum of 4 squares and not 3, that $G(2) = g(2) = 4$.

What about our context? i.e. that of homogeneous polynomials in a polynomial ring over a field? Using the same notation as in the Waring Problem for Integers, we can very naturally ask:

> Does there exist an integer $g(j)$ such that every element in S_j is a sum of $\leq g(j)$ j^{th} powers of linear forms?

The answer we can give is an immediate YES.

$$g(j) \text{ exists.} \quad \text{Moreover,} \quad g(j) \leq \dim_k S_j \ .$$

(This is immediate from Corollary 3.7.)

We can also consider an analogue to Waring's "Big" problem in the following way: let $N = \binom{n+j}{n}$, then we can think of S_j as an $\mathbb{A}^N(k)$, i.e. an affine space over k of dimension $N = \binom{n+j}{j}$ and let

$$\mathbb{A}^N \supseteq U_t(j) = \{F \in S_j | \ F = L_1^j + \cdots + L_t^j \text{ for } L_i \in S_1\} \ .$$

Definition: Let $G(j) =$ the least integer t such that $\overline{U_t} = \mathbb{A}^N$.

I am thinking of "closure in the Zariski-topology is the whole space" as the polynomial analogue to "all sufficiently large integers" i.e. "all but a finite number of integers".

What can we say about these numbers $G(j)$ and $g(j)$? The theorems of Gauss and Lagrange for squares of integers have analogous statements for polynomials.

Theorem 4.1: Let $S = k[y_0, y_1, \ldots, y_n]$, where k is algebraically closed and the characteristic of k is not 2, then $G(2) = g(2) = n + 1$.

(Note that since $\dim_k S_2 = \binom{n+2}{2} = (n^2+3n+2)/2$, this is a substantial improvement over the trivial bound for $g(2)$ given by Corollary 3.7.)

Proof: The proof is an immediate application of some standard facts from linear algebra.

Recall that every quadratic form in S_2 can be associated to a symmetric $(n + 1) \times (n + 1)$ matrix and that every symmetric matrix can be diagonalized. The classification of quadratic forms, over an algebraically closed field of characteristic not 2, is particularly simple: the only invariant is the rank of the associated matrix, i.e. the number of non-zero diagonal entries (which in the case of an algebraically closed field, can all be chosen as 1).

Thus, after diagonalizing the associated matrix we see that every quadratic form is a sum of $\leq n+1$ squares of linear forms, and the quadratic forms which are the sum of $< n+1$ squares of linear forms are described as the symmetric matrices of rank $\leq n$. But, a (symmetric) matrix has rank $t \leq n \Leftrightarrow$ all minors of size $(t+1) \times (t+1)$ vanish. For these minors to vanish, the entries of the matrix must satisfy certain polynomial equations in the entries of the matrix. These equations define a proper closed (non-empty) subset of S_2. Thus, also $G(2) = n+1$.

Remark: Notice that, in the notation above, the sets $U_t(2)$ are closed for every t. My impression is that this happens very infrequently. In fact, I'm not aware of any counterexample to the following:

Conjecture 4.2: Suppose that $j > 2$. The set $U_t(j)$ is closed $\Leftrightarrow t \geq g(j)$ or $t = 1$.

The fact that $U_t(j)$ is closed for $t \geq g(j)$ is simply the definition i.e. in this case $U_t(j) = S_j$. I will now explain why the statement is true when $t = 1$. This will give me an opportunity to introduce some simple ideas about linear systems that we will need later.

Recall, we have $S = k[y_0, \ldots, y_n]$ and

$$U_1(j) = \{F \in S_j \mid F = L^j \text{ for some } L \in S_1\} \ .$$

But, since k is algebraically closed, $F = L^j \Leftrightarrow cF = (c^{1/j}L)^j$. So, when looking at the set of j^{th} powers of linear forms in $A^N(k) = S_j$ ($N = \binom{n+j}{n}$), we may as well pass to the same question in the projective space based on S_j, i.e. on $\mathbb{P}(S_j)$.

Once we think of doing that, we recall the following: every $F \in S_j$ defines a hypersurface of degree j in $\mathbb{P}^n$; F and G define the same hypersurface in $\mathbb{P}^n \Leftrightarrow F = cG$ for some $c \in k, c \neq 0$. Thus, $\mathbb{P}(S_j)$ can also be thought of as representing (i.e parametrizing) the hypersurfaces in $\mathbb{P}^n$ of degree j.

Let's look at a specific example.

Example: Let $n = 2$, i.e. $S = k[y_0, y_1, y_2]$. Then $S_2 \simeq \mathbb{A}^6(k)$. What should we use as a basis for this vector space?

Recall that a quadratic form in S corresponds to a symmetric 3×3 matrix. e.g. if $F = y_0^2 + y_0y_1 + y_1^2 + y_1y_2 + y_2^2$ then

$$F \leftrightarrow \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{pmatrix} .$$

So, a "natural" basis for S_2 is

$$\{y_0^2, 2y_0y_1, 2y_0y_2, y_1^2, 2y_1y_2, y_2^2\}$$

since these correspond (in the same order) to the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which are a basis for the space of 3×3 symmetric matrices and have entries only 0 and 1.

So,

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix} \leftrightarrow F = a_{00}y_0^2 + a_{01}(2y_0y_1) + a_{02}(2y_0y_2) + a_{11}y_1^2 + a_{12}(2y_1y_2) + a_{22}y_2^2$$

$$\leftrightarrow [a_{00} : a_{01} : a_{02} : a_{11} : a_{12} : a_{22}] .$$

Now, suppose that $F = L^2$ where $L = \alpha_0y_0 + \alpha_1y_1 + \alpha_2y_2$. Then

$$L^2 = \alpha_0^2y_0^2 + \alpha_0\alpha_1(2y_0y_1) + \alpha_0\alpha_2(2y_0y_2) + \alpha_1^2y_1^2 + \alpha_1\alpha_2(2y_1y_2) + \alpha_2^2y_2^2$$

$$\leftrightarrow [\alpha_0^2 : \alpha_0\alpha_1 : \alpha_0\alpha_2 : \alpha_1^2 : \alpha_1\alpha_2 : \alpha_2^2]$$

$$\leftrightarrow \begin{pmatrix} \alpha_0^2 & \alpha_0\alpha_1 & \alpha_0\alpha_2 \\ \alpha_0\alpha_1 & \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_0\alpha_2 & \alpha_1\alpha_2 & \alpha_2^2 \end{pmatrix} = \mathcal{A}_F = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix} .$$

Note: $rk\mathcal{A}_F = 1$. This corresponds to the fact, which we already know, that with a change of bases in S_1 (a new basis which has L as one of the basis vectors) the matrix $\mathcal{A}_F$ is congruent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which has rank = 1.

Conversely, any 3×3 symmetric matrix of rank $= 1$ corresponds to a quadratic form F which is the square of a linear form. Thus, at least in the case of 3 variables, the set $U_1(2)$ is a closed subset of the space $\mathbb{P}^5$, since if we use the indeterminates $Z_{00}, Z_{11}, Z_{22}, Z_{01}, Z_{02}, Z_{12}$ in $\mathbb{P}^5$, then $U_1(2)$ is nothing more than the closed subset of $\mathbb{P}^5$ defined by the vanishing of the equations which make the matrix

$$\begin{pmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{01} & Z_{11} & Z_{12} \\ Z_{02} & Z_{12} & Z_{22} \end{pmatrix} \text{ have rank 1.}$$

I.e. which make the principal 2×2 minors vanish.

But, there is another way to view this set! If we order the monomials of S_2 as

$$y_0^2, y_0y_1, y_0y_2, y_1^2, y_1y_2, y_2^2$$

we can use them to define a function

$$\Phi : \mathbb{P}^2 \to \mathbb{P}^5$$

by

$$\Phi([a : b : c]) = [a^2 : ab : ac : b^2 : bc : c^2] \ .$$

This map is the so-called *Veronese Embedding* of $\mathbb{P}^2$ into $\mathbb{P}^5$. (The image of $\mathbb{P}^2$ in $\mathbb{P}^5$, the Veronese surface in $\mathbb{P}^5$, was the first surface (not in $\mathbb{P}^3$) which was studied in great depth.) So, what we have seen is that the Veronese surface in $\mathbb{P}^5$ "IS" the set of squares of linear forms from $S = k[y_0, y_1, y_2]$.

I won't go through all the details here, but what we did in this example is completely general.

I.e. Let $S = k[y_0, y_1, \ldots, y_n]$ and consider the space S_j and a basis for it (say given by the monomials of degree j in S – in some order $M_1, \ldots, M_{\binom{n+j}{n}}$). Define a map

$$\nu_j : \mathbb{P}^n \longrightarrow \mathbb{P}^N \qquad (N = \binom{n+j}{n} - 1)$$

by

$$\mathbf{x} \longrightarrow [M_1(\mathbf{x}) : \ldots : M_{\binom{n+j}{n}}(\mathbf{x})] \ .$$

One proves that this map is an embedding and that the image is a closed subvariety of $\mathbb{P}^N$ defined by a collection of quadratic polynomials. This image, denoted $\nu_j(\mathbb{P}^n)$, is also called a **Veronese variety**. (Hartshorne, in his book, calls this variety the j-uple embedding of $\mathbb{P}^n$.) In a fashion analogous to what we did for the quadrics, we can show that:

the Veronese variety $\nu_j(\mathbb{P}^n)$ is exactly the set of j^{th}-powers of linear forms from S_1 .

(Take care, in this identification one has to "scale" the coordinates in $\mathbb{P}^N$ by multinomial coefficients, like we did with the "2's" in the case when $j = 2$.)

Having said this about the Veronese varieties, $\nu_j(\mathbb{P}^n)$, the question arises as to how we should think about "sums" of j^{th}-powers.

Recall that, quite generally, if $P_1 = [a_0 : a_1 : \ldots : a_r]$ and $P_2 = [b_0 : b_1 : \ldots : b_r]$ are two distinct points of $\mathbb{P}^r$ then we can use the notation $P_1 + P_2$ to refer to <u>*all*</u> the points on the line in $\mathbb{P}^r$ which connects P_1 and P_2. The way we do this, of course, is to say

$$P_1 + P_2 = \{Q \in \mathbb{P}^r \mid \; Q = [\lambda a_0 + \mu b_0 : \ldots : \lambda a_r + \mu b_r] \text{ where } [\lambda : \mu] \in \mathbb{P}^1\} \; .$$

This is exactly the set of points on a line of $\mathbb{P}^r$ (or a "plane" in $\mathbb{A}^{r+1}$, the set of all linear combinations of the vectors $\mathbf{v_1} = (a_0, \ldots, a_r)$ and $\mathbf{v_2} = (b_0, \ldots, b_r)$ except for $\mathbf{0}$).

So, if we want to speak about sums of two j^{th} powers of linear forms in S, this is the same thing as speaking about the "*chord*" to $\nu_j(\mathbb{P}^n)$ which connects the two points, i.e. a (non-degenerate) secant line of $\nu_j(\mathbb{P}^n)$.

Similarly, if we want to speak about a sum of three j^{th} powers, we will then be speaking about a plane (i.e. a $\mathbb{P}^2$) in $\mathbb{P}^N$ which contains 3 distinct points of $\nu_j(\mathbb{P}^n)$. These are called $3-$ *secant planes* or $3-$ *secant* $\mathbb{P}^2$'s. If there is space enough in $\mathbb{P}^N$ we can speak about $s-$ *secant* $\mathbb{P}^{s-1}$'s, i.e. $\mathbb{P}^{s-1}$'s which contain s distinct points of $\nu_j(\mathbb{P}^n)$.

Thus, we can rephrase our two questions above in a geometric manner:

"The Little Waring Problem": What is the least integer s such that every point of $\mathbb{P}^N$ ($N = \binom{j+n}{n} - 1$) is on a $t-$ *secant* $\mathbb{P}^{t-1}$ to t distinct points of $\nu_j(\mathbb{P}^n)$ for some $t \leq s$?

"The Big Waring Problem": What is the least integers s such that the Zariski closure (in $\mathbb{P}^N$) of the points on

$$\cup_{t=1}^{s}(t - \textit{secant } \mathbb{P}^{t-1})\text{'s}, \;\; (\text{based on } t \text{ distinct points of } \nu_j(\mathbb{P}^n)\;) = \mathbb{P}^N \; .$$

After making some preliminary investigations of these problems in some special cases, I will then give the results I know. Interestingly enough, there is an answer to the "BIG" problem (which is due to Alexander - Hirschowitz) but I know of no answer to the "Little" problem. (Although, in a recent paper of Ehreborg and Rota they speak of coming back to this problem in a future paper.)

There is an enormous classical literature on these problems, with the names of Clebsch (1860's), Terracini, Lasker, Bertini, Severi, Palatini (1900's), Wakeford (1920's), Bronowski (1930's), and Reznick, Ehreborg and Rota (1990's) figuring prominently in this study. I will also try to say something about these works.

But, before I begin to get into a more detailed discussion of the $t-\textit{secant}$ $\mathbb{P}^{t-1}$'s to the Veronese varieties, it is necessary (and interesting) to get familiar with some of the basic properties of these varieties. I'll first do that.

In considering the Veronese varieties, $\nu_j(\mathbb{P}^n)$, it is very important not to forget their connecton with $\mathbb{P}^n$ itself. Let me illustrate what I mean by continuing with the concrete example above of the Veronese surface, $\nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$. We continue to use $k[y_0, y_1, y_2]$ for the homogeneous coordinate ring of $\mathbb{P}^2$ and $k[Z_{00}, Z_{01}, Z_{02}, Z_{11}, Z_{12}, Z_{22}]$ for the homogeneous coordinate ring of $\mathbb{P}^5$.

Consider the quadratic form, $F = 2y_0^2 + y_1^2 + y_1y_2 + y_2^2$ and the conic $\mathcal{C}$ that it defines in $\mathbb{P}^2$. What about the image of this conic under the map ν_2?

Let $[a:b:c] \in \mathcal{C}$, i.e. $2a^2 + b^2 + bc + c^2 = 0$. Then

$$\nu_2[a:b:c] = [a^2 : ab : ac : b^2 : bc : c^2] \ .$$

> *Claim:* The point $\nu_2[a:b:c]$ lies on the hyper$\underline{\textit{plane}}$ of $\mathbb{P}^5$ defined by the equation
>
> $$H_F = 2Z_{00} + Z_{11} + Z_{12} + Z_{22} = 0 \ .$$
>
> (which is obvious, since $2a^2 + b^2 + bc + c^2 = 0$.)

I.e. $\nu_2(\mathcal{C}) = \nu_2(\mathbb{P}^2) \cap H_F$ where H_F is the hyperplane in $\mathbb{P}^5$ determined by F.

But, this example is completely general: if we are considering $\nu_j(\mathbb{P}^n) \subseteq \mathbb{P}^N$ $(N = \binom{n+j}{j} - 1)$ then $F \in S_j$ defines a hypersurface $V(F) \subseteq \mathbb{P}^n$ and

$$\nu_j(V(F)) = \nu_j(\mathbb{P}^n) \cap H_F$$

where H_F is the <u>*hyperplane*</u> of $\mathbb{P}^N$ determined by F.

Thus, ν_j "converts" problems concerning intersections of hypersurfaces of degree j in $\mathbb{P}^n$ into problems concerning intersections of hyperplanes in $\mathbb{P}^N$ with the variety $\nu_j(\mathbb{P}^n)$, and conversely.

To see how this works in practice, let's apply these ideas to make a calculation of the degree of the Veronese <u>$surfaces$</u>, $\nu_j(\mathbb{P}^2)$.

> *Claim:* The degree of the surfaces $\nu_j(\mathbb{P}^2) = j^2$.
>
> *Pf.* Let $\mathcal{S} = \nu_j(\mathbb{P}^2) \subseteq \mathbb{P}^N$ ($N = \binom{j+2}{2} - 1$). Like any surface in $\mathbb{P}^N$, the degree of $\mathcal{S}$ is the number of distinct points in which a general $\mathbb{P}^{N-2} \subseteq \mathbb{P}^N$ neets $\mathcal{S}$.
>
> But, a $\mathbb{P}^{N-2} \subseteq \mathbb{P}^N$ is the zeroes of two independent linear forms in $\mathbb{P}^N$. So, we want to know
>
> $$H_{F_1} \cap H_{F_2} \cap \nu_j(\mathbb{P}^2)\ . \qquad (*)$$
>
> But, $H_{F_i} \cap \nu_n(\mathbb{P}^2)$ corresponds to the zeroes of F_i in $\mathbb{P}^2$. So, the common points to the three varieties in $(*)$ are: the points in $\mathbb{P}^2$ (corresponding to $\nu_j(\mathbb{P}^2)$) at which $V(F_1)$ meets $V(F_2)$. I.e. $V(F_1) \cap V(F_2)$ in $\mathbb{P}^2$. But, for F_1, F_2 general forms of degree j in $\mathbb{P}^2$, Bezout's theorem gives that the intersection consists of j^2 distinct points.

(Note: The same kind of argument will give that the degree of $\nu_j(\mathbb{P}^n) = j^n$.)

Lecture 5: Veronese Varieties

We saw, in the last lecture, that there is a connection between Waring's problems concerning the representation of a homogeneous form of degree j as a sum of j^{th} powers of linear forms and the geometry of certain secant varieties to the Veronese varieties. As I mentioned then, I would like to spend a little time recovering some simple results about these secant varieties.

Proposition 5.1: Let $\mathcal{S} = \nu_j(\mathbb{P}^n) \subseteq \mathbb{P}^N$ $(N = \binom{j+n}{n} - 1)$. A line of $\mathbb{P}^N$ can meet $\mathcal{S}$ in <u>*at most*</u> two points, i.e. the intersection is a subscheme (of $\mathbb{P}^1$) of multiplicity ≤ 2.

Proof: Since $\dim(\mathcal{S}) + \dim(\mathbb{P}^1) = n+1$ is (usually) much smaller than N, we don't expect a line to meet $\mathcal{S}$ at all!

Now, a $\mathbb{P}^1$ in $\mathbb{P}^N$ is the intersection of $N-1$ linearly independent linear forms which define hyperplanes $H_{F_1}, \ldots, H_{F_{N-1}}$. The intersection of this line with $\mathcal{S}$ corresponds to the intersection, in $\mathbb{P}^n$, of the hypersurfaces of degree j, $V(F_1), \ldots, V(F_{N-1})$.

If we let $\mathbb{X}$ denote the subscheme of $\mathbb{P}^n$ in which these varieties meet, i.e. the subscheme defined by the ideal $(F_1, \ldots, F_{N-1})^{sat} = I$, then

$$\dim_k\left((k[y_0, \ldots, y_n]_j / I_j\right) \leq \binom{n+j}{n} - \left[\binom{n+j}{n} - 1 - 1\right] = 2 \ .$$

What can we say about $\mathbb{X}$, given that $H_{\mathbb{X}}(j) \leq 2$ for $j \geq 2$? Quite a bit!

First of all, by Macaulay's Theorem (which describes the growth of the Hilbert function of an ideal) we know that $H_{\mathbb{X}}(j+t) \leq 2$ for all $t \geq 0$. So,

> 1) if for some t we have $H_{\mathbb{X}}(j+t) = 0$, then this means that the line we began with doesn't meet $\nu_j(\mathbb{P}^n)$ at all.
> (So, we may as well suppose that $H_{\mathbb{X}}(j+t) \neq 0$ for any $t \geq 0$.)
>
> 2) if $H_{\mathbb{X}}(j+t_0) = 1$ for some $t_0 \geq 0$ then, again by Macaulay's Theorem, this would imply that $H_{\mathbb{X}}(j+t) = 1$ for all $t \geq t_0$. Thus, the Hilbert polynomial of $\mathbb{X}$ is a constant $(= 1)$ and so $\mathbb{X}$ is a single (reduced) point.
> (This can occur: just take F_1, F_2, F_3, F_4 to be four independent conics which have only one point in common and where two of the F_i pass

through that point transversally. Then, the line in $\mathbb{P}^5$ obtained as the intersection of the hyperplanes H_{F_i} intersects $\nu_2(\mathbb{P}^2)$ simply.)

3) now suppose that $H_{\mathbb{X}}(j+t) = 2$ for all $t \geq 0$. In this case, $\mathbb{X}$ is a subscheme of $\mathbb{P}^N$ of multiplicity 2 (the Hilbert polynomial must stay at 2). Since I is a saturated ideal, S/I contains a non-zerodivisor of degree 1 and so the Hilbert function of $\mathbb{X}$ has to be:

$$H(S/I,-) : 1 \quad 2 \quad 2 \quad \cdots$$

(and indeed $(F_1, \ldots, F_{N-1})^{sat}$ is much bigger than $(F_1, \ldots, F_{N-1})$ although they agree in degree j).

It follows that $\dim_k I_1 = (n+1) - 2 = n - 1$. Thus, by making a linear change of variables in S we may as well assume that $I_1 = (y_1, \ldots, y_{n-1})$. Thus, $S_2/S_1I_1 =< \overline{y_0}^2,\ \overline{y_n}^2,\ \overline{y_0y_n} >$. Since $\dim_k(S_2/I_2) = 2$ there are $a,\ b,\ c\ \in k$ such that

$$ay_0^2 + by_n^2 + cy_0y_n \in I_2 \ .$$

Since the latter is a quadratic form in two variables we can write it as L_1L_2 (where L_1 and L_2 are independent linear forms in y_0 and y_n) or as L_1^2. In the first case we might as well assume that $L_1L_2 = y_0y_n$ and the second that $L_1^2 = y_n^2$. So, up to a linear change of coordinates in $\mathbb{P}^n$ either:

$$I = (y_1, \ldots, y_{n-1}, y_0y_n)$$

(the ideal of 2 distinct points,)

or

$$I = (y_1, \ldots, y_{n-1}, y_n^2)$$

(the ideal of a scheme of multiplicity 2 supported at one point.)

In either case, the proposition is now complete.

Things get only slightly more complicated if we consider intersections with planes.

Proposition 5.2: Let $\mathcal{S} = \nu_j(\mathbb{P}^n) \subseteq \mathbb{P}^N$ $(N = \binom{j+n}{n} - 1)$.

A plane in $\mathbb{P}^N$, if it meets $\mathcal{S}$,

1) meets it in a plane conic (i.e. a rational normal curve in $\mathbb{P}^2$) (in which case $j = 2$); or

2) meets it in a zero-dimensional scheme of multiplicity ≤ 3.

Proof: As above, a plane in $\mathbb{P}^N$ (i.e. a $\mathbb{P}^2$ in $\mathbb{P}^N$) is the intersection of $N-2$ linearly independent linear forms which define hyperplanes $H_{F_1}, \dots, H_{F_{N-2}}$. The intersection of this $\mathbb{P}^2$ with $\nu_j(\mathbb{P}^n)$ is the subscheme of $\mathbb{P}^n$ defined by $(F_1, \dots, F_{N-2})^{sat} = I$ and we know that $\dim_k(S_j/I_j) \leq 3$.

Case 1: If $j \geq 3$ then Macaulay's Theorem says that

$$\dim_k(S_{j+t}/I_{j+t}) \leq 3$$

for all $t \geq 0$. Since we are assuming that the $\mathbb{P}^2$ does meet $\mathcal{S}$ then I defines a zero-dimensional subscheme of multiplicity ≤ 3. The same argument works if $j = 2$ and $\dim_k(S_2/I_2) \leq 2$.

Case 2: If $j = 2$ and $\dim_k(S_2/I_2) = 3$. Then, since $3_{(2)} = \binom{3}{2}$ we have $3^{<2>} = \binom{4}{3} = 4$. So, there seem to be several possibilities for the Hilbert function of S/I.

a). *If the Hilbert function is eventually constant*, then I defines a zero dimensional scheme. Since I is a saturated ideal we either have:

$$\alpha) \quad H(S/I, -) = 1 \quad 3 \quad 3 \quad \cdots$$

or

$$\beta) \quad H(S/I, -) = 1 \;\; 2 \;\; 3 \quad \cdots s-1 \;\; s \;\; s \quad \cdots .$$

In case α) we have that I defines a scheme of multiplicity 3 and we are done. In case β), I is the ideal of a complete intersection subscheme of multiplicity s, supported on a line L and $I_2 = (F_1, \dots, F_{N-2})_2$. But, $I = (L_1, \dots, L_{n-1}, G)$, where $L_1, \dots, L_{n-1}$ define the line L and G is a form of degree $s \geq 3$. But then, $I_2 = (L_1, \dots, L_{n-1})_2$ and so $(F_1, \dots, F_{N-2})^{sat} = I$ is the ideal of a **line**, not a set of points. So $H(S/I, -)$ is <u>not</u> eventually constant.

b). The Hilbert function of S/I is **not** *eventually constant.*

Since I is saturated the Hilbert function of S/I never decreases, and if it is constant once it is constant ever after. This gives us only one possible Hilbert function for S/I (since we know the Hilbert function in degree 2), namely:

$$H(S/I,-) =: 1 \ \ 2 \ \ 3 \ \ \cdots \ \ s \ \ s+1 \ \ \cdots$$

But, in this case, I defines a scheme of dimension 1 and degree 1 (all of that deducible from the Hilbert polynomial). But, from the fact that $2 = H(S/I, 1)$ we find that I defines a line. The image of this line under ν_2 gives a plane conic in $\mathbb{P}^N$ as we wanted to show.

Things get sucessively more complicated!

Proposition 5.3: Let $\mathcal{S} = \nu_j(\mathbb{P}^n) \subseteq \mathbb{P}^N$ where ($N = \binom{j+n}{n} - 1$).

A $\mathbb{P}^3$ in $\mathbb{P}^N$, if it meets $\mathcal{S}$, meets it either in

a) a zero dimensional scheme of multiplicity ≤ 4;

or

b) a rational normal curve of degree 3 (in which case, $j = 3$);

or

c) a rational normal curve of degree 2, i.e. a plane conic (in which case, $j = 2$).

Proof: As before, we find $F_1, \ldots, F_{N-3} \in S_j$ which are linearly independent and such that, if $I = (F_1, \ldots, F_{N-3})^{sat}$ then $\dim_k(S_j/I_j) \leq 4$.

Since $4_4 = \binom{4}{4} + \binom{3}{3} + \binom{2}{2} + \binom{1}{1}$ we have $4^{<4>} = 4$, and it follows from Macaulay's Theorem that

Case 1: If $j \geq 4$ or if $j = 3$ and $\dim_k(S_3/I_3) \leq 3$ or if $j = 2$ and $\dim_k(S_2/I_2) \leq 2$ then (assuming that our original $\mathbb{P}^3$ meets $\mathcal{S}$), I defines a zero dimensional subscheme of multiplicity ≤ 4.

Case 2: If $j = 3$. Then we may assume that $I_3 =< F_1, \ldots, F_{N-3} >$ and $\dim_k(S_3/I_3) = 4$. Now, $4_3 = \binom{4}{3}$ and so $4^{<3>} = \binom{5}{4} = 5$. Since the Hilbert function cannot decrease, we have that:

$$H(S/I,-) = 1 \ \ 2 \ \ 3 \ \ 4 \ \ a \ \ ? \ \ ?$$

where the only possibilities for a are 4 or 5.

If $a = 4$ then I defines a zero dimensional scheme of multiplicity 4 and we are done.

If $a = 5$: then either $H(S/I, -)$ continues to grow by 1 – in which case I is the ideal of a line in $\mathbb{P}^n$, which is taken by ν_3 into a rational normal curve of degree 4 (and we are done) – or the Hilbert function stops growing at some point.

In that case I would describe a set of ≥ 5 points on a line of $\mathbb{P}^n$. If $(L_1, \ldots, L_{n-1}) = J$ is the ideal of this line, then (since the points are on a line)

$$I_3 =< F_1, \ldots, F_{N-3} >= (L_1, \ldots, L_{n-1})_3$$

and so $(F_1, \ldots, F_{N-3})^{sat} = J$, not I (the ideal of points!). Thus, this case does not occur.

Case 3: $j = 2$.

α) We first consider the situation when $H(S/I, 2) = 3$. Then

$$H(S/I, -) =: 1 \quad 2 \quad 3 \quad a \quad ?$$

where $a = 3$ or $a = 4$. If $a = 3$ then I describes a zero dimensional sheme of multiplicity ≤ 3 and we are done. If $a = 4$ there are two possibilities: either the Hilbert function continues to grow by 1 (the maximum possible) or it does not. In the first instance we obtain that I is the ideal of a line in $\mathbb{P}^n$ and then ν_2 of that line is a plane conic (and we are done). In the second instance we obtain that I describes a set of ≥ 4 points on a line. If, as in case 2), we let J denote the ideal of that line, then:

$$I_2 = J_2 \supsetneq < F_1, \ldots, F_{N-3} >$$

and so $(F_1, \ldots, F_{N-3})^{sat} = J$ and not I, so this is impossible.

β) Let $H(S/I, 2) = 4$. Since $4_2 = \binom{3}{2} + \binom{1}{1}$ we have $4^{<2>} = \binom{4}{3} + \binom{2}{2} = 5$. So, we must have:

$$H(S/I, -) = 1 \quad 3 \quad 4 \quad a \quad ?$$

where $a = 4$ or $a = 5$.

If $a = 4$ then I describes a zero dimensional scheme of multiplicity 4 in $\mathbb{P}^n$ and we are done.

If $a = 5$ there are two possibilities: either $H(S/I, -)$ continues to grow by 1 or it stops growing at some point. In the first instance we obtain, by examining the Hilbert polynomial, that I describes a curve of degree 1, i.e. a line. But, there are not enough linear forms in I to give a line! So, this case cannot occur and we must assume that I describes a zero dimensional subscheme of multiplicity $s \geq 5$.

Now, the linear forms in I describe a plane, and we can factor out those forms and assume that our scheme lives in $\mathbb{P}^2$. From the Hilbert function we see that the ideal of the scheme contains two quadrics. These two quadrics either have a linear factor in common (in which case the saturation of the ideal generated by the quadrics $F_1, \ldots, F_{N-3}$ is the ideal of that line, and so cannot occur with our hypothesis.) or the two quadrics have no common linear form. In the later case, the scheme defined by those two quadrics cannot have multiplicity > 4. Since we have that the multiplicity is at least 5, this case cannot occur either and we are finished with the proof. $\square$

(che fatica!)

Warning: Notwithstanding the apparent pattern that the two results above seem to demonstrate, it is not necessarily true that if we have $\nu_j : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ and write $\mathcal{S} = \nu_j(\mathbb{P}^n)$ that a $\mathbb{P}^t \subseteq \mathbb{P}^N$, if it meets $\mathcal{S}$ in a zero dimensional scheme, meets it in a scheme of multiplicity $\leq t + 1$. Consider the case (among many) of $\nu_3 : \mathbb{P}^3 \to \mathbb{P}^{19}$ and consider a $\mathbb{P}^{16} \subset \mathbb{P}^{19}$ which corresponds to a 3 dimensional subspace of cubics which are a regular sequence. Such a $\mathbb{P}^{16}$ meets $\mathcal{S}$ in exactly 27 points of $\mathbb{P}^3$.

I don't know, for a given t, what the maximum multiplicity of intersection a $\mathbb{P}^t$ in $\mathbb{P}^N$ can have with $\mathcal{S}$ (always assuming that the intersection is zero dimensional). It seems to me one should be able to do this for $\mathbb{P}^2$, but perhaps it is even known in general.

Remark: Before I finish off this circle of ideas and explain the solution to the "Big" Waring problem, I want to show that $U_2(3)$ is not a closed set, i.e. the polynomials in

$S = k[y_0, y_1, y_2]$ of degree three which are a sum of two cubes of linear forms do not fill up the chordal variety to $\nu_3(\mathbb{P}^2) \subset \mathbb{P}^9$.

The idea here (due to Marvi Catalisano) is to show that the points of the tangent planes to $\nu_3(\mathbb{P}^2) = \mathcal{S}$ are all in the closure of the set of "true" secant lines to $\mathcal{S}$, but are not themselves on any "true" secants.

So, let $P \in \nu_3(\mathbb{P}^2) = \mathcal{S}$ and let Π be the tangent plane to $\mathcal{S}$ at P. It is not difficult to see that Π meets $\mathcal{S}$ at P in a subscheme of $\mathbb{P}^9$ of multiplicity ≥ 3 and that any line in Π, through P, meets $\mathcal{S}$ at P in a subscheme of $\mathbb{P}^9$ of multiplicity at least 2.

Now, let $Q \in \Pi$ $(Q \neq P)$. I want to show that Q is not on a "true" secant line, i.e. the cubic form in S_3 which corresponds to Q is not the sum of two cubes of linear forms.

Suppose it were. Then we could find $P_1,\ P_2 \in \mathcal{S}$ such that the line through P_1 and P_2 meets Π at Q. Let Π' be the plane containing the intersecting lines: $\overline{P_1P_2}$ and $\overline{PQ}$. Π' meets $\mathcal{S}$ at least once at P_1 and at P_2 and at least twice at P. I.e. its intersection with $\mathcal{S}$ is a subscheme of $\mathbb{P}^9$ of multiplicity at least 4. But, by Proposition 5.2 (note that $j = 3$ in our case), this is impossible. That contradiction finishes off the remark.

Aside: For those of you who have a problem with the "intersection" argument I offer the following extra remarks which may make the argument above more palatable!

> Make a change of coordinates so that $P = [1 : 0 : \ldots : 0] \in \mathbb{P}^9$ and the tangent plane Π has defining ideal $(y_1, \ldots, y_7)$. Let $\wp \subseteq S = k[y_0, \ldots, y_9]$ be the prime ideal which defines $\mathcal{S}$ and let $\tilde{\wp}$ be the dehomogenization of $\wp$ with respect to y_0. If $f \in \tilde{\wp}$ then $f = f_1 + \cdots + f_r$ where $f_i \in R = k[y_1, \ldots, y_9]$ has degree i (there is no f_0 since f vanishes at $P = (0, \ldots, 0)$). The statement that Π has defining ideal $(y_1, \ldots, y_7)$ means that the vector subspace of R_1 spanned by the linear parts of the $f \in \tilde{\wp}$ is that generated by $y_1, \ldots, y_7$.
>
> So, to see "how much" Π meets $\mathcal{S}$ at P, we consider
>
> $$\frac{k[y_1, \ldots, y_9]_{(y_1, \ldots, y_9)}}{(y_1, \ldots, y_9, \tilde{\wp})}$$
>
> This is easily seen to be isomorphic to
>
> $$\frac{k[y_8, y_9]_{(y_8, y_9)}}{(\tilde{\wp}')} = B$$

where $\tilde{\wp}'$ is obtained from $\tilde{\wp}$ by setting $y_1 = \cdots = y_7 = 0$ in $\tilde{\wp}$. By our assumption about the tangent plane, this kills all the linear parts of elements of $\tilde{\wp}$ and so $\dim_k B \geq 3$ (with $\overline{1}$, $\overline{y_8}$, $\overline{y_9}$ definitely linearly independent and outside $\tilde{\wp}'$.

Now suppose we take a line L in the plane Π (w.l.o.g. assume the line is defined by the ideal $(y_1, \ldots, y_7, y_8)$. Then

$$\frac{k[y_1, \ldots, y_9]_{(y_1,\ldots,y_9)}}{(y_1, \ldots, y_8, \tilde{\wp})} \simeq \frac{k[y_8, y_9]_{(y_8,y_9)}}{(y_8, \tilde{\wp}')}$$

and this latter clearly has $\dim_k \geq 2$ (with $\overline{1}, \overline{x_9}$ clearly independent and outside $(y_8, \tilde{\wp}')$.

Now take any other plane, besides Π, which contains L (again, with no loss of generality we can assume this other plane is defined by the ideal $(y_1, \ldots, y_6, y_8)$). Then,

$$\frac{k[y_1, \ldots, y_9]_{(y_1,\ldots,y_9)}}{(y_1, \ldots, y_6, y_8, \tilde{\wp})}$$

When we now set $y_1 = \cdots = y_6 = y_8 = 0$ in $\tilde{\wp}$, and call the resulting ideal $\tilde{\wp}''$, then it has elements all of the form

$$\alpha y_7 + \tilde{f}_2 + \cdots + \tilde{f}_r \text{ where } \tilde{f}_i \text{ has degree } i \text{ in } k[y_7, y_9]$$

Thus

$$\dim_k \frac{k[y_7, y_9]_{(y_7,y_9)}}{(\tilde{\wp}'')} \geq 2$$

(with $\overline{1}, \overline{y_9}$ linearly independent and outside the ideal.)

This is enough to justify the argument above once we note that if q is $\wp$-primary (where q and $\wp$ are homogeneous ideals in S) then

$$e(q) = e(\wp) \dim_{A_\wp/\wp A_\wp}(A_\wp/qA_\wp) \ .$$

I now want to explain how to obtain the solution to Waring's Big Problem for homogeneous polynomials.

Let $S = k[y_0, \dots, y_n]$. We want to know which elements of S_j can be written as a sum of $s - j^{th}$ powers of linear forms. This is the same thing as understanding the image of the map:

$$\Phi : \underbrace{S_1 \times \cdots \times S_1}_{s-\text{times}} \longrightarrow S_j$$

given by

$$\Phi(L_1, \dots, L_s) = L_1^j + \cdots + L_s^j \ .$$

If we view S_1 as $\mathbb{A}^{n+1}(k)$ and S_j as $\mathbb{A}^N(k)$ (where $N = \binom{n+j}{n}$) then Φ can be seen as a polynomial map

$$\Phi : \mathbb{A}^{s(n+1)} \longrightarrow \mathbb{A}^N(k) \ .$$

We are most interested in knowing the dimension of the *image* of this map. The way we will do this is to consider the differential of the map Φ, i.e $d\Phi$.

Recall that $d\Phi$ is a function which, for every point $P \in \mathbb{A}^{s(n+1)}$, gives a linear transformation $(d\Phi)(P)$, from the tangent space of $\mathbb{A}^{s(n+1)}$ at P to the tangent space of $\mathbb{A}^N$ at $\Phi(P)$ i.e.

$$(d\Phi)(P) := d\Phi|_P : T_P(\mathbb{A}^{s(n+1)}) \longrightarrow T_{\Phi(P)}(\mathbb{A}^N)$$

Since the tangent space to $\mathbb{A}^t$ at any of its points is again $\mathbb{A}^t$, we have that

$$d\Phi|_P : \mathbb{A}^{s(n+1)} \longrightarrow \mathbb{A}^N \ .$$

Thus, if we know the (generic) rank of these linear transformations, we'll know the dimension of the image.

So, for a given point P, how do we calculate the differential of Φ at that point? i.e. given $\mathbf{v} \in T_P(\mathbb{A}^{s(n+1)})$ how do we find $\left[(d\Phi)|_P\right](\mathbf{v})$?

The usual way to do this is to find a curve $\mathcal{C}$ through the point P, whose tangent vector at P is $\mathbf{v}$, and then take the curve $\Phi(\mathcal{C})$ and find its tangent vector at $\Phi(P)$.

So, pick a point $P = (L_1, \dots, L_s) \in \mathbb{A}^{s(n+1)}$ and a vector $\mathbf{v} \in T_P(\mathbb{A}^{s(n+1)}) \simeq \mathbb{A}^{s(n+1)}$. We write $\mathbf{v} = (M_1, \dots, M_s)$ where we think of the M_i as elements of $\mathbb{A}^{n+1}$ for $i = 1, \dots, s$ (i.e. we think of the M_i as elements of S_1).

Consider the following (parametrized) curve in $\mathbb{A}^{s(n+1)}$ through P, with tangent vector $\mathbf{v}$ at P:

$$t \quad \longrightarrow \quad (L_1 + M_1 t, L_2 + M_2 t, \dots, L_s + M_s t)$$

(i.e. a straight line through P in the direction $\mathbf{v}$.)

What is the image of this curve?

$$\Phi(L_1 + M_1 t, L_2 + M_2 t, \ldots, L_s + M_s t) = \sum_{i=1}^{s} (L_i + M_i t)^j \ .$$

We can find the tangent vector, in $\mathbb{A}^N$, to $\Phi(\mathcal{C})$ at $\Phi(P)$ as follows:

$$\frac{d}{dt}(\sum_{i=1}^{s}(L_i + M_i t)^j) = \sum_{i=1}^{s} j(L_i + M_i t)^{j-1} M_i$$

and if we evaluate this when $t = 0$ we find that the tangent vector to $\Phi(\mathcal{C})$ at $\Phi(P)$ is $\sum_{i=1}^{s} j L_i^{j-1} M_i$.

Thus, as we let $\mathbf{v} = (M_1, \ldots, M_s)$ vary over the whole space $\mathbb{A}^{s(n+1)}$, the tangent vectors we get vary over all the forms in the vector space $< L_1^{j-1} S_1, \ldots, L_s^{j-1} S_1 >$. I.e. the rank of the differential at $P = (L_1, \ldots, L_s)$ is $\dim_k < L_1^{j-1} S_1, \ldots, L_s^{j-1} S_1 >$.

Putting together everything we have seen up to this point, we obtain the following:

Theorem 5.5: Let $L_1, \ldots, L_s$ be linear forms in $S = k[y_0, \ldots, y_n]$ where

$$L_i = a_{i0} y_0 + a_{i1} y_1 + \ldots + a_{in} y_n$$

and let $P_1, \ldots, P_s \in \mathbb{P}^n(k)$ where

$$P_i = [a_{i0} : \ldots : a_{in}] \ .$$

Moreover, let $P_i \leftrightarrow \wp_i \subset S$.

Let

$$\Phi : \underbrace{S_1 \times \cdots \times S_1}_{s-\text{times}} \longrightarrow S_j \text{ be given by } \Phi(L_1, \ldots, L_s) = L_1^j + \cdots + L_s^j \ .$$

then

$$rk\ (d\Phi)|_{(L_1,\ldots,L_s)} = \dim_k < L_1^{j-1} S_1, \ldots, L_s^{j-1} S_1 >= H\left(\frac{S}{\wp_1^2 \cap \ldots \cap \wp_s^2}, j\right) \ .$$

Lecture 6: The Big Waring Problem

As we saw in the last lecture, if we want to know the dimension of the variety in $\mathbb{P}(S_j) = \mathbb{P}^N$, $(N = \binom{j+n}{n} - 1)$, which is the closure of the set of forms in $S = k[y_0, \ldots, y_n]$ of degree j which are the sums of $\leq s \quad j^{th}$ powers of linear forms, we need to know, for s general points $\{P_1, \ldots, P_s\}$ in $\mathbb{P}^n$, the Hilbert function $H(S/I, j)$, where $I = \wp_1^2 \cap \ldots \cap \wp_s^2$ $(\wp_i \leftrightarrow P_i, \wp_i \subset S)$.

Since this variety in $\mathbb{P}^N$ is a secant variety to the Veronese variety, (the closure of the $s-$secant $\mathbb{P}^{s-1}$'s to $\nu_j(\mathbb{P}^n)$), we shall introduce a notation for it which recognizes this fact.

Notation: Set $S = k[y_0, \ldots, y_n]$ and let $\mathbb{P}(S_j) = \mathbb{P}^N$. The subvariety of $\mathbb{P}^N$ which is the closure of the set of forms in S_j which are the sum of $\leq s \quad j^{th}$ powers of linear forms, will be denoted

$$Sec_{s-1}(\nu_j(\mathbb{P}^n)) \ .$$

So, Theorem 5.5 of the last lecture can be restated as follows:

Theorem 6.1: Let $S = k[y_0, \ldots, y_n]$ and let $P_1, \ldots, P_s$ be a generic set of s points in $\mathbb{P}^n$, where $P_i \leftrightarrow \wp_i \subseteq S$.

Then

$$\dim\ (\mathrm{sec}) = H\left(\frac{S}{\wp_1^2 \cap \ldots \cap \wp_s^2},\ j\right) - 1 = \dim_k\left(\frac{S_j}{(\wp_1^2 \cap \ldots \cap \wp_s^2)_j}\right) - 1 \ .$$

Let me begin with a simple, but interesting, application of this result.

The rational normal curve in $\mathbb{P}^n$:

In this case $\nu_j(\mathbb{P}^1) \subseteq \mathbb{P}^j$ and Theorem 6.1 gives:

$$\dim\ (Sec_{s-1}(\nu_j(\mathbb{P}^1)) = \dim_k\left(\frac{S_j}{(\wp_1^2 \cap \ldots \cap \wp_s^2)_j}\right) - 1$$

where $S = k[y_0, y_1]$ and $\wp_i \leftrightarrow P_i$ are generic points of $\mathbb{P}^1$.

Now the Hilbert function of a set of fat points on a line was described completely by Ed Davis and me in [Queen's Papers in Pure and Applied Mathematics, No. 67, The Curves Seminar, Vol. III - The Hilbert Function of a Special Class of 1-dimensional Cohen-Macaulay algebras] and then redone in a very elegant fashion by Brian Harbourne in [Canad.Math.Soc.Conf.Proc. **6** (1986) - The geometry of rational surfaces and Hilbert functions of points in the plane.]. The only part of those results that we need is the following:

$$e(\wp_1^2 \cap \ldots \cap \wp_s^2) = 2s \quad \text{and} \quad \dim_k\left(\frac{S_j}{(\wp_1^2 \cap \ldots \cap \wp_s^2)_j}\right) = 2s \Leftrightarrow j \geq 2s-1 \ .$$

So, let's first consider the variety $Sec_1(\nu_j(\mathbb{P}^1)) \subseteq \mathbb{P}^j$, i.e. the usual secant variety. The "expected" dimension of this variety is three (choose 2 points on the curve $\nu_j(\mathbb{P}^1)$ and then connect them with a $\mathbb{P}^1$).

By what we said above, we must consider the ideal $\wp_1^2 \cap \wp_2^2$ in $S = k[y_0, y_1]$. This ideal has multiplicity 4 and Hilbert function:

$$1 \quad 2 \quad 3 \quad 4 \quad 4 \quad \cdots$$

So,

$$\dim\,(Sec_1(\nu_j(\mathbb{P}^1)) = 3 \Leftrightarrow j \geq 3 \ .$$

I.e. the secant variety (of lines) for the rational normal curve in $\mathbb{P}^n$ (for $n \geq 3$) has dimension 3. Obviously the rational normal curve in $\mathbb{P}^2$ *cannot* have any secant variety with dimension 3! (Later in this lecture I will give the equations for some of these secant varieties.) It follows that the general form of degree 3 in $k[y_0, y_1]$ is a sum of two cubes of linear forms.

What about the variety $Sec_2(\nu_j(\mathbb{P}^1))$? Since $e(\wp_1^2 \cap \wp_2^2 \cap \wp_3^2) = 6$, this ideal has Hilbert function

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 6 \quad \cdots$$

The "expected" dimension for $Sec_2(\nu_j(\mathbb{P}^1))$ is 5 and, we see from the Hilbert function, that this *is* the dimension only for $j \geq 5$. Note that again, this could not happen sooner.

It follows that $Sec_2(\nu_5(\mathbb{P}^1))$ fills up $\mathbb{P}^5$ and that $Sec_2(\nu_6(\mathbb{P}^1))$ is a hypersurface in $\mathbb{P}^6$ while $Sec_2(\nu_7(\mathbb{P}^1))$ is of codimension 2 in $\mathbb{P}^7$ etc. We'll give, later, the equations of some of these varieties as well. Notice that, as before, we can therefore deduce that the general form of degree 5 in $k[y_0, y_1]$ is the sum of three 5th powers of linear forms.

It is clear from these considerations that the "**Big**" Waring problem is relatively easy for forms in $k[y_0, y_1]$. This appears to have been classically known. Ehrenborg and Rota [*Apolarity and Canonical Forms for Homogeneous Polynomials* European Jo.of Comb.,1993, Theorem 4.3] refer to it as a theorem of Sylvester. More precisely:

Sylvester's Theorem: A general form of degree $2j-1$ in $k[y_0, y_1]$ can be written as the sum of j $(2j-1)^{st}$ powers of linear forms.

(We saw this explicity for $j = 1,\ 2$ and the general case is evident from those two cases.)

There are many other classical facts that can be derived from Theorem 6.1. I have chosen some specific examples to illustrate this. These examples are all cited in the paper of Ehrenborg and Rota mentioned above (although the proofs there are given in a different way). I want to give here, the "fat points" version of the proofs of these classical facts.

To obtain the results I am referring to, I will use two strong (and surprisingly elementary) results of Catalisano, Trung and Valla (Proc. AMS, vol. 118, 1993, 717-724 - *A sharp bound for the regularity index of fat points in general position.*)

Recall that if $I = \wp_1^{\alpha_1} \cap \ldots \cap \wp_s^{\alpha_s} \subset S = k[y_0, \ldots, y_n]$ (where $\alpha_1 \leq \cdots \leq \alpha_s$) is an ideal of fat points in $\mathbb{P}^n$ then $e(I) = \sum_{i=1}^{s} \binom{\alpha_i+n-1}{n}$ is the *multiplicity* of I.

Theorem 6.2: (Catalisano-Trung-Valla)

Let I be as above and suppose that $\wp_i \leftrightarrow P_i$. Write $\mathbb{X} = \{P_1, \ldots, P_s\}$ and suppose that no $t+1$ points of $\mathbb{X}$ lie on a $\mathbb{P}^{t-1}$ for any $t \leq n$ (i.e. the points of $\mathbb{X}$ are in *linearly general position*).

Then

$$1)\ H(S/I,\ j) = e(I) \quad \text{for all} \quad j \geq \max\{\alpha_1 + \alpha_2 - 1,\ \left\lceil \frac{n-2+\sum_{i=1}^{s} \alpha_i}{n} \right\rceil\}\ ;$$

2) if $P_1, \ldots, P_s$ are points on a rational normal curve then

$$H(S/I, j) < e(I) \text{ for } j < \max\{\alpha_1 + \alpha_2 - 1,\ \left\lceil \frac{n-2+\sum_{i=1}^{s} \alpha_i}{n} \right\rceil\}\ .$$

As do Rota and Ehrenborg, I will follow the classic English style, adopting the brisk terminology that has (unfortunately) passed into disuse: a form of degree p in the polynomial ring in q variables will be called a "*q-ary p-ic*".

Proposition 6.3: A generic quarternary cubic can be written as a sum of 5 cubes. (i.e. a general form of degree 3 in $S = k[y_0, y_1, y_2, y_3]$ can be written as the sum of 5 cubes).

Proof: Since the Hilbert function of S begins:

$$1 \quad 4 \quad 10 \quad 20 \quad 35 \quad \cdots$$

to prove this proposition we need only show, in view of Theorem 6.1, that if $P_1, \ldots, P_5$ are 5 points of $\mathbb{P}^3$ that are chosen generically then, if we set

$$I = \wp_1^2 \cap \ldots \cap \wp_5^2 \ , \ \wp_i \leftrightarrow P_i \qquad \text{we have} \qquad H(S/I, 3) = 20 = \dim_k S_3 \ .$$

But, we know $e(I) = 20$ and Theorem 6.2 gives that $H(S/I, j) = 20$ for

$$j \geq \max\left\{2+2-1, \ \left\lceil \frac{3-2+10}{3} \right\rceil \right\} = \max\{3,3\} = 3$$

if the points are chosen in linear general position. This suffices to prove the proposition. □

Remark: I would like to know if **every** cubic form in S, above, is a sum of 5 cubes. I am almost sure that this is not the case, but I don't know either an algebraic or a geometric proof of this fact. It would be interesting to have one.

Proposition 6.4: (Clebsch, 1867) The general ternary quartic cannot be written as the sum of 5 fourth powers (i.e. the general form of degree 4 in $S = k[y_0, y_1, y_2]$ is not the sum of five 4^{th} powers).

Proof: Following the lines of the previous proposition, if we let $P_1, \ldots, P_5$ be any five points in $\mathbb{P}^2$ and let $\wp_i \leftrightarrow P_i$, then $e(\wp_1^2 \cap \ldots \cap \wp_5^2) = 15$. So, it will suffice to show that

$$H\left(\frac{S}{\wp_1^2 \cap \ldots \cap \wp_5^2}, \ 4\right) < 15 = \dim_k S_4 \ .$$

for 5 generically chosen points of $\mathbb{P}^2$. But, as we observed in an earlier lecture, the unique conic through 5 general points, doubled, gives a quartic in the ideal of fat points we are considering. Using Bezout's theorem, and the fact that the unique conic through 5 general points is irreducible, we see that this doubled conic is the only quartic in the ideal of fat points. Thus, in this latter case,

$$H\left(\frac{S}{\wp_1^2 \cap \ldots \cap \wp_5^2},\ 4\right) = 14$$

and we are done. □

Remark: This proof actually shows that the variety $Sec_4(\nu_4(\mathbb{P}^2))$ is a hypersurface in $\mathbb{P}(S_4) = \mathbb{P}^{14}$. It is interesting to think about the equation for this hypersurface.

I am deeply endebted to Prof. D. Gallarati (Genova) who explained to Marvi Catalisano and me the ideas behind the so-called "*notation of Clebsch*", which is particularly suited to dealing with the Veronese varieties and its secant varieties. Using this we were able to find the equation of the hypersurface above, and also to explain several other interesting facts about the Veronese varieties.

We shall have to make a small detour to deal with this notation and its implications.

First, consider (ordered) n-tuples of numbers, i.e.

$$\{(i_1, \ldots, i_n) \mid i_1 \leq \ldots \leq i_s \text{ where } i_j \in \{0, 1, \ldots, s\}\} \ .$$

Note that these tuples are in $1-1$ correspondence with the monomials of degree n in $k[y_0, \ldots, y_s]$ as follows:

$$(i_1, \ldots, i_n) \leftrightarrow y_{i_1} y_{i_2} \ldots y_{i_n} \ .$$

I.e.

$$(0, \ldots, 0, 1) \leftrightarrow y_0^{n-1} y_1$$

$$y_0 y_1^2 y_2 y_3 \leftrightarrow (0, 1, 1, 2, 3) \ \text{ etc.}$$

If $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_m)$ are two tuples as above, we form the $(m+n)-$tuple

$$(i_1, \ldots, i_n)(j_1, \ldots, j_m)$$

by interlacing the i_k's and j_ℓ's so that the numbers $i_1, \ldots, i_n, j_1, \ldots, j_m$ are again in order.

E.G. If $s = 4$, $n = 3$, $m = 2$, i.e. we choose from $\{0, 1, 2, 3, 4\}$. In this case

$$(0, 0, 3)(1, 2, 3, 4) = (0, 0, 1, 2, 3, 3, 4)$$

(Notice how this composition is related to multiplication of monomials.)

Now, fix s and for each integer n, we order the $n-$tuples lexicographically and consider the $1 \times \binom{s+n}{n}$ matrix $\mathcal{M}_{s,n}$ formed by the set of all the ordered $n-$tuples. For example, for $s = 2$, $n = 2$ we have:

$$\mathcal{M}_{2,2} = (\,(0,0) \quad (0,1) \quad (0,2) \quad (1,1) \quad (1,2) \quad (2,2)\,)\ .$$

For fixed s and any m and n we can form the matrix

$$\mathcal{M}^t_{s,m}\mathcal{M}_{s,n} = \begin{pmatrix} \underbrace{(0,\dots,0)}_{m-\text{tuple}} \\ \vdots \\ \underbrace{(s,\dots,s)}_{m-\text{tuple}} \end{pmatrix} \begin{pmatrix} \underbrace{(0,\dots,0)}_{n-\text{tuple}} & \cdots & \underbrace{(s,\dots,s)}_{n-\text{tuple}} \end{pmatrix}$$

Example: 1) Let $s = 2, m = n = 1$.

$$\begin{pmatrix} (0) \\ (1) \\ (2) \end{pmatrix} (\,(0) \quad (1) \quad (2)\,) = \begin{pmatrix} (0,0) & (0,1) & (0,2) \\ (0,1) & (1,1) & (1,2) \\ (0,2) & (1,2) & (2,2) \end{pmatrix}$$

If we now think of the symbols (i,j) as variables we obtain a 3×3 symmetric matrix of variables:

$$\begin{pmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{01} & Z_{11} & Z_{12} \\ Z_{02} & Z_{12} & Z_{22} \end{pmatrix} \tag{$\dagger$}$$

I'll do one more example:

2) Let $s = 2$, $m = n = 2$

$$\mathcal{M}^t_{2,2}\mathcal{M}_{2,2} = \begin{pmatrix} (0,0) \\ (0,1) \\ (0,2) \\ (1,1) \\ (1,2) \\ (2,2) \end{pmatrix} (\,(0,0) \quad (0,1) \quad (0,2) \quad (1,1) \quad (1,2) \quad (2,2)\,)$$

$$= \begin{pmatrix} (0,0,0,0) & (0,0,0,1) & (0,0,0,2) & (0,0,1,1) & (0,0,1,2) & (0,0,2,2) \\ - & (0,0,1,1) & (0,0,1,2) & (0,1,1,1) & (0,1,1,2) & (0,1,2,2) \\ - & - & (0,0,2,2) & (0,1,1,2) & (0,1,2,2) & (0,2,2,2) \\ - & - & - & (1,1,1,1) & (1,1,1,2) & (1,1,2,2) \\ - & - & - & - & (1,1,2,2) & (1,2,2,2) \\ - & - & - & - & - & (2,2,2,2) \end{pmatrix}$$

and writing these as variables

$$= \begin{pmatrix} Z_{0000} & Z_{0001} & Z_{0002} & Z_{0011} & Z_{0012} & Z_{0022} \\ - & Z_{0011} & Z_{0012} & Z_{0111} & Z_{0112} & Z_{0122} \\ - & - & Z_{0022} & Z_{0112} & Z_{0122} & Z_{0222} \\ - & - & - & Z_{1111} & Z_{1112} & Z_{1122} \\ - & - & - & - & Z_{1122} & Z_{1222} \\ - & - & - & - & - & Z_{2222} \end{pmatrix} \qquad (\dagger\dagger)$$

Notice that there are exactly $\binom{s+m+n}{s}$ different variables in this symmetric matrix (which correspond to the monomials of degree $m+n$ in $k[y_0, \dots, y_s]$) and that some of the variables appear more than once in this array.

The usefulness of this notation comes from the following observations: if we let $\mathcal{Z}_{n,m} = \mathcal{M}_{s,n}^t \mathcal{M}_{s,m}$ be the matrix of variables above, i.e. in $N = \binom{s+m+m}{s}$ variables.

> *Claim:* Consider $\mathcal{S} = v_{n+m}(\mathbb{P}^s) \subseteq \mathbb{P}^{N-1}$ and let $P \in \mathcal{S}$. Then $\mathcal{Z}_{m,n}(P)$ is a matrix of rank 1.
>
> *Pf:* The proof is a trick with the notation! Let $\mathbf{a} = [a_0 : \dots : a_s] \in \mathbb{P}^s$. If $(i_1, \dots, i_n)$ is an n-tuple as above, then write
>
> $$a_{(i_1,\dots,i_n)} = a_{i_1} a_{i_2} \dots a_{i_n} \ .$$
>
> So, if $L = a_0 y_0 + \dots + a_s y_s$, then the (scaled) coefficients of L^n are precisely
>
> $$(a_{(0,\dots,0)} \ a_{(0,0,\dots,0,1)} \ \cdots \ a_{(s,s,\dots,s)}) := \mathcal{M}_{s,n}(\mathbf{a})$$
>
> So, $L^{n+m} \leftrightarrow \mathcal{M}_{s,n}^t(\mathbf{a}) \mathcal{M}_{s,m}(\mathbf{a}) = \mathcal{Z}_{m,n}(L^{m+n})$ and it is then obvious that $\mathcal{Z}_{m,n}(L^{m+n})$ is a matrix of rank 1.

Proposition 6.5: For every m, n such that $m + n = j$, the Veronese variety is contained in the subvariety of $\mathbb{P}^{N-1}$ defined by the 2×2 minors of the matrix

$$\mathcal{M}^t_{s,n}\mathcal{M}_{s,m} = \mathcal{Z}_{n,m}$$

Remark: The formation of these matrices depends very much on the ordering chosen for the monomials, while the final result does not. We've wondered to what extent different orderings give different ideals of 2×2 minors. Note: nothing in this says that the 2×2 minors $\underline{define}$ the Veronese variety.

In exactly the same way we see that: if $\mathbf{a} \leftrightarrow L_1$ and $\mathbf{b} \leftrightarrow L_2$ then

$$L_1^{m+n} + L_2^{m+n} \leftrightarrow \mathcal{M}^t_{s,n}(\mathbf{a})\mathcal{M}_{s,m}(\mathbf{a}) + \mathcal{M}^t_{s,n}(\mathbf{b})\mathcal{M}_{s,m}(\mathbf{b}) = \mathcal{Z}_{m,n}(L_1^{m+n} + L_2^{m+n}) \ .$$

Since, if $A_1, \ldots, A_r$ are all $m \times n$ matrices of rank 1 and $r \leq \min\{m, n\}$ then $\sum_{i=1}^r A_i$ has rank $\leq r$, it follows that the determinantal variety defined by the vanishing of the 3×3 minors of $\mathcal{Z}_{m,n}$ contains the secant variety to $\mathcal{S}$. We can obviously continue on in this way and we obtain:

Theorem 6.6: Let $\mathcal{Z}_{n,m}$ be the $\binom{s+n}{s}$ by $\binom{s+m}{s}$ matrix defined above, and let ℓ be a positive integer so that $\ell < \min\{\binom{s+n}{s}, \binom{s+m}{s}\}$. Let $\mathcal{S} = \nu_{n+m}(\mathbb{P}^s)$ and let $\mathcal{I}_t$ be the ideal defined by the $t \times t$ minors of $\mathcal{Z}_{m,n}$

Then

$$Sec_{\ell-1}(\mathcal{S}) \subseteq \text{ the variety defined by } \mathcal{I}_{\ell+1} \ .$$

Remarks:

a) If we look at the two examples we made above, then in the first case (matrix (†)) we have the 3×3 matrix whose 2×2 minors actually define the Veronese surface in $\mathbb{P}^5$ (i.e $\nu_2(\mathbb{P}^2)$) and the determinant of that matrix defines the secant variety to this Veronese. Notice that from this representation we find that the Veronese variety is a singular subvariety of the secant variety and has multiplicity 2 in that variety.

b) In the same way, the determinant of the 6×6 matrix (††) gives the equation of the hypersurface $Sec_4(\nu_4(\mathbb{P}^2))$ in $\mathbb{P}^{14}$, which we wanted to find.

Using (††) we find that

i) $\nu_4(\mathbb{P}^2) \subseteq Sec_4(\nu_4(\mathbb{P}^2))$ is a singular subvariety of multiplicity 5;

ii) $Sec_1(\nu_4(\mathbb{P}^2)) \subseteq Sec_4(\nu_4(\mathbb{P}^2))$ is a singular subvariety of multiplicity 4;

iii) $Sec_2(\nu_4(\mathbb{P}^2)) \subseteq Sec_4(\nu_4(\mathbb{P}^2))$ is a singular subvariety of multiplicity 3;

iv) $Sec_3(\nu_4(\mathbb{P}^2)) \subseteq Sec_4(\nu_4(\mathbb{P}^2))$ is a singular subvariety of multiplicity 2.

c) If we look at the special case of $n = 1$ and $m = j - 1$ then we get an $(s+1) \times \binom{j-1+s}{s}$ matrix and this is the matrix whose 2×2 minors are known to generate the defining ideal of the Veronese variety $\nu_j(\mathbb{P}^s)$. We can also look at the minors of this matrix of size $\leq s + 1$. We are not aware of any theorem which says that these give the defining ideal of the appropriate secant variety, except for the case where $s = 1$, i.e. when the Veronese variety is the rational normal curve. In these cases it is easy to see that the secant varieties are all arithmetically Cohen-Macaulay and (using the Eagon-Nortcott resolution) one can even find the minimal free resolution of the defining ideal of these varieties. **One wonders if all the secant varieties are arithmetically Cohen-Macaulay?**

It may well be that many of these problems were solved over 100 years ago! It was difficult to find any references to the problems I have mentioned, so if known, the results do not seem to be in general circulation. (Although, just recently (April 1995) I was happy to receive a copy of the Brandeis thesis of Michael Catalano-Johnson with interesting theorems and some useful historical facts.)

Lecture 7: The Big Waring Problem - Continued

Recall: We have been considering the following two Waring problems for forms in S_j, $S = k[x_0, \dots, x_n]$.

a) **The "Little" Waring problem:** find the least integer $g(j)$ such that <u>*every*</u> form $F \in S_j$ is a sum of $\leq g(j)$ j^{th} powers of linear forms.

b) **The "Big" Waring problem:** find the least integer $G(j)$ such that the <u>*general*</u> form $F \in S_j$ is a sum of $\leq G(j)$ j^{th} powers of linear forms.

We also introduced the various secant varieties to the Veronese varieties, i.e. the varieties $Sec_{s-1}(\nu_j(\mathbb{P}^n))$, which are the closure (in $\mathbb{P}^N$, where $N = \binom{j+n}{j} - 1$) of the set

$$\cup\{\mathbb{P}^{s-1} \subset \mathbb{P}^N | \mathbb{P}^{s-1} \text{ contains a set of } s \text{ linearly independent points of } \nu_j(\mathbb{P}^n)\} \ .$$

We have seen that $G(j) =$ the least integer s such that $Sec_{s-1}(\nu_j(\mathbb{P}^n)) = \mathbb{P}^N$ and that

$$\dim(Sec_{s-1}(\nu_j(\mathbb{P}^n)) = H\left(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_s^2}, j\right) - 1$$

where $\wp_i \leftrightarrow P_i \in \mathbb{P}^n$ and $\{P_1, \cdots, P_s\}$ is a generic set of s points of $\mathbb{P}^n$.

We also saw that $Sec_{s-1}(\nu_j(\mathbb{P}^n))$ has an "expected" dimension; namely

$$\min\{sn + (s-1) = s(n+1) - 1, N\}$$

but that this is not always acheived. (In such a case we shall say that the secant variety is *deficient* .)

By the same token we have seen that

$$H\left(\frac{S}{\wp_1^2 \cap \dots \wp_s^2}, j\right)$$

has an expected value for every j; namely $\min\{\binom{j+n}{n}, s(n+1)\}$, but that this value is not always acheived either. The relationship between these "expectations" (and their failures) was discussed above.

We looked at one such failure in detail: specifically, we saw (in fact it was first noted by Clebsch over a hundred years ago) that the general ternary quartic cannot be written as a sum of 5 fourth powers of linear forms i.e $Sec_4(\nu_4(\mathbb{P}^2))$ has dimension 13 (inside $\mathbb{P}^{14}$) rather than being all of that enveloping space. Moreover, this fact comes from the simple observation that 5 general points $P_1, \ldots, P_5$ in $\mathbb{P}^2$ with corresponding ideal $I = \wp_1^2 \cap \cdots \cap \wp_5^2 \subseteq S = k[x_0, x_1, x_2]$ has $H(S/I, 4) = 14$ (rather than the 15 we expected). In fact, we found a degree 6 equation (the determinant of the 6×6 symmetric matrix (††) of the previous lecture) which contained the variety of 5-secant $\mathbb{P}^4$'s of $\nu_4(\mathbb{P}^2)$.

To finish the discussion of that example, we should actually show that the determinant is indeed the defining equation for $Sec_4(\nu_4(\mathbb{P}^2)) \subseteq \mathbb{P}^{14}$ by showing, for example, that the determinant is irreducible. Fortunately, Michael Catalano-Johnson has recently made a lovely observation which asserts (in a special case) that $Sec_t(\nu_j(\mathbb{P}^n))$ (if it is not all of its enveloping projective space) cannot lie on a hypersurface of degree $t+1$. In particular $Sec_4(\nu_4(\mathbb{P}^2))$ cannot lie on a hypersurface of $\mathbb{P}^{14}$ of degree ≤ 5 and that finishes off the discussion of that example.

> **Aside:** Catalano-Johnson did not give us a proof for his observation, but Catalisano has found a simple argument (which is probably what Catalano-Johnson had in mind). For completeness I will include Catalisano's argument.
>
> *Lemma:* (M. Catalano-Johnson) Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a nondegenerate variety and suppose that
>
> $$Sec_t(\mathbb{X}) \subseteq V(F) \nsubseteq \mathbb{P}^n \ .$$
>
> Then $\deg F \geq t+2$.
>
> *Proof:* (M. Catalisano) Since $\mathbb{X}$ is non-degenerate we can choose
>
> $$\mathbb{Y} = \{P_1, \ldots, P_n, P_{n+1}\} \subseteq \mathbb{X}$$
>
> which are linearly independent. Suppose that we can find an F (as above) for which $\deg F \leq t+1$.
>
> Now $t < n$ (otherwise $Sec_t(\mathbb{X}) = \mathbb{P}^n$) and so $t+1 < n+1$. Consider
>
> $$\mathbb{Z} = \{P_1, \ldots, P_{t+1}, P_{t+2}\} \subseteq \mathbb{Y} \ .$$

Note that $\mathbb{Z}$ determines a unique $\mathbb{P}^{t+1} := \mathbb{P}^{t+1}_{\mathbb{Z}} \subseteq \mathbb{P}^n$.

If $\mathbb{Z}_i = \{P_1, \ldots, P_i^*, \ldots, P_{t+2}\}$ (i.e. eliminate P_i from $\mathbb{Z}$), then $\mathbb{Z}_i$ determines a unique $\mathbb{P}^t_i \simeq \mathbb{P}^t \in Sec_t(\mathbb{X})$. Thus

$$\cup_{i=1}^{t+2}(\mathbb{P}^t_i) = \mathcal{L}$$

is a t-dimensional subvariety of $Sec_t(\mathbb{X})$ which has degree $t+2$.

Since $\mathbb{P}^{t+1}_{\mathbb{Z}} \cap V(F)$ has dimension t and contains $\mathcal{L}$, we get that

$$\deg(\mathbb{P}^{t+1}_{\mathbb{Z}} \cap V(F)) \geq t+2 \ .$$

Since $\deg F = t+1$ we have a contradiction (Bezout) unless $V(F) \supseteq \mathbb{P}^{t+1}_{\mathbb{Z}}$.

But, in this latter case we can repeat this argument for any set of $n+1$ independent points of $\mathbb{X}$. We either get a contradiction or we obtain that $V(F) \supseteq Sec_{t+1}(\mathbb{X})$. Since we eventually have $Sec_s(\mathbb{X}) = \mathbb{P}^n$ the proof is complete.

There are other "deficient" secant varieties to the Veronese varieties, and I would like to discuss three more of them.

Example 7.1: The variety $Sec_8(\nu_4(\mathbb{P}^3))$ does not fill $\mathbb{P}^{34}$.

This is unexpected since choosing 9 points on the 3-fold $\nu_4(\mathbb{P}^3)$ gives a 27-dimensional choice plus the "connecting" $\mathbb{P}^8$ gives an expected 35 dimensional choice. But, this variety is extremely deficient. In fact, we'll see that the dimension of $Sec_8(\nu_4(\mathbb{P}^3))$ is 33 and hence is a hypersurface in $\mathbb{P}^{34}$, and so has dimension 2 less than expected. (In the Clebsch example, the dimension was 1 less than expected.)

Showing that the dimension of $Sec_8(\nu_4(\mathbb{P}^3))$ is 33 is equivalent to showing that

$$H\left(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_9^2}\ ,4\right) = 34 < 35 = \dim_k S_4$$

where $P_1, \ldots, P_9$ (with $P_i \leftrightarrow \wp_i$) are generically chosen points of $\mathbb{P}^3$ and $S = k[y_0, \ldots, y_3]$.

Now, any 9 points of $\mathbb{P}^3$ always lie on a quadric hypersurface Q (unique and irreducible if the points are chosen generically enough). So, there is always a form of degree 4 in $\wp_1^2 \cap \ldots \cap \wp_9^2$ and so

$$H\left(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_9^2}\ ,4\right) \leq 34 \ .$$

That is enough to prove that $Sec_8(\nu_4(\mathbb{P}^3))$ doesn't fill $\mathbb{P}^{34}$. Proving that this variety is a hypersurface in $\mathbb{P}^{34}$ or, equivalently, proving that the Hilbert function above has value exactly 34 in degree 4 (i.e. showing that Q^2 is the unique quartic in the ideal $\wp_1^2 \cap \cdots \cap \wp_9^2$) is a bit more delicate. Catalisano has a very nice proof of this latter fact which I won't go into here.

Once this is established, however, we can use the methods of the last section to form the 10×10 matrix

$$\mathcal{M}_{3,2}^t \mathcal{M}_{3,2} = \mathcal{Z}_{2,2} \ .$$

The determinant of this matrix vanishes on the variety $Sec_8(\nu_4(\mathbb{P}^3))$ and so (again using the Catalano-Johnson result) we get that the polynomial $det(\mathcal{Z}_{2,2})$ is the defining equation of $Sec_8(\nu_4(\mathbb{P}^3)) \subseteq \mathbb{P}^{34}$.

Note that, in the language of Waring's Problem, we obtain from this example that:

the generic quarternary quartic is not a sum of 9 fourth powers of linear forms.

This is not one of the classical results that Rota and Ehrenborg refer to in their paper, but it is mentioned in the article of Terracini (Annali di Matematica Pura ed Applicata, Serie III, t.24, p.1-10, 1915). I am not sure if Terracini was the first to notice it.

Example 7.2: The variety $Sec_{13}(\nu_4(\mathbb{P}^4))$ does not cover $\mathbb{P}^{69}$.

Note that the choice of 14 points from a 4-fold plus a "connecting" $\mathbb{P}^{13}$ should give a space of dimension 69. To show that this secant variety is deficient, it is enough to prove that

$$H\left(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_{14}^2}\ ,4\right) = 69 < 70 = \dim_k S_4$$

where $P_1, \ldots, P_{14}$ are 14 general points of $\mathbb{P}^4$ and $P_i \leftrightarrow \wp_i \subseteq S = k[y_0, \ldots, y_4]$.

Now $\dim_k S_2 = 15$, so there is always a quadric Q through any 14 points of $\mathbb{P}^4$ (unique and irreducible if the points are chosen generally enough.)

Thus $Q^2 \in (\wp_1^2 \cap \cdots \cap \wp_{14}^2)_4$ and so $H(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_{14}^2}\ ,4) \leq 69$. As before, this is enough to prove that $Sec_{13}(\nu_4(\mathbb{P}^4))$ is deficient. I have not been able to find a direct proof that the value of this Hilbert function is exactly 69, in degree 4, (i.e. that Q^2 is the unique quartic in $(\wp_1^2 \cap \cdots \cap \wp_{14}^2)_4$ but I am sure that this is correct. **I'd like to have a proof of this fact.**

Assuming that the Hilbert function in degree 4 is as I asserted, then $Sec_{13}(\nu_4(\mathbb{P}^4))$ is a hypersurface of $\mathbb{P}^{69}$ and the determinant of the matrix

$$\mathcal{M}_{4,2}^t \mathcal{M}_{4,2} = \mathcal{Z}_{2,2}$$

(this time of size 15 × 15) gives the defining equation (again, thanks to the Catalano-Johnson result.)

Note that, in the language of Waring's Problem, this result says (even without the *precise* value of the Hilbert function above) that:

> the generic quinary quartic is not the sum of 14 fourth powers of linear forms.

(Again, Ehrenborg and Rota don't mention this example - which is in Terracini's article cited above. I don't know when it was first noticed.)

Example 7.3: The variety $Sec_6(\nu_3(\mathbb{P}^4)) \subseteq \mathbb{P}^{34}$ does not fill $\mathbb{P}^{34}$.

The expected dimension of this secant variety is $7 \cdot 4 + 6 = 34$, but for 7 general points $P_1, \dots, P_7$ in $\mathbb{P}^4$ ($P_i \leftrightarrow \wp_i \subseteq S = k[y_0, \dots, y_4]$)

$$H\left(\frac{S}{\wp_1^2 \cap \cdots \cap \wp_7^2}, 3\right) = 34 < 35 = \dim_k S_3$$

(which is enough to prove the result.)

Now $e(\wp_i^2) = 5$ and so if $I = \wp_1^2 \cap \cdots \cap \wp_7^2$ then $e(I) = 35$. Recall that 7 points of $\mathbb{P}^4$ are always on a rational normal curve in $\mathbb{P}^4$, so, using Theorem 6.2 above, we find that

$$H(S/I, j) < 35 \quad \text{for} \quad j < \max\{\ 3,\ [(2+14)/4]\ \} = 4\ .$$

Hence $H(S/I, 3) \leq 34$. The proof that the value of the Hilbert function is exactly 34 is, again, a bit delicate, but Catalisano has a proof for this case also.

It follows that $Sec_6(\nu_3(\mathbb{P}^4))$ is a hypersurface of $\mathbb{P}^{34}$.

Remark: Unfortunately, the Clebsch method does not work to give the equation of this variety and, at this point, I have no idea what the defining equation is (except, that from the Catalano-Johnson result, it must have degree ≥ 8.)

In fact, I don't even know the degree of this variety (nor do I know a synthetic way to find the degrees of the varieties $Sec_t(\nu_j(\mathbb{P}^n))$ in general). It is hard to believe that these degrees are not known, but that seems to be the case.

The reason that I have spent so much time on these exceptions is that, thanks to the following wonderful theorem of J.Alexander and A.Hirschowitz, **these are the only exceptions**. Thus we have a complete answer to the Big Waring Problem for Homogeneous Forms.

Theorem 7.4: (J. Alexander, A. Hirschowitz) Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a general set of s points in $\mathbb{P}^n$. Let $P_i \leftrightarrow \wp_i \subseteq k[y_0, \ldots, y_n] = S$ and let $j \geq 3$.

Then

$$H\left(\frac{S}{\wp_1^2 \cap \ldots \cap \wp_s^2}\ , j\right) = \min\{\ (n+1)s,\ \dim_k S_j\}$$

except for

a) $n = 2,\ j = 4,\ s = 5$ (Proposition 6.4)
b) $n = 3,\ j = 4,\ s = 9$ (Example 7.1)
c) $n = 4,\ j = 4,\ s = 14$ (Example 7.2)
d) $n = 4,\ j = 3,\ s = 7$ (Example 7.3) .

Translating this into the language of Secant Varieties to the Veronese Varieties we get:

Corollary 7.5: (see Iarrobino's paper: Inverse Systems of a Symbolic Power II)

Let $X = Sec_t(\nu_j(\mathbb{P}^n))$ $(j \geq 3)$. Then

$$\text{the dimension of } X = \min\{(t+1)n + t\ ,\ \binom{n+j}{j} - 1\}$$

except for

a) $j = 3,\ n = 4,\ t+1 = 7$ (Example 7.3), (deficienty 1);
b) $j = 4,\ n = 2,\ t+1 = 5$ (Proposition 6.4), (deficiency 1);
c) $j = 4,\ n = 3,\ t+1 = 9$ (Example 7.1), (deficiency 1);
d) $j = 4,\ n = 4,\ t+1 = 14$ (Example 7.2), (deficiency 1).

Remark: The proof of this theorem is spread over severl papers and a hundred journal pages. It would be wonderful to have a more direct proof of this important theorem.

(See the relatively elementary proof of M. Catalano-Johnson, in this volume, for the case $n = 2$.)

To finish off this circle of ideas concerning the Waring Problems for Homogeneous Forms, I want to make some small comments on the "Little Waring Problem" (i.e. what is the least integer $g(j)$ for which EVERY form in $k[y_0, \ldots, y_n]$ of degree j is a sum of $\leq g(j)$ j^{th} powers of linear forms?) I said earlier in these notes that I didn't know any case, apart from the case where $j = 2$ (any n), where this problem was solved. I have since found a bit more information in the book of J.Harris (*Algebraic Geometry* Exc.11.35). The exercise considers the case $n = 1$, i.e. homogeneous forms in $k[y_0, y_1]$.

Sylvester's Theorem (Lecture 6, just before 6.2) gave us the answer to the "generic" problem. Recall that that theorem says that a general form of degree n is the sum of d n^{th} powers of linear forms if and only if $2d - 1 \geq n$. Harris adds: "... moreover, if $2d - 1 = n$ it is uniquely so expressible.", i.e. roughly $(n + 1)/2$ n^{th} powers are needed, generically.

Since the rational normal curve, $\mathcal{C} \subset \mathbb{P}^n$, has degree n and through every point of $\mathbb{P}^n$ we can find a hyperplane which meets $\mathcal{C}$ in n distinct points, we obtain that every form of degree n in $k[y_0, y_1]$ is the sum of $\leq n$ n^{th} powers of linear forms. Moreover, it is not hard to show (and this is the exercise in Harris' book) that if P is a point of $\mathbb{P}^n$ that is on a tangent line to $\mathcal{C}$ then P requires n n^{th} powers in its expresssion as a sum of powers of linear forms. So, the "little" Waring Problem for $k[y_0, y_1]$ is completely solved.

Let me state the result formally.

Theorem 7.6: Let $S = k[y_0, y_1]$ and let $F \in S_n$. Then F can be written as a sum of n n^{th} powers of linear forms.

Moreover, $F = y_0^{n-1}y_1$ cannot be written as the sum of fewer than n n^{th} powers of linear forms.

More generally, if we consider $\nu_j(\mathbb{P}^n) \subseteq \mathbb{P}^N$ $(N = \binom{j+n}{j} - 1)$ and P any point of $\mathbb{P}^N$ off $\nu_j(\mathbb{P}^n)$, then a general $\mathbb{P}^{N-n} = \mathbb{P}^s$ through P meets $\nu_j(\mathbb{P}^n)$ in $\deg(\nu_j(\mathbb{P}^n)) = j^n$ distinct points. By the Uniform Position Lemma of Harris, every $s + 1$ subset of these j^n points is linearly independent. Thus the form of degree j in $k[x_0, \ldots, x_n]$ which corresponds to P can be written as a sum of $s + 1 = N - n + 1$ j^{th} powers of linear forms (in $\binom{j^n}{s+1}$-ways, using just this $\mathbb{P}^s$).

Notice also, that since we can choose infinitely many different $\mathbb{P}^s$'s through P, there are infinitely many such representations. This is in marked contrast to the case when a form of degree j can be expressed as a sum of $\binom{n+j}{j} - n - 1$ j^{th} powers of linear forms. In that case (generically) the representation is unique (as has been shown by Iarrobino and Kanev in their paper "The Length of a homogeneous form, Determinantal Loci of Catalecticants and Gorenstein Algebras" - henceforth called their "Length" paper!) (Thanks, by the way, to Iarrobino for an interesting exchange on this Bertini argument and for his clearing up an obscurity (!) in my original remarks to him.)

One wonders how good this bound is! The first place to try it out is for cubic forms in $S = k[x_0, x_1, x_2]$. Using Cor. 7.5 (with $j = 3,\ n = 2$) we find that generically a cubic form in S is a sum of 5 cubes of linear forms and, by the remarks above, <u>*every*</u> cubic form in S is a sum of $\leq \binom{2+3}{3} - 2 = 8$ cubes of linear forms.

However Bruce Reznick, in his preprint "Sums of Powers of Complex Linear Forms" (Thm. 7.6)), says that $F = x_0(x_0x_1 - x_2^2)$ is the **only** cubic in $\mathbb{C}[x_0, x_1, x_2]$ requiring 5 cubes of linear forms i.e. all others require 4 or less!

Clearly there is much more to be said about this problem (understatement!!!).

Beyond Waring!

As I mentioned early on in this discussion, the attempt to express homogeneous forms as a sum of powers of linear forms was an attempt to simplify (and organize) the forms of a given degree. The theorem on the polarization of quadratic forms (or the diagonalization of symmetric matrices) - a complete and beautiful theorem in itself - no doubt contributed to the attempt at expressing forms as sums of powers of linear forms. Perhaps we human beings are especially attracted to "powers" (Fermat's and Catalan's problems being two examples that come immediately to mind as ones that have attracted many people's interest).

Nevertheless, there were many other attempts at canonical forms attempted, and Ehrenborg and Rota, in their previously cited paper, mention several of these (as does Bronowski in his series of papers in the 1930's on "Canonical expressions ... ").

I'm just going to look at one of these other results now as I want to move on to the fascinating work of Iarrobino and Kanev on Catalecticants and Gorenstein (artinian) rings and explain the connection with what has gone on above.

For example, Ehrenborg and Rota mention:

Proposition 7.7: The general ternary quartic can be written

$$h_1h_2 + h_3^2 \qquad \text{where the} \quad h_i \quad \text{are quadratic forms.}$$

(i.e. If $S = k[y_0, y_1, y_2]$, then the general element of S_4 can be so written.)

Proof: We start by considering the map

$$\Phi : S_2 \times S_2 \times S_2 \longrightarrow S_4$$

described by

$$\Phi(h_1, h_2, h_3) = h_1h_2 + h_3^2$$

Following our earlier example we want to find the maximum rank of the differential of this map. We consider the line through (h_1, h_2, h_3) parametrized by

$$(h_1, h_2, h_3) + t(Q_1, Q_2, Q_3)$$

whose image under Φ is: $(h_1 + tQ_1)(h_2 + tQ_2) + (h_3 + tQ_3)^2$.

The derivative of Φ along this line is

$$(h_1 + tQ_1)Q_2 + (h_2 + tQ_2)Q_1 + 2(h_3 + tQ_3)Q_3$$

which, when $t = 0$ gives

$$h_1Q_2 + h_2Q_1 + 2h_3Q_3$$

i.e. as we vary Q_1, Q_2, Q_3 we obtain tangent vectors in the vector space which is the degree 4 part of the ideal (h_1, h_2, h_3).

Thus, we want to know the size of this vector space for general h_1, h_2, h_3. There are several ways to do this:

a) Three general quadrics form a regular sequence in S, so the Hilbert function of the ideal they generate is:

$$1 \quad 3 \quad 3 \quad 1 \quad 0 \cdots$$

Thus, three general quadrics generate the space of all the fourth degree forms in S and we are done.

b) If we let $h_1 = L_1^2, h_2 = L_2^2, h_3 = L_3^2$ (where L_1, L_2 and L_3 are linearly independent linear forms) , then $(L_1^2, L_2^2, L_3^2)_4 =< L_1^2 S_2, L_2^2 S_2, L_3^2 S_2 >$ and we saw in Theorem 3.2, this vector space is $(I^{-1})_4$ where $I = \wp_1^3 \cap \wp_2^3 \cap \wp_3^3$. So, it would be enough to prove that

$$H\left(\frac{S}{\wp_1^3 \cap \wp_2^3 \cap \wp_3^3}\ ,\ 4\right) = 15 = \dim_k S_4$$

and that is easy since there is no plane quartic with 3 (non-colinear) triple points (by Bezout).

Lecture 8:... *and now for something completely different?*

In this, and in the suceeding sections, I will be using (extensively) several preprints: one by Iarrobino and Kanev that I referred to earlier as "Length"; one by Susan J. Diesel - Irreducibility and Dimension Theorems for Families of Height 3 Gorenstein Algebras; and one by Iarrobino - Inverse system of a symbolic power, II: the Waring problem for forms (revised form of 11/93).

I will not always give references for specific facts that I use from these works, but most of what I say can be deduced from what is in those papers (with the notable exception of the material on divided power rings). I want to make very clear my debt to these authors and especially to Tony Iarrobino for his generosity in giving me an advance look at his work and for responding to my many queries on the contents of these papers.

Recall our original notation:

$$R = k[x_0, \dots, x_n] \qquad\qquad S = k[y_0, \dots, y_n]$$

where the elements of R are considered as partial differential operators acting on the elements of S. Unless we specifically state otherwise, all ideals will be homogeneous and all modules graded.

Recall also that if I is a homogeneous ideal of R then the graded R-submodule of S annihilated by I is denoted I^{-1} and called the *inverse system* of I.

Definition 8.1: The ring R/I is an *artinian* ring if and only if $\dim_k R/I < \infty$ if and only if $I_j = R_j$ for all $j >> 0$. ($\Leftrightarrow I \supseteq (x_0, \dots, x_n)^t$ for some t.)

Notation: We let m denote the (irrelevant) unique homogeneous maximal ideal of R, i.e. $m = (x_0, \dots, x_n)$. If no confusion can occur, we also let m denote the image of $(x_0, \dots, x_n)$ in any (homogeneous) quotient, A of R.

Definition 8.2: The *socle* of the ring A, denoted $Soc(A)$) is:

$$Soc(A) := (0 : m) = \{g \in A \mid gm = 0\}\ .$$

(Note: Since m is homogeneous, $Soc(A)$ is a homogeneous ideal of A.)

Examples 8.3:
1). Let $A = k[x_0, x_1]/(x_0^2, x_1^2)$, then

$$A = k \oplus (k\overline{x_0} \oplus k\overline{x_1}) \oplus (k\overline{x_0 x_1}) \ .$$

Clearly $\overline{x_0 x_1} \in Soc(A)$ and, in fact, $Soc(A) = (\overline{x_0 x_1})$.

2). Let $A = k[x_0, x_1]/(x_0^3, x_0 x_1, x_1^2)$, then

$$A = k \oplus (k\overline{x_0} \oplus k\overline{x_1}) \oplus k\overline{x_0}^2 \ .$$

Then $\overline{x_1}$ and $\overline{x_0}^2$ are in $Soc(A)$. In fact, $Soc(A) = (\overline{x_0}^2, \overline{x_1})$.

Remarks:
1). If $A = k[x_0, \dots, x_n]/I = \oplus A_i$ and $F \in A_t$, then

$$F \in Soc(A) \Leftrightarrow F\overline{x_i} = 0 \text{ for } i = 0, \dots, n \ .$$

2). Let A be an artinian ring as above, and write

$$A = k \oplus A_1 \oplus \dots \oplus A_\ell \quad (A_\ell \neq 0) \ .$$

Then we always have $A_\ell \subseteq Soc(A)$.

Definition 8.4: Let

$$A = k[x_0, \dots, x_n]/I = k \oplus A_1 \oplus \dots \oplus A_\ell \text{ with } A_\ell \neq 0 \ .$$

Then ℓ is called the *socle degree* of A.

Note that the socle degree of A is the least integer ℓ for which $m^{\ell+1} \subseteq I$.

Definition 8.5: The graded artinian ring A is called a *Gorenstein ring* if $\dim_k Soc(A) = 1$.

Thus, if A is an artinian ring having socle degree ℓ then A is Gorenstein if and only if $Soc(A) = A_\ell$ and $\dim A_\ell = 1$.

Remark: In Example 8.3 above, 1) is a Gorenstein ring and 2) is not. Notice that these two rings have the same Hilbert function.

Proposition 8.6 The Hilbert function of an artinian Gorenstein ring A is symmetric.

More precisely, if ℓ is the socle degree of A then

$$H(A,t) = H(A, \ell - t) \text{ for all } t \ .$$

Proof: The result follows immediately from the following

Claim: The pairing

$$A_t \times A_{\ell - t} \longrightarrow A_\ell \simeq k$$

(induced by the multiplication of A) is a perfect pairing.

(Hence $A_t \simeq A^*_{\ell - t}$ ($* =$ vector space dual) and hence both have the same dimension.)

Proof of claim: We need to show that if $a \in A_t$ and $ab = 0$ for all $b \in A_{\ell - t}$ then $a = 0$.

Now $A_{\ell-t}$ is generated by the monomials $\overline{x}^\beta$ where $\deg \beta = \ell - t$ and, by assumption, $a\overline{x}^\beta = 0$ for all such β.

Moreover, $a\overline{x}^{\beta'} = 0$ for all β' where $\deg \beta' = \ell - t - 1$. This is so because

$$(a\overline{x}^{\beta'})\overline{x}_i = 0 \text{ for all } i = 0, \ldots, n \ .$$

Thus $a\overline{x}^{\beta'} \in Soc(A)$. But, $\deg a\overline{x}^{\beta'} = t + (\ell - t - 1) = \ell - 1$, so this cannot be a non-zero element of $Soc(A)$.

Hence $a\overline{x}^{\beta'} = 0$ for all β' of $\deg = \ell - t - 1$. We can thus continue this process until we obtain that $a\overline{x}_i = 0$ for all $i = 0, \ldots, n$. Thus, $a \in Soc(A)$. But $\deg a = t \neq \ell$ and so $a = 0$ and we are done. $\square$

Remark: In fact, if A is an artinian ring, with socle degree ℓ and $\dim_k A_\ell = 1$ then

$$A \text{ is a Gorenstein ring} \Leftrightarrow \text{the pairing } \ A_t \times A_{\ell - t} \to A_\ell$$

$$\text{is a perfect pairing for every } 0 \leq t \leq \ell \ .$$

To see why this is so just note that the Claim in the Proposition above gives half of the result, while if there was a non-zero socle element in degree t (for $t < \ell$) then it would annihilate everything in $A_{\ell - t}$ contradicting the fact that the pairing is perfect.

This characterization of Gorenstein rings is only one of many interesting such characterizations. The characterization I will spend the most time discussing, however, comes from a consideration of the action of R on S as above.

We've seen that if $I \subseteq R$ then R/I is artinian iff I^{-1} is a finitely generated R-submodule of S. The (graded) Gorenstein (artinian) rings fit very nicely into this equivalence.

Theorem 8.7: (Macaulay) Let $R = k[x_0, \ldots, x_n]$ and let $A = R/I$ (I homogeneous) be artinian.

A is a Gorenstein ring with socle degree $j \Leftrightarrow I^{-1}$ is a principal submodule of S generated by a form of degree j.

I.e. R/I is Gorenstein $\Leftrightarrow\ I = ann(F),\ F \in S_j$.

Remark: For those "in the know", the *Cohen-Macaulay type* of the graded artinian ring $A = R/I$ is the same as the (minimal) number of generators of the R-submodule I^{-1}. I won't enter into that here.

In order to prove Macaulay's theorem I will follow a route proposed by Iarrobino. To understand that approach it is useful to introduce a concept which was first baptized by Iarrobino – the notion of the *ancestor ideal.*

If $R = k[x_0, \ldots, x_n]$ and $V \subseteq R_j$ is a subspace then

$$R_{j-i} \supseteq V : R_i := \{G \in R_{j-i} \mid GR_i \subseteq V\}$$

is a vector subspace of R_{j-i}.

Definition-Proposition 8.8: With the notation above, the set

$$\overline{V} = [\sum_{i=j}^{1} V : R_i] \oplus (V)$$

is a homogeneous ideal of R called the *ancestor ideal* of V.

It is the largest ideal J of R for which $J_{j+t} = (V)_{j+t}$ for all $t \geq 0$.

Proof: We have:

$$\overline{V} = < V : R_j > \oplus < V : R_{j-1} > \oplus \cdots \oplus < V : R_1 > \oplus V \oplus R_1 V \oplus \cdots$$

and clearly $\overline{V}$ is closed under addition. Also, multiplying anything in $\overline{V}$ of degree $\geq j$ by anything in R clearly is back again in $\overline{V}$. So, the only multiplication to consider is when

$$G \in R_t \ (t \in \mathbb{N}) \text{ and } H \in < V : R_i > , \ 1 \leq i \leq j \ \ (\text{so } \deg H = j - i) \ .$$

In that case, write $\deg GH = t + j - i = s$.

Case 1: $t \geq i$.

Then $G \in R_t \Rightarrow G = \sum_\alpha F_\alpha G_\alpha$ where $\deg G_\alpha = i$ and $\deg F_\alpha = t - i$.

But then

$$GH = (\sum_\alpha F_\alpha G_\alpha) H = \sum_\alpha F_\alpha (G_\alpha H) \ .$$

Since $H \in < V : R_i >$ and $G_\alpha \in R_i$ we get that $G_\alpha H \in V$ and so $F_\alpha(G_\alpha H) \in (V)$.

Thus, $GH \in (V)$ and we are done.

Case 2: $t < i$.

Then $\deg GH = t + (j - i) = (t - i) + j < j$ i.e $GH \in R_{t+j-i}$ and we need to show that

$$GH \in < V : R_{j-(t+j-i)} > = < V : R_{i-t} > \ .$$

But $(GH)R_{i-t} = H(GR_{i-t})$ and since we always have $GR_{i-t} \subseteq R_i$, in order to show that $GH \in < V : R_{i-t} >$ it will suffice to show that $HR_i \subseteq V$. But, this is exactly how H was chosen.

To finish off the proof we want to show why $\overline{V}$ is the biggest homogeneous ideal J of R for which $J_{j+t} = (V)_{j+t}$ for all $t \geq 0$.

So, suppose that $J \supseteq \overline{V}$ and that $J_i \supset \overline{V}_i$ for some $i < j$. Then there is an element $G \in J_i$ such that $G \notin < V : R_{j-i} >$. I.e. there is an $H \in R_{j-i}$ such that $GH \notin V$. But, $H \in R_{j-i}$ and $G \in J_i$ implies that $HG \in J_j = V$, and that is the contradiction which establishes the result.

Note: Recall that the saturation of a homogeneous ideal is the largest ideal which agrees with the given ideal in all sufficiently high degrees.

Thus, if $V \subseteq R_j$ then $(V) \subseteq \overline{V} \subseteq (V)^{sat}$. All of these containments can be proper, as the following example shows:

Example 8.9: Let $V = < x_1^4, x_1x_2^3, x_1^3x_2 > \subseteq R_4$, where $R = k[x_1, x_2]$.

Then $R_1V = < x_1^5, x_1^4x_2, x_1^3x_2^2, x_1^2x_2^3, x_1x_2^4 > = (x_1)_5$ and thus, $(V)^{sat} = (x_1)$.

On the other hand, $V : R_1 = < x_1^3 >$ and $V : R_2 = (0)$, so

$$\overline{V} = < x_1^3 > \oplus V \oplus R_1V \oplus \cdots$$

Thus,

$$(V) \underset{\neq}{\subseteq} \overline{V} \underset{\neq}{\subseteq} (V)^{sat} \ ,$$

and all three ideals agree in high enough degrees.

Early in these lectures we saw the following: Let

$$R = k[x_0, \ldots, x_n] \qquad\qquad S = k[y_0, \ldots, y_n]$$

and let $I \subseteq R$ an ideal. Then it is easy to describe I^{-1} using the following important fact (see Proposition 2.5):

$$(I^{-1})_j = I_j^{\perp}$$

(where the $\perp$ is with respect to the pairing

$$R_j \times S_j \longrightarrow k \ ).$$

So, in particular, if I is an artinian ideal then I^{-1} is finitely generated and easily constructed.

But, how do we go in the other direction? Specifically: if we let $F \in S_j$ and let $I = ann(F)$, $I \subseteq R$, how do we go about constructing I?

Clearly, since $F \in S_j$, we can use the pairing

$$R_j \times S_j \longrightarrow k$$

to find that I_j has codimension 1 in R_j and it is $< F >^{\perp}$. Also, clearly, $I_{j-t} = (R_tF)^{\perp}$. But, that is not a particularly useful description of I. The following proposition gives us a useful way to describe $I = ann(F)$.

Proposition 8.10: If $F \in S_j$ and $I = ann(F)$, then

$a)$ $I_j =< F >^{\perp}$ (in the pairing $R_j \times S_j \longrightarrow k$.)

$b)$ $I = \overline{< F >^{\perp}} + m^{j+1}$.

Proof: $a)$ is obvious from our remarks above.

$b)$ We'll first show that

$$I \supseteqq \overline{< F >^{\perp}} + m^{j+1} \ .$$

Now, m^{j+1} certainly annihilates $F \in S_j$ and also $< F >^{\perp} = I_j$ as we have already remarked on above. So, we only need to prove the containment in degrees $< j$.

So, let $G \in \overline{< F >^{\perp}}$ where $\deg G = t < j$. By the definition of the ancestor ideal $\overline{< F >^{\perp}}$ we must then have $GR_{j-t} \in < F >^{\perp}$, i.e. $Gx^{\alpha} \in < F >^{\perp}$ for any monomial x^{α} of degree $j - t$.

We want to show that $G \circ F = 0$. But, by the definition of the action, $G \circ F \in S_{j-t}$ and (see the beginning of Section 2) we have

$$x^{\alpha} \circ (G \circ F) = (x^{\alpha} G) \circ F \ .$$

But, $x^{\alpha} G \in I_j =< F >^{\perp}$ and so $(x^{\alpha} G) \circ F = 0$. Thus, $x^{\alpha} \circ (G \circ F) = 0$ for every monomial $x^{\alpha} \in R_{j-t}$. Since the pairing $R_{j-t} \times S_{j-t} \longrightarrow k$ is nondegenerate, this implies that $G \circ F = 0$, as was to be shown.

As for the other inclusion, i.e.

$$I \subseteq \overline{< F >^{\perp}} + m^{j+1} \ .$$

there is no question about this inclusion in degrees $\geq j$. So, let $G \in I$, $\deg G = t < j$ and let $H \in R_{j-t}$.

Since I is an ideal, $GH \in < F >^{\perp}$, i.e. $GR_{j-t} \subseteq < F >^{\perp}$ i.e. $G \in \overline{< F >^{\perp}}$, as we wanted to show. □

There is one more proposition we shall need to prove Macaulay's theorem.

Proposition 8.11: Let $A = R/I$ be an artinian graded ring with socle degree j and for which $\dim_k A_j = 1$.

Then

$$A \text{ is a Gorenstein ring } \Leftrightarrow I = \overline{I_j} + m^{j+1} \ .$$

Proof: $:\Rightarrow$

We first prove the easy inclusion $I \subseteq \overline{I_j} + m^{j+1}$.

Since the socle degree of R/I is j, we certainly have $I_\ell = (m^{j+1})_\ell$ for any $\ell \geq j+1$. Also, the ideals I and $\overline{I_j} + m^{j+1}$ certainly agree in degree j.

In degree $t < j$, let $H \in I_t$. Then, since I is an ideal, $HR_{j-t} \subseteq I_j$ and hence, by definition, $H \in (\overline{I_j})_t$. Thus, this inclusion is obvious.

We now consider the other inclusion, i.e. $\overline{I_j} + m^{j+1} \subseteq I$.

Again, since R/I has socle degree j, $m^{j+1} \subseteq I$. Also, both $\overline{I_j} + m^{j+1}$ and I agree in degree j. So, it is enough to show that $(\overline{I_j})_t \subseteq I_t$ in all degrees $t < j$.

To see this, recall (see the Claim in Proposition 8.6) that the pairing

$$A_t \times A_{j-t} \longrightarrow A_j \simeq k$$

is a perfect pairing.

Regard this pairing as

$$R_t/I_t \times R_{j-t}/I_{j-t} \longrightarrow R_j/I_j$$

and choose $G \in (\overline{I_j})_t$. Then $GR_{j-t} \subseteq I_j$. I.e. in the pairing $\overline{G}\overline{x}^\beta = 0$ for every β, where $\deg \beta = j - t$.

By the perfectness of the pairing we then get that $\overline{G} = 0$, i.e. $G \in I_t$, which is what we wanted to show.

$\Leftarrow$:

Let's suppose that $I = \overline{I_j} + m^{j+1}$. We want to show that $A = R/I$ is Gorenstein. In view of the remark after Prop. 8.6, it will be enough to show that the pairings

$$R_t/I_t \times R_{j-t}/I_{j-t} \longrightarrow R_j/I_j$$

are all perfect.

So, let $\overline{G} \in (R/I)_t$ and suppose that $\overline{G}\overline{x}^\beta = 0$ for every β with $\deg \beta = j - t$. But then, $GR_{j-t} \subseteq I_j$ and this implies that $G \in (\overline{I_j})_t$. Since $I = \overline{I_j} + m^{j+1}$ we get that $G \in I_t$ i.e. $\overline{G} = 0$ and we are done.

We are now ready to prove Macaulay's Theorem.

Proof: (Theorem 8.7)

So, suppose that $A = R/I$ and $I = ann(F),\ F \in S_j$. Then, since the apolarity pairing

$$R_j \times S_j \longrightarrow k$$

is perfect and $F \in S_j$, we have $I_j =< F >^{\perp}$ and so $\dim_k(R_j/I_j) = 1$.

Also, since $\deg F = j$ we must have $m^{j+1} \subseteq I$. Thus, A is an artinian ring of socle degree j for which $\dim_k A_j = 1$. We can now apply Proposition 8.11 and so we get that A is Gorenstein $\Leftrightarrow I = \overline{I_j} + m^{j+1}$.

But, we just saw that $I_j =< F >^{\perp}$ and by Proposition 8.10 $I = \overline{< F >^{\perp}} + m^{j+1}$ and so we are done.

Conversely, suppose that A is a Gorenstein ring with socle degree j. By Proposition 8.11 we have $I = \overline{I_j} + m^{j+1}$. But, since A is Gorenstein, $\dim_k(R_j/I_j) = 1$. Thus, there must be an $F \in S_j$ such that $I_j =< F >^{\perp}$. It remains to show that $I = ann(F)$

Let $J = ann(F)$. Then $J_j = I_j$ by construction. But, by Proposition 8.10, the ideal $ann(F)$ is completely determined by its degree j piece and so is $\overline{J_j} + m^{j+1}$. But then $I = J$ and we are done. □

Lecture 9: Parameter Spaces for Gorenstein Artinian Ideals

From Macaulay's Theorem (Theorem 8.7) we saw : if $R = k[x_0, \ldots, x_n]$ and $A = R/I$ is a Gorenstein ring with socle degree of $A = j$, then $I = ann(F)$ where $F \in S_j$ $(S = k[y_0, \ldots, y_n])$. Obviously,

$$ann(F) = ann(\lambda F) \text{ for any } F \in S_j \ , \lambda \neq 0 \ , \lambda \in k \ .$$

With this theorem (and observation) in hand we immediately obtain the following:

> *The projective space* $\mathbb{P}(S_j) \simeq \mathbb{P}^N$ *(where* $N = \binom{j+n}{n} - 1$*) is a parameter space for all the Gorenstein (artinian) quotients of* $R = k[x_0, \ldots, x_n]$.

This parameter space gives us a natural place in which to view, geometrically, the family of all (artinian) Gorenstein quotients of $R = k[x_0, \ldots, x_n]$ having socle degree j, as well as certain specific subfamilies of such rings. In particular, it will be natural to think of the geometric properties of families of such Gorenstein rings with specified invariants (Hilbert function, graded Betti numbers for example).

Since I will be interested in the Hilbert function (first) of such Gorenstein rings, I want to explain quickly how one goes about calculating the Hilbert function of $A = R/I$ when $I = ann(F)$, $F \in S_j$.

First observe that the R-submodule of S generated by F (which we shall denote (F)) is:

$$k \oplus R_{j-1}F \oplus \cdots \oplus R_1 F \oplus < F > \ ,$$

i.e.

$$(F)_t = \begin{cases} R_{j-t}F & \text{for } t \leq j; \\ 0 & \text{for } t > j. \end{cases}$$

Moreover, since $I = ann(F)$ we have:

$$\dim_k I_t = \begin{cases} \dim_k (R_{j-t}F)^{\perp} & \text{for } t \leq j \ ; \\ \dim_k R_t & \text{for } t > j \ . \end{cases}$$

i.e.

$$\dim_k (R_t/I_t) = \begin{cases} \dim_k (R_{j-t}F) & \text{for } t \leq j \ ; \\ 0 & \text{for } t > j \ . \end{cases}$$

So, we have proved the following Proposition:

Proposition 9.1: If $R = k[x_0, \ldots, x_n]$ and $I \subseteq R$, $I = ann(F)$ where $F \in S_j$ ($S = k[y_0, \ldots, y_n]$) and if we set $A = R/I$, then

$$H(A,t) = \dim_k(R_{j-t}F) = \dim_k < \left(\frac{\partial}{\partial y^B}\right)F \mid \deg B = j-t > \ .$$

(Perhaps the only thing remaining to comment on in this proposition is the last equality. But, that is nothing more than a restatement of how the ring R acts on the ring S.)

Before I go on to work out some examples, I would like to have another way to look at the action of $R = k[x_1, \ldots, x_n]$ on $S = k[y_1, \ldots, y_n]$ when the characteristic of k is 0.

We already saw that if $\alpha = (\alpha_1, \ldots, \alpha_n)$ then, in the pairing,

$$R_j \times S_j \longrightarrow k$$

we have

$$x^\alpha \times y^\alpha \longrightarrow \alpha_1!\alpha_2!\ldots\alpha_n!$$

which is not, in general, 1. I.e. the basis "vectors" x^α and y^α are not dual bases. I would like to get around this situation so that certain calculations can be made simpler.

We first introduce some notation: if $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, where $\alpha_i\ , \beta_i \in \mathbb{Z}_{\geq 0}$, then we write:

$$\alpha! := \prod_{i=1}^{n} \alpha_i! \quad \text{and} \quad \binom{\alpha+\beta}{\alpha} := \prod_{i=1}^{n} \binom{\alpha_i+\beta_i}{\alpha_i} \ .$$

It is a simple exercise to show that:

$$\binom{\alpha+\beta}{\alpha} = \frac{(\alpha+\beta)!}{\alpha!\beta!} = \binom{\alpha+\beta}{\beta} \ .$$

We proceed somewhat formally: We start with $R = k[x_1, \ldots, x_n] = \oplus_{i=o}^{\infty} R_i$, which we think of as a (graded) infinite dimensional vector space over k. We form the (graded) dual vector space, which we call $\mathcal{D}$:

$$\mathcal{D} = k \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots$$

where $\mathcal{D}_i$ is the vector space dual to R_i.

If we let $\{x^\alpha\}$ be the set consisting of the standard monomial basis of R_j then we write $\{Y^{(\alpha)}\}$ for the set which consists of the dual basis. I.e. $Y^{(\alpha)}$ is the linear functional on R_j which takes x^α to 1 and all other basis vectors of R_j to 0.

If we write $e_1 = (1, 0, \ldots, 0),\ \cdots, e_n = (0, 0, \cdots, 1)$ then we shall denote this "dual" (infinite dimensional) vector space by

$$\mathcal{D} = k\{Y^{(e_1)}, \ldots, Y^{(e_n)}\} \ .$$

So, as a graded vector space, $\mathcal{D}$ has as basis $\{Y^{(\alpha)} \mid \alpha \in \mathbb{Z}^n_{\geq 0}\}$. I would like to put a ring structure on $\mathcal{D}$.

We define:

$$(aY^{(\alpha)})(bY^{(\beta)}) = ab\binom{\alpha+\beta}{\alpha}Y^{(\alpha+\beta)}$$

and extend this linearly to all of $\mathcal{D}$.

It is easy to see that the only thing we need to check to see if this makes $\mathcal{D}$ into a commutative ring with 1, is:

Claim: $Y^{(\alpha)}(Y^{(\beta)}Y^{(\gamma)}) = (Y^{(\alpha)}Y^{(\beta)})Y^{(\gamma)}$.

Proof: It is easy to see that verifying the claim amounts to showing that

$$\binom{\alpha+\beta+\gamma}{\alpha}\binom{\beta+\gamma}{\beta} = \binom{\alpha+\beta}{\alpha}\binom{\alpha+\beta+\gamma}{\alpha+\beta} \ .$$

But, both of these are easily seen to be

$$\frac{(\alpha+\beta+\gamma)!}{\alpha!\beta!\gamma!}$$

Remark 9.2: *a*) I'll leave, as a simple induction exercise, that

$$Y^{(\alpha_1)}Y^{(\alpha_2)}\ldots Y^{(\alpha_s)} = \frac{(\alpha_1+\cdots+\alpha_s)!}{\alpha_1!\cdots\alpha_s!}Y^{(\alpha_1+\cdots+\alpha_s)}$$

in particular

$$(Y^{(\alpha)})^d = \frac{(d\alpha)!}{(\alpha!)^d}Y^{(d\alpha)} \ \text{ and if } \alpha = (e_i) \text{ then } (Y^{(e_i)})^d = d!Y^{(de_i)} \ .$$

It follows that if $\alpha = (\alpha_1, \ldots, \alpha_n)$ and we let

$$Y^\alpha = (Y^{(e_1)})^{\alpha_1} \cdots (Y^{(e_n)})^{\alpha_n}$$

then

$$Y^\alpha = \alpha! Y^{(\alpha)} \ .$$

b) The ring $\mathcal{D}$, as constructed above, is sometimes referred to as the *ring of divided powers* or *divided power* ring. The notation I am using, $\mathcal{D} = k\{Y^{(e_1)}, \cdots, Y^{(e_n)}\}$ is completely nonstandard! I will explain what the "divided powers" are later.

c) The multiplication I've defined above arises very naturally as the dual to the comultiplication on R given by the diagonal map

$$\Delta : R \longrightarrow R \otimes_k R$$

where Δ is the unique map out of the polynomial ring R which takes 1 to $1 \otimes_k 1$ and x_i to $x_i \otimes 1 + 1 \otimes x_i$. It would take me too far afield to go into the details about this right now.

d) The reader should notice that the coefficients of the multiplication are in $\mathbb{Z}$ and so make sense in a ring of any characteristic. I.e. the divided power ring can be defined with k any base ring.

Example 9.3:
1) Let's consider the case of one variable in characteristic 0. So, $\mathcal{D} = k\{Y^{(e_1)}\}$ is the vector space dual to $R = k[x_1]$. We have

$$\mathcal{D} = k \oplus < Y^{((1))} > \oplus < Y^{((2))} > \oplus \cdots$$

Now

$$Y^{((1))} Y^{((1))} = \binom{(1)+(1)}{(1)} Y^{((2))} = 2Y^{((2))}$$

and, more generally

$$(Y^{((1))})^d = d! Y^{((d))} \ .$$

Since, in characteristic 0, $d!$ is never 0, we see that, as an algebra, $\mathcal{D}$ is generated by $Y^{((1))}$.

2) Let's consider the same ring, but this time let k have characteristic 2.

As before,

$$\mathcal{D} = k \oplus < Y^{((1))} > \oplus < Y^{((2))} > \oplus \cdots$$

But now, $Y^{((1))}Y^{((1))} = (Y^{((1))})^2 = 0$ and so $(Y^{((1))})^d = 0$ for all $d \geq 2$.

But, $Y^{((2))}Y^{((1))} = \binom{(2)+(1)}{(2)} Y^{((3))}$. Since

$$\binom{(2)+(1)}{(2)} = \binom{3}{2} = 3 \equiv 1 (mod\ 2)$$

we have

$$Y^{((2))}Y^{((1))} = Y^{((3))} \ .$$

It seems fairly clear that a knowledge of the multiplication in this ring is heavily dependent on the divisibility of the binomial coefficients by various primes (in our case the prime 2).

The most useful result that I know of in this direction is a theorem of Lucas.

Theorem 9.4: Let $a = \sum_{i=0}^{\infty} a_i p^i$, $b = \sum_{i=0}^{\infty} b_i p^i$ (where $0 \leq a_i, b_i < p$) (i.e. the base p expansions of a and b respectively.)

Then

$$\binom{b}{a} \equiv \prod_{i=0}^{\infty} \binom{b_i}{a_i} (mod\ p) \ .$$

Note: Since both a_i, b_i are $< p$ the only way that $\binom{b_i}{a_i} \equiv 0 (mod\ p)$ is if $\binom{b_i}{a_i} = 0$, i.e. $b_i < a_i$.

If you want to play with this a bit, consider the following examples:

in char = 2: 12 = 4 + 8 (is the base 2 expansion of 12) and

$$Y^{((12))} = Y^{((4))}Y^{((8))} \ .$$

in char = 3: 15 = 2(3) + 1(9) is the base 3 expansion of 15.

$$(Y^{((3))})^2 Y^{((9))} = cY^{((15))}$$

where $c \neq 0$ modulo 3. (There is a pattern here which the reader might try to unravel.)

From foolings around like this, one can eventually show that, in characteristic p, the ring $k\{Y^{(e_1)}\}$ is (infinitely) generated by the elements $\{Y^{((p^e))}\}$ for all the prime powers

p^e. Thus, this nice ring, with the same Hilbert function as the polynomial ring in one variable, is not a noetherian ring.

For us, since we will usually work with characteristic 0, the most important result about the ring $\mathcal{D}$ is the following:

Theorem 9.5: Let k be a field of characteristic 0. As above, we let $R = k[x_1, \ldots, x_n]$, $S = k[y_1, \ldots, y_n]$ and $\mathcal{D} = k\{Y^{(e_1)}, \ldots, Y^{(e_n)}\}$.

Let $\phi : S \longrightarrow \mathcal{D}$ be the k-algebra homomorphism given by letting ϕ be the identity on k and $\phi(y_i) = Y^{(e_i)}$.

Then ϕ is an isomorphism of k-algebras.

Moreover, if

$$R_i \times S_j \to S_{j-i}$$

is the differentiation action of R on S and

$$R_i \times \mathcal{D}_j \to \mathcal{D}_{j-i}$$

is the contraction operation given by:

$$x^\alpha \times Y^{(\beta)} \to \begin{cases} 0 & \text{if } \alpha \not\leq \beta \\ Y^{(\beta-\alpha)} & \text{if } \alpha < \beta \end{cases}$$

then the following diagram commutes

$$\begin{tikzcd}
R_i \arrow[d, "id."'] & \times & S_j \arrow[r] \arrow[d, "\phi_j"] & S_{j-i} \arrow[d, "\phi_{j-i}"] \\
R_i & \times & \mathcal{D}_j \arrow[r] & \mathcal{D}_{j-i}
\end{tikzcd}$$

i.e. S and $\mathcal{D}$ are also isomorphic as R-modules.

Proof: Since S is a polynomial algebra there is an algebra homomorphism as defined in the statement of the theorem.

Be careful, however! We have that $\phi(y_i) = Y^{(e_i)}$ but (for example)

$$\phi(y_i^2) = (Y^{(e_i)})^2 = 2Y^{(2e_i)} = 2Y^{((2,0,\ldots,0))} \ .$$

Observe also that if $y^\alpha \in S_j$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, then

$$\phi(y^\alpha) = \phi(y_1^{\alpha_1} \cdots y_n^{\alpha_n}) = \phi(y_1)^{\alpha_1} \cdots \phi(y_n)^{\alpha_n}$$

$$= (Y^{(e_1)})^{\alpha_1} \cdots (Y^{(e_n)})^{\alpha_n} = Y^\alpha = \alpha! Y^{(\alpha)} \ .$$

Now, in characteristic 0, $\alpha! \neq 0$ and since the $Y^{(\alpha)}$ are a basis for the vector space $\mathcal{D}_j$ we see that ϕ is 1-1 and onto, i.e. ϕ is an isomorphism of rings.

Finally let's see what happens with the various bilinear mappings:

if $x^\alpha \in R_i$ and $y^\beta \in S_j$ then, if $\alpha \leq \beta$,

$$x^\alpha \times y^\beta \longrightarrow \frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha}$$

while

$$x^\alpha \times \phi(y^\beta) = x^\alpha \times (\beta! Y^{(\beta)}) \longrightarrow \beta! Y^{(\beta-\alpha)} \ .$$

Now notice that

$$\phi\left(\frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha}\right) = \frac{\beta!}{(\beta-\alpha)!}\phi(y^{\beta-\alpha})$$

$$= \frac{\beta!}{(\beta-\alpha)!}(\beta-\alpha)! Y^{(\beta-\alpha)} = \beta! Y^{(\beta-\alpha)} \ .$$

and that completes the argument.

Remarks 9.6:

1) The inverse isomorphism $\phi^{-1} : \mathcal{D} \to S$ (of course, in characteristic 0) is given by

$$Y^{(\alpha)} \longrightarrow \frac{y^\alpha}{\alpha!}$$

2) If $L = a_1 y_1 + \cdots + a_n y_n$ is in S_1, then I would like to record what $\phi(L^d)$ looks like in $\mathcal{D}$.

Now,

$$\phi(L^d) = (\phi(L))^d = (a_1 Y^{(e_1)} + \cdots + a_n Y^{(e_n)})^d$$

$$= \sum_{(\alpha_1, \dots, \alpha_n), \sum \alpha_i = d} a_1^{\alpha_1} \cdots a_n^{\alpha_n} \binom{d}{\alpha_1 \ \alpha_2 \ \cdots \alpha_n} (Y^{(e_1)})^{\alpha_1} \cdots (Y^{(e_n)})^{\alpha_n}$$

$$= \sum a_1^{\alpha_1} \cdots a_n^{\alpha_n} \frac{d!}{\alpha_1! \cdots \alpha_n!} (\alpha_1! Y^{(\alpha_1 e_1)}) \cdots (\alpha_n! Y^{(\alpha_n e_n)})$$

$$= d! \sum_{(\alpha_1,\ldots,\alpha_n), \sum \alpha_i = d} a_1^{\alpha_1} \cdots a_n^{\alpha_n} Y^{((\alpha_1,\ldots,\alpha_n))} \ .$$

Example 9.7: Let $F = Y^{((2,2,2))}$ in $\mathcal{D}_6$. Then

$$R_1F =< Y^{((1,2,2))}, Y^{((2,1,2))}, Y^{((2,2,1))} > \text{ so } \dim_k R_1F = 3;$$

$$R_2F =< Y^{((0,2,2))}, Y^{((1,1,2))}, Y^{((1,2,1))}, Y^{((2,0,2))}, Y^{((2,1,1))}, Y^{((2,2,0))} > \text{ so } \dim_k R_2F = 6;$$

$$R_3F =< Y^{((0,1,2))}, Y^{((0,2,1))}, Y^{((1,0,2))}, Y^{((1,1,1))}, Y^{((1,2,0))}, Y^{((2,0,1))}, Y^{((2,1,0))} >$$

so $\dim_k R_3F = 7$.

We can now use the symmetry of the Hilbert function of a Gorenstein ring to assert that $\dim_k R_4F = 6$; $\dim_k R_5F = 3$; $\dim_k R_6F = 1$. So, if $I = ann(F)$ where $I \subseteq k[x_0, x_1, x_2] = R$ then

$$H(R/I, -) = 1 \ \ 3 \ \ 6 \ \ 7 \ \ 6 \ \ 3 \ \ 1 \ \ 0 \ \ \cdots$$

I now want to explain why the ring $\mathcal{D}$ is called the ring of *divided powers.*

Let R be a non-negatively graded R_0-algebra,

$$R = R_0 \oplus R_1 \oplus \cdots \ .$$

Definition 9.8: A system of *divided powers* on R is a family of functions

$$-^{[i]} : \cup_{j>0} R_j \to \cup_{j>0} R_j \text{ for } i = 0, \ 1, \ \ldots$$

such that the following rules are satisfied:

1) The function $-^{[0]}$ is the constant function 1, and the function $-^{[1]}$ is the identity function. Moreover, $\deg F^{[d]} = d \deg F$.

2) $F^{[d]}F^{[e]} = \binom{d+e}{d} F^{[d+e]}$;

3) $(F^{[d]})^{[e]} = \frac{(de)!}{e!(d!)^e} F^{[de]}$;

4) $(FG)^{[d]} = d! F^{[d]} G^{[d]} = F^d G^{[d]} = F^{[d]} G^d$;

5) $(\alpha F)^{[d]} = \alpha^d F^{[d]}$ for $\alpha \in R_0$;

6) $(F+G)^{[d]} = \sum_{e=0}^{d} F^{[e]}G^{[d-e]}$.

Proposition 9.9: If $k = R_0$ is a field of characteristic 0 then the functions

$$F^{[d]} = \frac{F^d}{d!}$$

is a system of divided powers on R.

Proof: The condition 1) is obvious. As for 2), just note that

$$\frac{F^d}{d!}\frac{F^e}{e!} = \frac{(d+e)!}{d!e!}\left(\frac{1}{(d+e)!}F^{d+e}\right) \ .$$

As for 3), note that

$$(F^{[d]})^{[e]} = \frac{1}{e!}(F^{[d]})^e = \frac{1}{e!}\left(\frac{F^d}{d!}\right)^e = \frac{1}{e!}\frac{1}{(d!)^e}F^{de}$$

$$= \frac{1}{e!}\frac{1}{(d!)^e}(de)!\left(\frac{1}{(de)!}F^{de}\right) = \frac{1}{e!}\frac{1}{(d!)^e}F^{[de]} \ .$$

For 4) we have:

$$(FG)^{[d]} = \frac{1}{d!}(FG)^d = \frac{F^d}{d!}G^d = F^{[d]}G^d = \frac{G^d}{d!}F^d = G^{[d]}F^d \ .$$

For 5) we have:

$$(\alpha F)^{[d]} = \frac{1}{d!}(\alpha F)^d = \alpha^d \frac{F^d}{d!} = \alpha^d F^{[d]} \ .$$

For the “hoped for” binomial theorem, we have:

$$(F+G)^{[d]} = \frac{1}{d!}(F+G)^d = \frac{1}{d!}\left(\sum_{e=0}^{d}\binom{d}{e}F^eG^{d-e}\right)$$

$$= \frac{1}{d!}\left(\sum_{e=0}^{d}\frac{d!}{e!(d-e)!}F^eG^{d-e}\right) = \sum_{e=0}^{d}\frac{F^e}{e!}\frac{G^{d-e}}{(d-e)!} = \sum_{e=0}^{d}F^{[e]}G^{[d-e]} \ .$$

□

It is worth noting that, not only does the “binomial” theorem have a nice form for divided powers but so also does the “multinomial” theorem. I.e.

Theorem 9.10: Suppose that the F_i, $i = 1, \ldots, r$ are homogeneous forms of the same degree. Then

$$(F_1 + \cdots + F_r)^{[d]} = \sum_{(\alpha_1,\ldots,\alpha_r), \sum \alpha_i = d} F_1^{[\alpha_1]} \cdots F_r^{[\alpha_r]} \ .$$

Proof: We know

$$(F_1 + \cdots + F_r)^{[d]} = \frac{1}{d!}(F_1 + \cdots + F_r)^d$$

$$= \frac{1}{d!} \sum_{(\alpha_1,\ldots,\alpha_r) \ \sum \alpha_i = d} \binom{d}{\alpha_1 \ \cdots \alpha_r} F_1^{\alpha_1} \cdots F_r^{\alpha_r} \ .$$

But, since

$$\binom{d}{\alpha_1 \ \cdots \alpha_r} = \frac{d!}{a_1! \cdots \alpha_r!}$$

we can distribute the factorials around to get the desired result.

Terminology: If R is a graded k-algebra with a system of divided powers and if F is homogeneous in R then we refer to $F^{[d]}$ as the d^{th} *divided power of* F.

Example 9.11: If we look back at Remark 9.6 we see that it is easy to deduce that if

$$\mathcal{L} = a_1 Y^{(e_1)} + \cdots + a_n Y^{(e_n)} \in \mathcal{D}_1$$

where $\mathcal{D} = k\{Y^{(e_1)}, \ldots, Y^{(e_n)}\}$ and k is a field of characteristic 0, then

$$\mathcal{L}^{[d]} = \sum_{(\alpha_1,\ldots,\alpha_n), \sum \alpha_i = d} a_1^{\alpha_1} \cdots a_n^{\alpha_n} Y^{((\alpha_1,\cdots,\alpha_n))} \ .$$

From this formula it appears as if this doesn't depend on the fact that k had characteristic 0. I.e. in the computation of the divided power, there was a part that involved the coefficients of the form $\mathcal{L}$ and there is a part that involves multinomial coefficients.

Let's look at another example.

Example 9.12: Let $\mathcal{D} = \mathbb{Q}\{Y^{(e_1)}, \ldots, Y^{(e_n)}\}$ and let $F \in \mathcal{D}_2$,

$$F = 3Y^{((2,0))} + 5Y^{((1,1))} + 7Y^{((0,2))} \ .$$

Then

$$F^{[2]} = \sum_{\alpha=(\alpha_1,\alpha_2,\alpha_3),\sum \alpha_i=2} \left(3Y^{((2,0))}\right)^{[\alpha_1]} \left(5Y^{((1,1))}\right)^{[\alpha_2]} \left(7Y^{990,2))}\right)^{[\alpha_3]} .$$

Since the possible α are in the set $\{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\}$ we'll know just about everything about $F^{[2]}$ if we know:

$$\left(3Y^{((2,0))}\right)^{[2]} = 9\left(\frac{1}{2!}\right)(Y^{((2,0))})^2 = 9\left(\frac{4!}{2!2!}\right)Y^{((4,0))}$$

and

$$\left(5Y^{((1,1))}\right)^{[2]} = 25\left(\frac{2!2!}{2!}\right)Y^{((2,2))}$$

and

$$\left(7Y^{((0,2))}\right)^{[2]} = 49\left(\frac{4!}{2!2!}\right)Y^{((0,4))} .$$

Notice that in each case the factor is in $\mathbb{Z}\{Y^{e_1},\ldots,Y^{(e_n)}\}$ i.e.

$$F^{[2]} \in \mathbb{Z}\{Y^{(e_1)},\ldots,Y^{(e_n)}\} .$$

This is no accident. In fact we have the following very useful fact.

Theorem 9.13: Let $F \in R = \mathbb{Z}\{Y^{(e_1)},\ldots,Y^{(e_n)}\} \subseteq \mathbb{Q}\{Y^{(e_1)},\ldots,Y^{(e_n)}\}$ where $F \in \cup_{i\geq 1} R_i$.

Then

$$F^{[d]} = \frac{F^d}{d!} \text{ is also in } R .$$

Proof: Let's write F as a sum of monomials of the form $a_\alpha Y^{(\alpha)}$. Then, by our previous observation about the multinomial theorem, we obtain that $F^{[d]}$ is a sum of products of terms of the form $(a_\alpha Y^{(\alpha)})^{[e]}$.

But since

$$(a_\alpha Y^{(\alpha)})^{[e]} = a_\alpha^e \left(\frac{1}{e!}\right)\left(Y^{(\alpha)}\right)^e$$

(where $a_\alpha^e \in \mathbb{Z}$ since $a_\alpha \in \mathbb{Z}$), it will be enough to show that

$$\frac{1}{e!}\left(Y^{(\alpha)}\right)^e \in \mathbb{Z}\{Y^{(e_1)},\ldots,Y^{(e_n)}\} .$$

Recall (Remark 9.2a) that

$$\left(Y^{(\alpha)}\right)^e = \left(\frac{(e\alpha)!}{(\alpha!)^e}\right) Y^{(e\alpha)}$$

so it will be enough to show that

$$\frac{1}{e!}\frac{(e\alpha)!}{(\alpha!)^e} \in \mathbb{Z} .$$

But, if $\alpha = (\alpha_1, \ldots, \alpha_t)$ then $(e\alpha)! = (e\alpha_1)!(e\alpha_2)! \cdots (e\alpha_t)!$ and so

$$\frac{(e\alpha)!}{(\alpha!)^e} = \frac{(e\alpha_1)!}{(\alpha_1!)^e} \cdots \frac{(e\alpha_t)!}{(\alpha_t!)^e} .$$

So, the theorem will follow from the following

Lemma: Let d, a be non-negative integers. Then

$$d! | \frac{(da)!}{(a!)^d} .$$

Proof: (Thanks to Peter Zion for this quickie!)

Now

$$\frac{(da)!}{(a!)^d} = \binom{da}{\underbrace{a \ \cdots \ a}_{d-\text{times}}} = \binom{da}{a}\binom{(d-1)a}{a} \cdots \binom{a}{a} .$$

So, it will be enough to show that

Claim: $d | \binom{da}{a}$ for any a and any d.

Proof: But, just note that

$$\binom{da}{a} = \frac{(da)(da-1)\cdots(da-a+1)}{a(a-1)!} = \frac{da}{a}\binom{da-1}{a-1} .$$

Since $\binom{da-1}{a-1}$ is an integer we are done.

We get the following corollary.

Corollary 9.14: Let $\mathcal{D} = k\{Y^{(e_1)}, \ldots, Y^{(e_n)}\}$, where k is any field. Then $\mathcal{D}$ admits a system of divided powers.

Proof: The thing to observe is that it will be enough to know what to make of $\left(Y^{(\alpha)}\right)^{[d]}$. But we can calculate that over $\mathbb{Z}$ and then take the image of the thing we get in $\mathcal{D}$ and that will be enough. □

We can now prove the following important theorem – important not so much because it is hard to prove but rather because of the direction in which it points.

Theorem 9.15: Let $R = k[x_0, \ldots, x_n]$ and $\mathcal{D} = k\{Y^{(e_0)}, \ldots, Y^{(e_n)}\}$ where k is a field of arbitrary characteristic.

Let $F \in \mathcal{D}_j$ and set $I = ann(F)$. Then

$$H(R/I, -) = \begin{matrix} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots \\ (0) & (1) & & (j-1) & (j) & (j+1) & \end{matrix} \qquad (*)$$

if and only if $F = \lambda\mathcal{L}^{[j]}$ where $\mathcal{L} \in \mathcal{D}_1$ and $\lambda \in k^*$.

Proof: $\Leftarrow$: Suppose that $F = \lambda\mathcal{L}^{[j]}$ where $\mathcal{L} = a_0Y^{(e_0)} + \cdots + a_nY^{(e_n)}$.

Then, as we saw in Example 9.11 and Corollary 9.14,

$$\mathcal{L}^{[j]} = \lambda^j \sum_{(\alpha_0,\ldots,\alpha_n),\sum \alpha_i = j} a_0^{\alpha_0} \ldots a_n^{\alpha_n} Y^{((\alpha_0,\ldots,\alpha_n))} \ .$$

Now

$$x_i \circ Y^{((\alpha_0,\ldots,\alpha_n))} = \begin{cases} 0 & \text{if } \alpha_i = 0, \\ Y^{((\alpha_0,\ldots,\alpha_i - 1,\ldots,\alpha_n))} & \text{if } \alpha_i \neq 0 \ . \end{cases}$$

Thus

$$x_i \circ \mathcal{L}^{[j]} = \lambda^j \sum_{(\alpha_0,\ldots,\alpha_n),\alpha_i \neq 0,\sum \alpha_t = j} a_0^{\alpha_0} \cdots a_i^{\alpha_i} \cdots a_n^{\alpha_n} Y^{((\alpha_0,\ldots,\alpha_i - 1,\ldots,\alpha_n))}$$

$$= a_i\lambda^j \sum_{(\beta_0,\ldots,\beta_n),\sum \beta_i = j-1} a_0^{\beta_0} \cdots a_n^{\beta_n} Y^{((\beta_0,\ldots,\beta_n))} = a_i\lambda^j\mathcal{L}^{[j-1]}$$

Thus, all first contractions of F are linearly dependent and hence $H(R/I, -)$ is as claimed.

$\Rightarrow$: Conversely, suppose that $H(R/I, -)$ has Hilbert functions $(*)$. Since $H(R/I, 1) = 1$ we have that $I_1 = (L_1, \ldots, L_n)$ where the L_i are linearly independent linear forms. We make a linear change of variables in R (and the analogous change in $\mathcal{D}$) so that $I_1 = (x_1, \ldots, x_n)$.

Since $H(R/I, j) = 1$ and I is a monomial ideal, we must have that I_j contains all the monomials of R_j except one, which is obviously seen to be x_0^j.

Since we know that $I = ann(F)$ for some $F \in \mathcal{D}_j$ we write that F as $F = \sum a_\alpha Y^{(\alpha)}$ where the sum is over those $\alpha = (\alpha_0, \ldots, \alpha_n)$ with $\sum \alpha_i = j$. Since $x^\alpha \circ F = a_\alpha = 0$ for every monomial $x^\alpha \in I_j$ we must have $F = \lambda Y^{((j,0,\ldots,0))}$ for some $\lambda \in k^*$.

Since

$$Y^{((j,0,\ldots,0))} = \left(Y^{(e_0)}\right)^{[j]} \quad \text{we have} \quad F = \lambda \left(Y^{(e_0)}\right)^{[j]}$$

as we wanted to show.

Remark:

1) If we wish, we can consider Theorem 9.15 in characteristic 0 directly and, instead of taking $F \in \mathcal{D}_j$ we could consider $F \in S_j$, and use differentiation instead of contraction. The theorem will then reach the same conclusion about the Hilbert function of $I = ann(F)$, but this time if $F = \lambda L^j$ where $L \in S_1$.

This is clear since if we use the isomorphism ϕ from S to $\mathcal{D}$ (in characteristic 0) then

$$\phi(\lambda L^j) = \lambda\phi(L^j) = \lambda j! L^{[j]} \ .$$

2) Notice that we only used that $H(R/I, 1) = 1$ to prove $\Rightarrow$. In fact, by Macaulay's theorem describing the growth of the Hilbert function, if $H(R/I, 1) = 1$ then $H(R/I, t) = 1$ or 0 for any t. So, the knowledge of $H(R/I, 1)$ was all that was really needed.

Theorem 9.16: The set of all Gorenstein quotients of $k[x_0, \ldots, x_n]$ having socle degree j and Hilbert function $(*)$ is the closed subvariety of $\mathbb{P}^N$ ($N = \binom{j+n}{n} - 1$) which is the Veronese variety $\nu_j(\mathbb{P}^n)$.

In particular, it is a smooth arithmetically Cohen-Macaulay subvariety of $\mathbb{P}^N$ which has dimension n and degree j^n.

Proof: The last remarks of the theorem are well known, and I won't go into that right now, but I do want to explain the connection between these special Gorenstein rings and the Veronese varieties.

In order to do that I should look again at the definition of the Veronese varieties (see also Lecture 4). The usual way to describe these varieties is to define them parametrically.

Let me do that in a particularly simple case and leave the (obvious) generalizations to the reader.

So, I will look at the above in the case of $R = k[x_0, x_1]$ and $\mathcal{D} = k\{Y^{(e_0)}, Y^{(e_1)}\}$.

For a fixed integer j we want to consider a map

$$\nu_j : \mathbb{P}^1 \longrightarrow \mathbb{P}(\mathcal{D}_j) \simeq \mathbb{P}^{\binom{j+1}{1}-1} \simeq \mathbb{P}^j \ .$$

The parameter map ν_j is then defined by ordering the monomials of degree j in R in some way (usually lexicographically):

$$x_0^j \ , x_0^{j-1}x_1 \ , \ldots , x_0x_1^{j-1} \ , x_1^j$$

then, if $P = [a_0 : a_1] \in \mathbb{P}^1$ we define

$$\nu_j(P) = \nu_j([a_0 : a_1]) := [a_0^j : a_0^{j-1}a_1 : \ldots : a_0a_1^{j-1} : a_1^j] \in \mathbb{P}^j \ ,$$

i.e. we "evaluate" all the monomials of degree j at the point P. (be careful since "evaluation" is not well-defined, in general, but note why we are OK in this case).

However, if we think of $\mathbb{P}^j$ as $\mathbb{P}(\mathcal{D}_j)$ - with coordinates - then we can think of it in the following way:

$$\text{let } F = \alpha_{j,0}Y^{((j,0))} + \alpha_{j-1,1}Y^{((j-1,1))} + \cdots + \alpha_{1,j-1}Y^{((1,j-1))} + \alpha_{0,j}Y^{((0,j))} \ .$$

But then

$$F \leftrightarrow [\alpha_{j,0} : \alpha_{j-1,1} : \ldots : \alpha_{1,j-1} : \alpha_{0,j}] \ .$$

So, in order to understand the image of the Veronese map, ν_j in this context, we must figure out which forms $F \in \mathcal{D}_j$ correspond to points of the form

$$[a_0^j : a_0^{j-1}a_1 : \ldots : a_0a_1^{j-1} : a_1^j] \ .$$

But we have already seen that if $\mathcal{L} = a_0Y^{(e_0)} + a_1Y^{(e_1)}$ then

$$\mathcal{L}^{[j]} = \sum_{(u_0,u_1),u_0+u_1=j} a_0^{u_0}a_1^{u_1}Y^{(u_0,u_1)}$$

i.e. $\mathcal{L}^{[j]}$ has exactly the coefficients we want. By Proposition 9.15, these are exactly the forms in $\mathcal{D}_j$ which give (by Macaulay duality) the Hilbert function we are considering. $\square$

As we saw above, if $R = k[x_0, \dots, x_n]$ then the artinian Gorenstein quotients of R with socle degree j and Hilbert function beginning 1 1 are parametrized by the Veronese variety $\nu_j(\mathbb{P}^n)$ in $\mathbb{P}^{\binom{j+n}{n}-1}$. But, as the next remark shows, this parametrization doesn't take into account the notion of isomorphism.

Remark 9.17: If $R = k[x_0, \dots, x_n]$ and R/I is a (graded) artinian quotient of R with socle degree j whose Hilbert function begins 1 1 then I must contain n linearly independent linear forms. With no loss of generality, we can assume those forms are $x_1, \dots, x_n$. Thus $I \supsetneqq (x_1, \dots, x_n)$ and

$$R/I \simeq \frac{(R/(x_1, \dots, x_n))}{(I/(x_1, \dots, x_n))} = \frac{k[x_0]}{J}$$

where $J = (x_0^{j+1})$.

Thus, all graded artinian rings having Hilbert function $(*)$ are isomorphic.

So, isomorphism is not what is at issue here. We are speaking about an "embedded" phenomena, i.e. a Gorenstein (artinian) quotient of a *fixed* polynomial ring.

Theorem 9.16 leads us naturally to the following questions:

> *a*) Suppose that we are given a positive integer j and we fix a finite sequence of non-zero integers, $T = (1, t_1, \dots, t_j = 1)$ which is symmetric, i.e. for which $t_s = t_{j-s}$ for all s. How can we describe artinian Gorenstein rings which are quotients of $k[x_0, \dots, x_n]$ having that sequence as Hilbert function? If the description is (at first) algebraic, what can we say geometrically about the family of such Gorenstein rings. Respecting the principle of the "*par condicio*", if the description is (at first) geometric, what can we say algebraically about the family of such Gorenstein rings.
>
> *b*) Suppose we impose additional algebraic invariants on these Gorenstein rings (e.g. we specify graded Betti numbers i.e. we fix the free resolution of the defining ideal) can we say anything geometric about the family of Gorenstein rings having these invariants?

In order to make some sense of these questions we need to have a way to decide when a given $F \in \mathcal{D}_j$ gives rise to $I = ann(F)$ with a given Hilbert function.

So, we have as a first sub-problem:

Describe R_1F for $F \in \mathcal{D}_j$.

Now R_1F is a subvector space of $\mathcal{D}_{j-1}$ spanned by the set of all $x^\alpha \circ F$, where $\deg \alpha = 1$. So, we can, once coordinates are chosen, display the coordinates of the $x^\alpha F$ as the rows of a matrix, each row corresponding to an x^α in R_1. The row space of that matrix will then describe the space R_1F and the rank of the matrix will describe the dimension of that space.

Example 9.18: Let $\mathcal{D} = k\{Y^{(e_0)}, Y^{(e_1)}, Y^{(e_2)}\}$ $F = Y^{(3,0,0)} + Y^{(0,2,1)} + Y^{(0,0,3)}$, then we obtain the following matrix

$$\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \end{array} \begin{array}{cccccc} Y^{(2,0,0)} & Y^{(1,1,0)} & Y^{(1,0,1)} & Y^{(0,2,0)} & Y^{(0,1,1)} & Y^{(0,0,2)} \\ \end{array}$$

$$\begin{array}{c|cccccc} & Y^{(2,0,0)} & Y^{(1,1,0)} & Y^{(1,0,1)} & Y^{(0,2,0)} & Y^{(0,1,1)} & Y^{(0,0,2)} \\ \hline x_0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 1 & 0 \\ x_2 & 0 & 0 & 0 & 1 & 0 & 1 \end{array} = \mathcal{C}_1$$

since

$$x_0 \circ F = Y^{(2,0,0)}$$

$$x_1 \circ F = Y^{(0,1,1)}$$

$$x_2 \circ F = Y^{(0,2,0)} + Y^{(0,0,2)} .$$

Since $rk\mathcal{C}_1 = 3$ we find that $\dim_k R_1F = 3$.

If we want to know the dimension of R_2F we proceed similarly; this time we take all the second contractions and express them as vectors in the (lexcicographically ordered) monomial basis of $\mathcal{D}_{j-2}$. Continuing with the example above, we find:

$$\begin{array}{c} \\ x_0^2 \\ x_0x_1 \\ x_0x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array}
\begin{array}{c} \begin{array}{ccc} Y^{(e_0)} & Y^{(e_1)} & Y^{(e_2)} \end{array} \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} = \mathcal{C}_2 \ .$$

Notice that $\mathcal{C}_1^t = \mathcal{C}_2$. Clearly the fact of the symmetry depended on the ordering we choose of the monomials and the consistency with which we choose the same ordering on the monomials of both R and $\mathcal{D}$.

We want to do this all somewhat more systematically.

Definition 9.19: Let $F \in \mathcal{D}_m$, $\mathcal{D} = k\{Y^{(e_0)}, \ldots, Y^{(e_n)}\}$ and write $F = \sum a_\alpha Y^{(\alpha)}$ where $\deg \alpha = m$. Choose two positive integers i and j so that $i + j = m$.

Then the matrix

$$\mathcal{C} = Cat_F(i; j : n+1)$$

is the $\binom{i+n}{n} \times \binom{j+n}{n}$ matrix formed as follows:

> let the rows of $\mathcal{C}$ be indexed by the monomials $x^\beta \in R_i$ and the columns indexed by the basis vectors $Y^{(\gamma)} \in \mathcal{D}_j$, then the (β, γ) entry of $\mathcal{C}$ is a_α where $\beta + \gamma = \alpha$.

The matrix is called the (i, j)-*catalecticant* matrix of F.

So, in the terminology we introduced earlier (in Section 6):

$$Cat_F(u; v : n+1) = \mathcal{M}_{n,u}^t \mathcal{M}_{n,v} = \mathcal{Z}_{u,v} \ .$$

(Unfortunately, in the earlier notation, there was no indication of the number of variables involved, i.e. of the "$n+1$". I'd like to correct that now and write

$$\mathcal{M}^t_{n,u}\mathcal{M}_{n,v} = \mathcal{Z}^{(n)}_{u,v} \ .)$$

So, in Lecture 6, the matrix (†) is $\mathcal{Z}^{(2)}_{1,1} = Cat_F(1;1:3)$.

Also, the matrix (††) of Lecture 6 is $\mathcal{Z}^{(2)}_{2,2} = Cat_F(2;2:3)$.

The matrix referred to in Example 7.1 is $\mathcal{Z}^{(3)}_{2,2} = Cat_F(2;2:4)$.

The matrix referred to in Example 7.2 is $\mathcal{Z}^{(4)}_{2,2} = Cat_F(2;2:5)$.)

These catalecticant matrices are critical for determining the Hilbert function of R/I when $I = ann(F)$.

Theorem 9.20: Let $\mathcal{D} = k\{Y^{(e_0)}, \ldots, Y^{(e_n)}\}$ and let $F \in \mathcal{D}_j$. Suppose $I \subset R = k[x_0, \ldots, x_n]$ and $I = ann(F)$.

Then

$$H(R/I, t) = rk.\ Cat_F(t; j-t : n+1) \ .$$

(This is simply a translation, into the language of catalecticants, of some of the things we saw above.)

Now that we have this way of looking at the Hilbert function of $A = R/I$, $I = ann(F)$, $F \in \mathcal{D}_j$, we can rephrase one of the questions we raised earlier.

Choose $T = (t_0 = 1, t_1, \ldots, t_{j-1}, t_j = 1)$ a sequence of positive integers for which $t_r = t_{j-r}$ for $r = 0, \ldots, j$ and such that $t_r \leq \dim_k \mathcal{D}_r$.

Aside: Note that, by the symmetry, if:

a) $j = 2s$, the important numbers in this sequence are just $t_0, \ldots, t_s$ (since $t_{s-1} = t_{s+1}$ etc.);

b) $j = 2s+1$, the important numbers in this sequence are still $t_0, \ldots, t_s$ (but now $t_s = t_{s+1}$ etc.).

Following the notation of Iarrobino and Kanev in “Length”, I shall use **bold-face** characters to describe the <u>*set*</u>:

$$\mathbf{Gor}(T) = \{F \in \mathbb{P}(\mathcal{D}_j) \mid rk(Cat_F(s; j-s : n+1)) = t_s\} \ .$$

There is (as yet) no scheme structure on this set. It might even be an empty set if there is no Gorenstein artinian quotient, A, of $R = k[x_0, \ldots, x_n]$ with $H(A, j) = t_j$ for all j.

In fact, it is an open problem to characterize those T for which $\mathbf{Gor}(T)$ is non-empty. (This is a problem which has been solved for $R = k[x_0, \ldots, x_n]$ only when $n = 1, 2$. I want to return to a discussion of this problem in a subsequent lecture.)

Lecture 10: Parameter Spaces for Gorenstein Artinian Ideals - Continued

In the last section I used the contraction operation of R on $\mathcal{D}$ to define the catalecticant matrices for an element $F \in \mathcal{D}_j$. This is not the classical method of doing things. It is more usual to see the catalecticant matrices defined by using the differentiation operation of R on S (and thus restrict to characteristic 0).

In view of Theorem 3.5 it doesn't matter which way we look at things in characteristic 0 (while in characteristic p we only have one method available to us).

In this section and the next I shall stay only with characteristic zero. I will do this because in certain places I use tangent space and limit arguments in my explanations and I've not had a chance to check to see if these arguments are formal enough to be modified for characteristic p. I suspect that many of them are.

On the other hand, the catalecticant matrices are simpler if I use $\mathcal{D}$ (and contraction) instead of S (and differentiation), and I am unwilling to give up that simplicity! So, I will stick with the definition of the catalecticants coming from contraction and hope this "mixing " of the two actions doesn't cause the reader undue confusion.

Because $\mathbf{Gor}(T)$ is described using rank conditions on catalecticant matrices, it is natural to consider the subschemes of $\mathbb{P}(S_j)$ defined by these rank conditions.

To describe these schemes let me first recall the following standard notation: if M is a matrix of size $r \times s$ with entries in the ring A, and if t is an integer which is $\leq \min\{r, s\}$, then we let $I_t(M)$ denote the ideal of A which is generated by the $t \times t$ minors of M, i.e. generated by the determinants of all the $t \times t$ submatrices of M.

Notation-Definitions 10.1: Let $R = k[x_0, \ldots, x_n],\ S = k[y_0, \ldots, y_n]$ be our usual starting rings and choose $j \in \mathbb{N},\ j \geq 2,\ j = 2\ell$ or $j = 2\ell + 1$. Let $\mathcal{F} \in S_j$ be the *generic* form in S of degree j. Let $T^{(n)} = (1, t_1, \ldots, t_{j-1}, 1)$ be a symmetric sequence of positive integers for which $t_r \leq \dim_k R_r$ (we use the ${}^{(n)}$ in the notation to recall that we are dealing with quotients of $R = k[x_0, \ldots, x_n]$). Let $\mathcal{R}$ denote the polynomial ring in the coefficients of $\mathcal{F}$ i.e. a homogeneous coordinate ring for $\mathbb{P}(S_j)$.

We define

$$\mathcal{I}_{\leq T^{(n)}} = I_{t_1+1}(Cat_{\mathcal{F}}(1; j-1 : n+1)) \cap \ldots \cap I_{t_\ell+1}(Cat_{\mathcal{F}}(\ell; j-\ell : n+1)) \subseteq \mathcal{R}$$

and then define

$$\mathcal{G}or(\leq T^{(n)}) := \text{ the subscheme of } \mathbb{P}(S_j) \text{ defined by the ideal } \mathcal{I}_{\leq T^{(n)}}\ .$$

Clearly, for any given $T^{(n)}$ there are only a finite number of possible other sequences $T'^{(n)} = (1, t'_1, \ldots, t'_{j-1}, 1)$, of the type we are discussing, with $t'_i \leq t_i$ for all i and, at least for one integer j, $t'_j < t_j$. In such a case we shall say that $T'(n) < T^{(n)}$.

Notice that if $t' < t$ then we have $I_t(M) \subseteq I_{t'}(M)$. It follows from this that if $T'^{(n)} < T^{(n)}$ then $\mathcal{I}_{\leq T^{(n)}} \subseteq \mathcal{I}_{\leq T'^{(n)}}$.

We then define

$$\begin{gathered}\mathcal{G}or(T^{(n)}) := \text{ the complement, in } \mathcal{G}or(\leq T^{(n)}) \text{ of the union} \\ \text{of the schemes } \mathcal{G}or(\leq T'^{(n)}) \text{ for every } T'^{(n)} < T^{(n)}.\end{gathered}$$

Thus $\mathcal{G}or(T^{(n)})$ is an open subscheme of $\mathcal{G}or(\leq T^{(n)})$.

Having made these definitions we can now identify:

$$\mathbf{Gor}(\leq T^{(n)}) \text{ with } \mathcal{G}or(\leq T^{(n)})^{red} \text{ and } \mathbf{Gor}(T^{(n)}) \text{ with } \mathcal{G}or(T^{(n)})^{red} \ .$$

One might want to concentrate on only one part of the sequence $T^{(n)}$. Thus, it is reasonable to define the sets:

$$\mathbf{U}_{\leq t}(u; j-u : n+1) = \{F \in \mathbb{P}(S_j) \mid rk(Cat_{\mathcal{F}}(u; j-u : n+1)) \leq t\}$$

and

$$\mathbf{V}_t(u; j-u : n+1) = \{F \in \mathbb{P}(S_j) \mid rk(Cat_{\mathcal{F}}(u; j-u : n+1) = t\} \ .$$

(Note again the use of bold face to denote *sets*.)

Coupled with these definitions are:

$$\mathcal{U}_{\leq t}(u; j-u : n+1) := \text{subscheme of } \mathbb{P}(S_j) \text{ defined by } I_{t+1}(Cat_{\mathcal{F}}(u; j-u : n+1)) \subseteq \mathcal{R}$$

and

$$\begin{gathered}\mathcal{V}_t(u; j-u : n+1) := \text{ the open subscheme of } \mathcal{U}_{\leq t}(u; j-u : n+1) \\ \text{whose complement is } \mathcal{U}_{\leq (t-1)}(u; j-u : n+1) \ .\end{gathered}$$

Note that

$$\mathbf{U}_{\leq t}(..) = \mathcal{U}^{red}_{\leq t}(..) \text{ and } \mathbf{V}_t(..) = \mathcal{V}^{red}_t(..) \ .$$

This has been a very heavy dose of notation; let's now look at some very specific examples.

Example 10.2: We will consider the possibilites for artinian Gorenstein quotients of $R = k[x_0, \ldots, x_n]$ which have socle degree 2.

In other words, take $\mathcal{F}$ a generic form of degree 2 in $S = k[y_0, \ldots, y_n]$.

If we write $\mathcal{F} = \sum Z_{ij} y_i y_j$ then the only catalecticant that enters into the discussion is:

$$Cat_{\mathcal{F}}(1; 1 : n+1) = \begin{array}{c} \\ x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{ccccc} y_0 & y_1 & \cdots & \cdots & y_n \end{array} \\ \begin{pmatrix} Z_{00} & Z_{01} & \cdots & \cdots & Z_{0n} \\ Z_{01} & Z_{11} & \cdots & \cdots & Z_{1n} \\ & & \ddots & & \\ & & & \ddots & \\ Z_{0n} & Z_{1n} & \cdots & \cdots & Z_{nn} \end{pmatrix} \end{array} .$$

Notice that this is the generic symmetric matrix of size $(n+1) \times (n+1)$. So, if $F \in S_2$ is a specialization of $\mathcal{F}$ and $I = ann(F)$ then $H(R/I, -) = 1 \ \ ? \ \ 1$.

Now the (?) in the Hilbert function above is exactly the rank of the matrix obtained by specializing the coefficients of $\mathcal{F}$ to those of F. But, from the theory of quadratic forms,

$$rk(Cat_F(1; 1 : n+1)) = r \ (\leq n+1) \Leftrightarrow F = L_1^2 + \cdots + L_r^2$$

where $L_1, \ldots, L_r$ are linearly independent linear forms in S_1.

Thus we get the following simple fact:

Proposition 10.3: Let $F \in S_2$ (as above) and let $I = ann(F)$. Then

$$H(R/I, -) = 1 \ \ r \ \ 1 \ \ (r \leq n+1) \Leftrightarrow F = L_1^2 + \cdots + L_r^2 \Leftrightarrow F \in Sec_{r-1}(\nu_2(\mathbb{P}^n))$$

where $L_1, \ldots, L_r$ are linearly independent linear forms in S_1.

Notice that in this case, the only Hilbert functions we get are totally ordered by inequality. I.e.

$$\text{if } T_s^{(n)} = (1, s, 1) \text{ then } \ T_1^{(n)} < T_2^{(n)} < \cdots < T_n^{(n)} < T_{n+1}^{(n)} .$$

Thus, $\mathcal{G}or(\leq T_s^{(n)})$ is the subscheme of $\mathbb{P}(S_2)$ defined by $I_{s+1}(Cat_{\mathcal{F}}(1; 1 : n+1))$. I.e. these are the subschemes of $\mathbb{P}^N$ ($N = \binom{n+2}{2} - 1$) defined by the vanishing of the minors of a generic symmetric matrix of size $n+1$. From 10.3 we see that $\mathbf{Gor}(\leq T_r^{(n)}) = Sec_{r-1}(\nu_2(\mathbb{P}^n))$. But, it is not clear if $\mathcal{G}or(\leq T_r^{(n)})$ is a reduced scheme.

In his book (*The Geometry of Determinantal Loci*) T.G. Room gives both the dimension and the degrees of these varieties. In Room's notation (see 8.6.3 or 8.6.4, pg. 141 of his book) the variety we are considering (i.e. $\mathbf{Gor}(\leq T_r^{(n)})$) is:

$$(S, |n+1, n+1|_r, [\binom{n+2}{2}]) \text{ or } \mathbb{Y}^r$$

and he gives the formula for the dimension of this variety as

$$\dim(\mathbf{Gor}(\leq T_r^{(n)})) = \frac{r(2n+3) - r^2 - 2}{2} \ .$$

In particular, when $r = 1$ we get that $\dim(\mathbf{Gor}(\leq T_1^{(n)})) = n$. This is in agreement with our earlier observation that $\mathbf{Gor}(T_1^{(n)}) = \nu_2(\mathbb{P}^n)$. (Note that there is no Hilbert function smaller that $T_1^{(n)}$.)

Also, when $r = n$ we get that $\dim(\mathbf{Gor}(\leq T_n^{(n)})) = \binom{n+2}{2} - 2$, i.e. this variety is a hypersurface in $\mathbb{P}(S_2)$. This corresponds to the fact that the equation of the hypersurface is nothing more than $\det(Cat_{\mathcal{F}}(1; 1 : n+1))$.

However, when $r = 2$ we get:

$$\dim(\mathbf{Gor}(\leq T_2^{(n)})) = \dim(Sec_1(\nu_2(\mathbb{P}^n))) = 2n \ .$$

Thus, $Sec_1(\nu_2(\mathbb{P}^n))$ has deficiency 1 since the "expected" dimension of this variety is $2n+1$.

Also, when $r = 3$ we get

$$\dim(\mathbf{Gor}(\leq T_3^{(n)})) = \dim Sec_2(\nu_2(\mathbb{P}^n))) = 3n - 1 \ .$$

Since the "expected" dimension is $3n+2$, the deficiency here is 3.

In general the expected dimension of $\mathbf{Gor}(\leq T_r^{(n)}) = Sec_{r-1}(\nu_2(\mathbb{P}^n))$ is $rn + r - 1$ while the actual dimension was given above. So, the deficiency is easily calculated to be $r(r-1)/2$ for every $r \leq n$. (I will leave it to the interested reader to use this information to write down the value of the Hilbert function of R/J in degree 2 when

$$J = \wp_1^2 \cap \wp_2^2 \cap \ldots \cap \wp_r^2$$

for $r \leq n+1$.)

> **Aside:** I believe that all the varieties $\mathcal{G}or(\leq T_r)$ above are reduced and are arithmetically Cohen-Macaulay and also that their resolutions as algebras are know. I recall some early work of Gulliksen and later work of Jozefiak, Pragacz and Weyman on this problem – which I have been unable to verify yet. My recollection is that they had a generic resolution for the ideals of minors of symmetric matrices. One should be able to give a more algebraic proof of Room's assertions using this approach, and also an independent proof of the next result, which gives the degrees of these varieties. (See the update in Lecture 11.)

Room also states a result (pg.133) which gives a formula for the degrees of the varieties $\mathbf{Gor}(\leq T_s^{(n)})$, namely

$$\deg(\mathbf{Gor}(\leq T_r^{(n)})) = \frac{\binom{n+1}{n+1-r} \cdots\cdots \binom{2(n+1)-(r+2)}{2}\binom{2(n+1)-(r+1)}{1}}{\binom{2(n+1)-(2r+1)}{n+1-r} \cdots\cdots \binom{3}{2}\binom{1}{1}} .$$

I've been unable to find a nicer expression for this, but I did do some calculations which I will share in an appendix to this section.

Remark: I am uncomfortable about putting too much credence in Room's calculations since it is not at all clear if he is calculating the dimensions and degrees of the schemes defined by the ideals of minors or he is finding the degrees and dimensions of the associated reduced schemes. Since the schemes are irreducible the dimension count is fine, but the degree count remains a conjecture until we are sure all the schemes above are reduced!

I should mention that, with respect to the calculation of the degree, Room uses the expression "We assume the order of $\mathbb{Y}^s$ is ...." (and then he gives the formula above and cites both C. Segre and H.F. Baker). I don't quite understand the use of the word "assume", unless he was unable to give his own demonstration of the result and wanted to make that clear. This is another reason for my unease over using Room as a proper reference for this result. (Again, see the update in Lecture 11.)

I'd like to now move onto a discussion of (artinian) Gorenstein graded rings of socle degree 3.

We continue with our usual notation: $R = k[x_0, \ldots, x_n]$, $S = k[y_0, \ldots, y_n]$ and $F \in S_3$, $I = ann(F) \subseteq R$. As in the case of socle degree 2, we still do not have many possibilities for the Hilbert function of R/I. Those possibilities are:

$$H(R/I, -) = 1 \quad r \quad r \quad 1 \text{ where } r \leq n+1 \ .$$

Again, as in the case of socle degree 2, the sequences $T^{(n)}$ for which $\mathbf{Gor}(T^{(n)})$ is <u>*potentially*</u> non-empty are linearly ordered. If $T_r^{(n)} = (1, r, r, 1)$ then

$$T_1^{(n)} < T_2^{(n)} < \cdots < T_{n+1}^{(n)} \ .$$

Proposition 10.4:

$$\mathbf{Gor}(T_r^{(n)}) \neq \emptyset \text{ for } r = 1,\ 2, \ldots,\ n+1.$$

Proof: It suffices to find forms F_t, all of degree 3, for which $\dim_k R_1 F_t = t$ for $(1 \leq t \leq n+1)$.

But, this is easy, just consider $F_t = y_0^3 + \cdots + y_{t-1}^3$. $\square$

Remark 10.5: Clearly, if $F_t = y_0^j + \cdots + y_t^j$ then $\dim_k R_i F_t = t+1$ for $i = 1, \ldots, j-1$. It follows that if

$$T^{(n)} = (\underbrace{1, t+1, t+1, \ldots t+1, t+1, 1}_{j+1-\text{ tuple}}) \quad (t+1 \leq n+1)$$

then $\mathbf{Gor}(T^{(n)}) \neq \emptyset$.

Continuing with this remark, suppose that $F \in S_j$ is a form with the property that $I = ann(F)$ gives a Gorenstein artinian ring $A = R/I$ for which

$$H(R/I, -) = 1 \quad r \quad r \quad \cdots \quad r \quad r \quad 1 \text{ where } r < n+1 \ .$$

Then I_1 contains $n+1-r$ linearly independent linear forms which, after a change of variables, we can assume are $x_r, \ \ldots, x_n$. But, if $x_i \circ F = 0$, this implies that y_i does not appear in F. Thus, there is no loss of generality in assuming that F is a polynomial in $k[y_0, \ldots, y_{r-1}]$.

Returning to the case of socle degree 3, we see that if $F \in S_3$ and $I = ann(F) \subseteq R$ then $H(R/I, -)$ is determined by the ranks of the two matrices:

$$Cat_F(1; 2 : n+1) \text{ and } Cat_F(2; 1 : n+1)\ .$$

But, since these matrices are the transposes of each other, we need only consider the first.

The generic $\mathcal{F}$ in S_3 may be written:

$$\mathcal{F} = \sum Z_{ijk} y_i y_j y_k \text{ where } 0 \leq i \leq j \leq k \leq n$$

and so $Cat_{\mathcal{F}}(1; 2 : n+1)$ is an $(n+1) \times \binom{n+3}{3}$ matrix:

$$Cat_{\mathcal{F}}(1; 2 : n+1) =$$

$$\begin{array}{c|cccccccccc|}
 & y_0^2 & y_0y_1 & \cdots & y_0y_n & y_1^2 & y_1y_2 & \cdots & y_1y_n & \cdots & y_n^2 \\
\hline
x_0 & Z_{000} & Z_{001} & \cdots & Z_{00n} & Z_{011} & Z_{012} & \cdots & Z_{01n} & \cdots & Z_{0nn} \\
x_1 & Z_{001} & & & & & & & & & Z_{1nn} \\
\vdots & & & & & & & & & & \\
x_n & Z_{00n} & Z_{01n} & \cdots & Z_{0nn} & Z_{11n} & Z_{12n} & & Z_{1nn} & \cdots & Z_{nnn}
\end{array}$$

(recall we also called this matrix $\mathcal{Z}_{1,2}^{(n)}$).

We start by considering $T_1^{(n)} = (1, 1, 1, 1)$.

Now $\mathcal{G}or(\leq T_1^{(n)})$ is defined by the ideal $I_2(Cat_{\mathcal{F}}(1; 2 : n+1))$, so (to use the earlier notation) $\mathcal{G}or(\leq T_1^{(n)}) = \mathcal{U}_{\leq 2}(1; 2 : n+1)$. We saw, last time, that $\mathbf{Gor}(T_1^{(n)}) = \nu_3(\mathbb{P}^n)$, and so the first question that comes to mind is:

Problem 10.6: Is $\mathcal{G}or(\leq T_1^{(n)}) = \nu_3(\mathbb{P}^n)$ also?

Note that this is a problem for every n and for every j, i.e. not only for $T_1^{(n)} = (1, 1, 1, 1)$, but also for $T^{(n)} = \underbrace{(1, 1, \ldots, 1, 1)}_{j+1-\text{tuple}}$.

So, our question really amounts to asking if the ideal $I_2(Cat_{\mathcal{F}}(1; j-1 : n+1))$ is the defining (prime) ideal of $\nu_j(\mathbb{P}^n)$. I don't know the answer to this.

Also interesting would be a proof that

$$I_2(Cat_{\mathcal{F}}(1; j-1 : n=1)) = I_2(Cat_{\mathcal{F}}(u; v : n+1)) \text{ when } u+v=j \ .$$

Let's now move on to $\mathcal{G}or(\leq T_2^{(n)})$, $T_2^{(n)} = (1,2,2,1)$. This is the subscheme of $\mathbb{P}^N$ ($N = \binom{n+3}{3} - 1$) defined by $I_3(Cat_{\mathcal{F}}(1; 2 : n+1))$.

As we saw earlier, if L_1 and L_2 are linearly independent linear forms in S_1 and $F = L_1^3 + L_2^3$ then the forms in $I_3(\mathcal{Z}_{1,2}^{(n)})$ all vanish on F. Thus

$$\mathbf{Gor}(\leq T_2^{(n)}) \supseteq Sec_1(\nu_3(\mathbb{P}^n)) \ .$$

Problem 10.7

$$\text{Is } \mathbf{Gor}(\leq T_2^{(n)}) = \mathcal{G}or(\leq T_2^{(n)}) = Sec_1(\nu_3(\mathbb{P}^n)) \ ?$$

I made a calculation (on a small computer) with the computer programme **Macaulay** (for the case $n=2$) and found that the answer to Problem 10.7 is Yes, in that case. (My computer took a while to make the calculation of $Sec_1(\nu_3(\mathbb{P}^2))$ and that is what stopped me from checking the case $n=4$.)

I can give an answer to the "reduced" part of Problem 10.7, but for me to do that I'll need to take a small (but interesting) detour. First, though, the promised calculations.

Appendix

$\nu_2(\mathbb{P}^3) \subseteq \mathbb{P}^9$:

	dimension	**degree**
Sec_1	6	10
Sec_2	8	4

$\nu_2(\mathbb{P}^4) \subseteq \mathbb{P}^{14}$:

	dimension	**degree**
Sec_1	8	35
Sec_2	11	20
Sec_3	13	5

$\nu_2(\mathbb{P}^5) \subseteq \mathbb{P}^{20}$:

	dimension	**degree**
Sec_1	10	18
Sec_2	14	112
Sec_3	17	35
Sec_4	19	6

$\nu_2(\mathbb{P}^6) \subseteq \mathbb{P}^{27}$:

	dimension	**degree**
Sec_1	12	562
Sec_2	17	672
Sec_3	21	294
Sec_4	24	56
Sec_5	26	7

Lecture 11: Some final words – for now!

Updates: I distributed these notes to some friends who are not attending the seminar and I received some remarks from them about some of the things that I was questioning. I want to share those comments with all of the readers of these notes.

As we saw in Lecture 10, the study of the varieties $\mathbf{Gor}(\leq T)$ and $\mathcal{G}or(\leq T)$ for Gorenstein artinian quotients of $R = k[x_0, \ldots, x_n]$ having socle degree 2 is equivalent to the study of the scheme defined by the ideal of all fixed size minors of the generic $(n+1) \times (n+1)$ symmetric matrix.

I voiced (if one can do that in print!) some doubts about taking Room's calculations of degrees (for the varieties so defined) too seriously since it wasn't clear if Room was speaking of the scheme defined by these minors or of the reduced scheme with the same support.

A note from Tony Iarrobino (with a reference to the book of Arbarello, Cornalba, Griffiths and Harris - Exercises on page 100-101) makes clear that the ideal generated by the $t \times t$ minors of the generic symmetric matrix is prime and so all of Room's calculations are placed on a firm footing.

The paper of Jozefiak, Pragacz and Weyman that is relevant here, is in Asterique (87-88), 1983, 109-189. They give a resolution of this ideal of minors, which has (theoretically) all of Room's calculations as a consequence – plus more! – since one can calculate the Hilbert polynomial of this variety from any resolution and the coefficients contain information on the degree, dimension and other invariants.

Bruce Reznick (Urbana) was kind enough to point out some historical points (which are contained in his book - Sums of Even Powers of Real Linear Forms - AMS Memoir, No. 463, 1992). I quote from page 49 of that book:

> "Sylvester was an excellent prosodist, and a "catalectic" line of verse is one which is lacking part of the last foot.
>
> A form which is a sum of fewer m^{th} powers than is canonically required thereby exhibits catalecticism."

(From Iarrobino I learned that the word is derived from the Greek - *Katalektikos* - meaning *cut-off* or *incomplete.*)

Reznick, with a straight face, (if you can do that in print!) goes on to point out that

Sylvester was not completely happy with his choice of term. Sylvester is quoted as follows: “Meicatalecticizant would more completely express the meaning of that which, for the sake of brevity, I denominate the catalecticant.”

I also learned from Reznick (see pgs 59-60 of the book mentioned above) that Sylvester also discovered Examples 7.1 and 7.2 while discussing Clebsch’s example (see the beginning of our Lecture 6) in a paper. (Sylvester’s paper is: Sur une extension d’un theoreme de Clebsch relatif aux courbes du quatrieme degre, C.R.Acad.Sci.(102) 1886, 1532-34 – Paper 47 in the Collected Papers, vol. 4, Cambridge Univ. Press, 1912). Apparently Sylvester found these examples “paradoxal” and used them to make a warning about excessive “counting of constants”.

With all of this help I am still without an historical reference to Example 7.3 (cubics in 5 variables).

****

Recall that last time we were looking at the case of socle degree 3 Gorenstein artinian quotients of $R = k[x_0, \ldots, x_n]$. We had noted that such quotients all had Hilbert function

$$T_r^{(n)} = 1 \quad r \quad r \quad 1 \quad \text{where } r \leq n+1$$

and that $\mathbf{Gor}(T_r^{(n)}) \neq \emptyset$ for $r = 1, \ldots, n+1$.

For this socle degree, the only catalecticant matrix that comes into play is:

$$Cat_{\mathcal{F}}(1; 2 : n+1) =$$

$$\begin{array}{cc} & \begin{array}{cccccccccc} y_0^2 & y_0y_1 & \cdots & y_0y_n & y_1^2 & y_1y_2 & \cdots & y_1y_n & \cdots & y_n^2 \end{array} \\ \begin{array}{c} x_0 \\ x_1 \\ \vdots \\ x_n \end{array} & \left(\begin{array}{cccccccccc} Z_{000} & Z_{001} & \cdots & Z_{00n} & Z_{011} & Z_{012} & \cdots & Z_{01n} & \cdots & Z_{0nn} \\ Z_{001} & & & & & & & & & Z_{1nn} \\ & & & & & & & & & \\ Z_{00n} & Z_{01n} & \cdots & Z_{0nn} & Z_{11n} & Z_{12n} & & Z_{1nn} & \cdots & Z_{nnn} \end{array}\right) \end{array} .$$

So, $\mathcal{G}or(\leq T_r^{(n)})$ is defined by $I_{r+1}(Cat_{\mathcal{F}}(1; 2 : n+1))$. We had discussed this for $r = 1$ and were in the middle of a discussion of this problem for $r = 2$.

We had already seen that:

$$\mathbf{Gor}(\leq T_2^{(n)}) \supseteq Sec_1(\nu_3(\mathbb{P}^n))$$

(and in Problem 10.7 we had asked if these were equal to each other and, in turn, equal to $\mathcal{G}or(\leq T_2^{(n)})$.) We now turn to that "reduced" problem.

Trying to decide if a given variety is a secant variety to the Veronese would certainly be easier if we knew more about what the elements of $Sec_{s-1}(\nu_j(\mathbb{P}^n)) \subseteq \mathbb{P}(S_j)$ look like. We know that forms like $F = L_1^j + \cdots + L_s^j$ (the L_i linear forms in S_1) are in $Sec_{s-1}(\nu_j(\mathbb{P}^n))$ but it's not clear what the "limiting" positions of such F's look like.

There is, however, for the case of the chordal variety (of a smooth variety) a complete description of the elements of $Sec_1(X)$ (where X is smooth inside $\mathbb{P}^r$). It is the best one could hope for:

> $Sec_1(X)$ consists of all the points on all the secant lines of X **plus**
> all the points on all the tangent spaces to points of $X \subseteq \mathbb{P}^r$. (see e.g.
> Harris - Prop. 15.10, pg. 191).

So, continuing with our thinking of $\nu_j(\mathbb{P}^n) \subseteq \mathbb{P}(S_j)$, we should ask if there is a nice characterization of those $F \in \mathbb{P}(S_j)$ which correspond to points in the tangent space to $\nu_j(\mathbb{P}^n)$ at one of its points. The answer is *YES*.

Lemma 11.1: $F \in S_j$ is in the tangent space to $\nu_j(\mathbb{P}^n) \subseteq \mathbb{P}(S_j)$ at the point L_1^j if and only if there is a linear form $L_2 \in S_1$ such that $F = L_1^{j-1}L_2$.

Proof: If $L_1 = a_0y_0 + \cdots + a_ny_n$, let $P_1 = [a_0 : \ldots : a_n] \in \mathbb{P}^n$. The points in the tangent space to $\nu_j(\mathbb{P}^n)$ at $\nu_j(P_1)$ come by considering all tangent vectors to curves in $\nu_j(\mathbb{P}^n)$ which are smooth at $\nu_j(P_1)$.

Let $P_2 = [b_0 : \ldots : b_n]$ be any other point of $\mathbb{P}^n$ and let $\mathcal{L}$ be the line in $\mathbb{P}^n$ which joins P_1 to P_2. Then $\nu_j : \mathcal{L} \to \mathcal{C}$, where $\mathcal{C} \subseteq \nu_j(\mathbb{P}^n)$ is a rational normal curve of degree j which is in some $\mathbb{P}^j \subseteq \mathbb{P}(S_j)$.

We can parametrize the points in $\mathcal{L}$ by $P_1 + tP_2$ and then, if $L_2 = b_0y_0 + \cdots + b_ny_n$, we have

$$\nu_j(P_1 + tP_2) = (L_1 + tL_2)^j \ .$$

Thus,

$$\frac{d}{dt}(L_1 + tL_2)^j = j(L_1 + tL_2)^{j-1}L_2 \ .$$

To find the tangent vector at $\nu_j(P_1)$ we just need to evaluate this derivative when $t = 0$. In this way we get

$$\frac{d}{dt}(L_1 + tL_2)^j|_{t=0} = jL_1^{j-1}L_2 \ .$$

Thus, the point in the tangent space at L_1^j is:

$$L_1^j + jL_1^{j-1}L_2 = L_1^{j-1}(L_1 + jL_2) = L_1^{j-1}L_2'$$

for some linear form L_2'.

If we let P_2 vary over all directions from P_1 we get the entire tangent space to $v_j(\mathbb{P}^n)$ at $\nu_j(P_1) = L_1^j$. That completes the proof.

Note: If $F = L_1^{j-1}L_2 = M_1^{j-1}M_2$ (and $j > 1$) then $L_1 = M_1$ and $L_2 = M_2$. Thus, a point of $\mathbb{P}(S_j)$ can be on the tangent space to at most one point of $\nu_j(\mathbb{P}^n)$.

Now for the promised portion of a solution to Problem 10.7.

Proposition 11.2:

$$\mathbf{Gor}(\leq T_2^{(n)}) = Sec_1(\nu_3(\mathbb{P}^n)) \ .$$

Proof: If R/I has Hilbert function with $H(R/I, 1) = 2$, then, by Remark 10.5, we can assume that F is a form of degree 3 which only involves y_0 and y_1. Now we showed earlier that $Sec_1(\nu_3(\mathbb{P}^1)) = \mathbb{P}^3 = \mathbb{P}(k[y_0, y_1]_3)$ and so every form of degree 3 in $k[y_0, y_1]$ is in that secant variety, i.e. every form of degree 3 in $k[y_0, y_1]$ can be written either as $L_1^3 + L_2^3$, or as $L_1^2L_2$ with L_1 and L_2 linear forms in $k[y_0, y_1]$. In view of Proposition 11.1, that is enough to prove the result.

The next case to consider, in socle degree 3, is $T_3^{(n)} = (1, 3, 3, 1)$.

Suppose first that $n = 2$*:* In this case we have that

$$\mathbf{Gor}(T_1^{(2)}) \subseteq \mathbf{Gor}(T_2^{(2)}) \subseteq \mathbf{Gor}(T_3^{(2)}) \subseteq \mathbb{P}^9 = \mathbb{P}(S_3) \ .$$

We've already seen that for $T_2^{(2)} = (1, 2, 2, 1)$ we have

$$\mathbf{Gor}(\leq T_2^{(2)}) = Sec_1(\nu_3(\mathbb{P}^2)) \subseteq \mathbb{P}^9 \ .$$

This is a variety of dimension 5 in $\mathbb{P}^9$.

Clearly, $\mathbf{Gor}(\leq T_3^{(2)}) = \mathbb{P}^9$ since every form in 3 variables can have at most 3 linearly independent first derivatives!

Notice that $Sec_2(\nu_3(\mathbb{P}^2))$ is a hypersurface in $\mathbb{P}^9$ and so is strictly smaller than $\mathbb{P}^9 = \mathbf{Gor}(\leq T_3^{(2)})$. Thus, not all these varieties $\mathbf{Gor}(\leq T)$ are secant varieties of appropriate Veronese varieites.

(By the way, I have no idea where the equation of that hypersurface really comes from. One can compute it, with either CoCoA or Macaulay, (but I have been unable to get my little computer to do the work required!) and get its degree, but that won't really explain where it comes from! There are lots of examples like this one, where a secant variety of $\nu_j(\mathbb{P}^n)$ is a hypersurface in its enveloping space and j is odd! In which case there is no obvious candidate for a determinant to explain the equation of the hypersurface. I think that finding these equations, **and where they really come from**, is a very interesting problem!)

> **Note added:** On June 12, 1995, using a larger computer than my laptop, a group of mathematicians in Genova computed the equation for this hypersurface. It is an equation of degree 4 (which was also not clear!).

The equation is:

$$\begin{gathered}
x_4^4 - 2x_3x_4^2x_5 + x_3^2x_5^2 + x_2x_4x_5x_6 - x_1x_5^2x_6 - 2x_2x_4^2x_7 - x_2x_3x_5x_7 + 3x_1x_4x_5x_7 \\
+x_2^2x_7^2 + x_0x_5x_7^2 + 3x_2x_3x_4x_8 - 2x_1x_4^2x_8 - x_1x_3x_5x_8 + x_2^2x_6x_8 + x_0x_5x_6x_8 \\
-x_1x_2x_7x_8 + x_0x_4x_7x_8 + x_1^2x_8^2 + x_0x_3x_8^2 - x_2x_3^2x_9 + x_1x_3x_4x_9 + x_1x_2x_6x_9 \\
-x_0x_4x_6x_9 - x_1^2x_7x_9 + x_0x_3x_7x_9 \ .
\end{gathered}$$

Now suppose that $n \geq 3$:

This case points out a very general situation which occurs not only in socle degree 3 but in any socle degree j when $H(R/I, 1) = r < n + 1$ (the number of variables). I would thus like to deal with this very general situation at this time.

Recall that we saw, in Lecture 10 (Remark 10.5), that if $I = ann(F)$, $\deg F = j$ and $H(R/I, 1) = r < n + 1$ then we could find $L_0, \ldots, L_{r-1}$, linearly independent linear forms in $S = k[y_0, \ldots, y_n]$ such that $F \in k[L_0, \ldots, L_{r-1}]$. What I didn't mention then was that these linear forms are "essentially" unique!

To be more precise about that, let me state a simple linear algebra fact that is at the heart of the matter.

Lemma 11.3: Let V be a vector space of dimension n and let W be a subspace of dimension $m < n$.

Suppose that $\mathcal{B} = \{e_1, \ldots, e_m\}$ is a fixed basis for W and that

$$\mathcal{E} = \{e_1, \ldots, e_m, v_1, \ldots, v_{n-m}\} \text{ and } \mathcal{E}' = \{e_1, \ldots, e_m, v'_1, \ldots, v'_{n-m}\}$$

are two bases for V which extend $\mathcal{B}$.

If $\mathcal{E}^* = \{f_1, \ldots, f_m, g_1, \ldots, g_{n-m}\}$ and $\mathcal{E}'^* = \{f'_1, \ldots, f'_m, g'_1, \ldots, g'_{n-m}\}$ are dual bases to $\mathcal{E}$ and $\mathcal{E}'$ in V^*, then

$$< g_1, \ldots, g_{n-m} > = < g'_1, \ldots, g'_{n-m} > \quad .$$

Proof: One need only observe that both spaces are exactly $W^\perp$.

Corollary 11.4: Let $R = k[x_0, \ldots, x_n]$, $S = k[y_0, \ldots, y_n]$ where $F \in S_j$ and $I = ann(F)$. Suppose that $H(R/I, 1) = r < n + 1$.

If $F \in k[L_0, \ldots, L_{r-1}] \cap k[L'_0, \ldots, L'_{r-1}]$ where the L_i and L'_i are (individually) linearly independent sets of linear forms in S_1, then

$$< L_0, \ldots, L_{r-1} > = < L'_0, \ldots, L'_{r-1} > \quad .$$

I.e. the polynomial ring in r variables to which F belongs is uniquely determined by F.

Proof: In view of Lemma 11.3 we need only observe that $F \in k[W]$ where $W = I_1^\perp$ and that is enough to prove the corollary.

Now let $\mathcal{F} \in k[y_0, \ldots, y_n]_j$ be a generic form of degree j and consider

$$Cat_{\mathcal{F}}(1; j-1 : n+1) \text{ an } (n+1) \times \binom{j+n}{n} - \text{matrix}$$

and

$$\mathbf{U}_{\le r}(1;j-1:n+1)=\{F\in\mathbb{P}(S_j)\mid\ \mathrm{rank}_k Cat_{\mathcal{F}}(1;j-1:n+1)\le r\}$$

(which is defined by $\sqrt{I_{r+1}(Cat_{\mathcal{F}}(1;j-1:n+1))}$.)

If we suppose that $r\le n+1$ then what we have just shown is that

$$\mathbf{U}_{\le r}(1;j-1:n+1)$$

$$=\{F\in\mathbb{P}(S_j)\mid F\in k[L_0,\dots,L_{r-1}],\ L_0,\dots,L_{r-1}\text{ linearly independent linear forms }\}$$

Proposition 11.5: $\mathbf{U}_{\le r}(1;j-1:n+1)$ is an irreducible projective variety of dimension

$$r(n+1-r)+\binom{j+r-1}{r-1}-1\ .$$

Proof: Let $\mathcal{G}$ be the Grassmanian of $\mathbb{P}^{r-1}$'s in $\mathbb{P}^n$ (so $\mathcal{G}$ is a projective variety of dimension $r(n+1-r)$). Then, the points of $\mathcal{G}$ parametrize the r dimensional subspaces of S_1 and hence each point $P\in\mathcal{G}$, ($P\leftrightarrow V$, V an r-dimensional subspace of S_1) describes a polynomial subring $k[L_0,\dots,L_{r-1}]\subset S$ where $<L_0,\dots,L_{r-1}>=V$. Then $\mathbb{P}(k[L_0,\dots,L_{r-1}]_j)$ is a projective space of dimension $N=\binom{j+r-1}{r-1}-1$ which parametrizes the forms of degree j (up to scalar multiples) in $k[L_0,\dots,L_{r-1}]$.

This gives us a regular function (in fact a surjection),

$$\phi:\mathcal{G}\times\mathbb{P}^N\longrightarrow\mathbf{U}_{\le r}(1;j-1:n+1)=\mathbf{U}_{\le r}$$

where

$$\phi:<L_0,\dots,L_{r-1}>\times F(Z_0,\dots,Z_{r-1})\longrightarrow F(L_0,\dots,L_{r-1})\ .$$

Now, if $F\in\mathbb{P}(S_j)$ is such that $\mathrm{rank}_k Cat_{\mathcal{F}}(1;j-1:n+1)=r$ (exactly) then we saw that F determines $<L_0,\dots,L_{r-1}>$. Since $\mathrm{rk}_k Cat_{\mathcal{F}}(1;j-1:n+1)=r$ on a non-empty open subset O in $\mathbf{U}_{\le r}$ we obtain that the fibres of ϕ over O consist of exactly one point. Thus,

$$\text{the dimension of }\mathbf{U}_{\le r}=\text{ the dimension of }\mathcal{G}\times\mathbb{P}^N=r(n+1-r)+\binom{j+r-1}{r-1}-1$$

as we wanted.

Since both $\mathcal{G}$ and $\mathbb{P}^N$ are irreducible, so is $\mathbf{U}_{\leq r}$.

This Proposition makes for some very obvious questions.

Problem 11.6:

1) Is $I_{r+1}(Cat_{\mathcal{F}}(1; j-1 : n+1))$ a prime ideal for $r < n+1$?
2) Is $\sqrt{I_{r+1}(Cat_{\mathcal{F}}(1; j-1 : n+1))} = \wp$ a perfect ideal? i.e. is $R/\wp$ an arithmetically Cohen-Macaulay variety.
3) Is $\mathbf{U}_{\leq r}(1; j-1 : n+1)$ a well-known variety? (we saw, when $r = 2$ and $j = 3$ that it was $Sec_1(\nu_3(\mathbb{P}^n))$.
4) What are some numerical invariants of $\mathbf{U}_{\leq r}$ (e.g. degree, Hilbert function, Hilbert polynomial, graded Betti numbers, etc...) (We see, from above, that it is a rational variety.)

If we return now to the case of socle degree 3 we see that everything is determined by the one matrix $Cat_{\mathcal{F}}(1; 2 : n+1)$ where $\mathcal{F}$ is a generic element of S_3. So, we have

$$\begin{array}{ccccc} \mathbf{Gor}(\leq T_1^{(n)}) & \subseteq & \mathbf{Gor}(\leq T_2^{(n)}) & \subseteq \cdots \subseteq & \mathbf{Gor}(\leq T_n^{(n)}) \\ \| & & \| & & \| \\ \nu_3(\mathbb{P}^n) & & Sec_1(\nu_3(\mathbb{P}^n)) & & \mathbb{P}(S_3) \end{array}$$

where all of the varieties are irreducible and rational and

$$\text{the dimension of } \mathbf{Gor}(\leq T_r^{(n)}) = r(n+1-r) + \binom{r+2}{r-1} - 1 \ .$$

Moreover, when $r > 3$, $Sec_{r-1}(\nu_3(\mathbb{P}^n)) \subsetneq \mathbf{Gor}(\leq T_r^{(n)})$. E.G. when $r = 3$ we know that $Sec_2(\nu_3(\mathbb{P}^n))$ has dimension $3n+2$ but $\mathbf{Gor}(\leq T_3^{(n)})$ has dimension $3n+3$.

Remark: Iarrobino has shown that the singular locus of $\mathbf{Gor}(\leq T_r^{(n)})$ is exactly $\mathbf{Gor}(\leq T_{r-1}^{(n)})$ for the case of $j = 3$. This might easily be true for any j, if we just look at the vanishing locus for the ideals of minors of the first catalecticants, as above. Iarrobino also remarks that, in general, the answer to 11.6 1), is no. He and Kanev have examples. We must then ask when the answer to 1) is yes.

The case of socle degree 4

We continue with our usual notation:

$$R = k[x_0, \ldots, x_n],\ S = k[y_0, \ldots, y_n],\ F \in S_4,\ R \supseteq I = ann(F),\ A = R/I\ .$$

Then

$$H(R/I, -) := 1 \quad a \quad b \quad a \quad 1 \quad 0 \quad \cdots$$

where

$$\begin{array}{l} 1 \leq a \leq n+1 \\ a \leq b \leq \binom{n+2}{2} \end{array} .$$

First Observations:

1) Note that this time we have <u>*two*</u> catalecticant matrices to consider. If $\mathcal{F}$ is a generic form of degree 4 "in" S_4 we write

$$\mathcal{C}_1^{(n)} = Cat_{\mathcal{F}}(1; 3 : n+1) \text{ (an } (n+1) \times \binom{n+3}{3} \text{ matrix)}$$

and

$$\mathcal{C}_2^{(n)} = Cat_{\mathcal{F}}(2; 2 : n+1) \text{(an } \binom{n+2}{2} \times \binom{n+2}{2} \text{ symmetric matrix) .}$$

2) If $a = n+1$ then the matrix $\mathcal{C}_1^{(n)}$ never enters into the discussion! and everything rests on the square symmetric matrix $\mathcal{C}_2^{(n)}$.

This is the second time we have come across j even and a symmetric matrix! Classically, it was this "central" matrix (and its determinant) which occupied people's attention. Some people even refer to the determinant of this central matrix as the *catalectic invariant of* F.

3) of course, if $a < n+1$ then our earlier discussion comes into play and $\mathrm{rk}\mathcal{C}_1^{(n)} \leq a$ takes place on the irreducible variety $\mathbf{U}_{\leq a}(1; 3 : n+1)$ which we discussed earlier.

Hence, if we let

$$T_{a,b}^{(n)} = (1, a, b, a, 1)$$

then if $a < n+1$ we have

$$\mathbf{Gor}(\leq T_{a,b}^{(n)}) = \mathbf{U}_{\leq b}(2; 2 : n+1) \cap \mathbf{U}_{\leq a}(1; 3 : n+1)$$

while if $a = n+1$ we have

$$\mathbf{Gor}(\leq T^{(n)}_{a,b}) = \mathbf{U}_{\leq b}(2; 2 : n+1) \ .$$

It is interesting to look at the special case of $n = 1$, i.e. $R = k[x_0, x_1]$, $S = k[y_0, y_1]$, $F \in S_4$, $\mathbb{P}(S_4) \simeq \mathbb{P}^4$, and

$$F = Z_1 x_0^4 + Z_2 x_0^3 x x_1 + Z_3 x_0^2 x_1^2 + Z_4 x_0 x_1^3 + Z_5 x_1^4 \ .$$

In this case there are only a few possibilities for $T^{(1)}_{a,b}$, namely:

$$\begin{aligned} (1,1,1,1,1) &= T^{(1)}_{1,1} \\ (1,2,2,2,1) &= T^{(1)}_{2,2} \\ (1,2,3,2,1) &= T^{(1)}_{2,3} \end{aligned}$$

(Exercise: Show that $(1,2,1,2,1)$ is not possible.)

The matrices in question are:

$$\mathcal{C}^{(1)}_1 = \begin{pmatrix} Z_1 & Z_2 & Z_3 & Z_4 \\ Z_2 & Z_3 & Z_4 & Z_5 \end{pmatrix}$$

and

$$\mathcal{C}^{(1)}_2 = \begin{pmatrix} Z_1 & Z_2 & Z_3 \\ Z_2 & Z_3 & Z_4 \\ Z_3 & Z_4 & Z_5 \end{pmatrix} \ .$$

It is well-known (see the new book of Harris mentioned earlier and the paper of J. Watanabe in this volume) that:

a) the ideal $I_2(\mathcal{C}^{(1)}_1)$ is the ideal of the rational normal curve in $\mathbb{P}^4$, i.e. of $\nu_4(\mathbb{P}^1)$), and

b) the ideal $I_2(\mathcal{C}^{(1)}_2)$ is the prime ideal which defines the rational normal curve in $\mathbb{P}^4$; and the ideal $I_3(\mathcal{C}^{(1)}_2) = det(\mathcal{C}^{(1)}_2)$ is the equation of the hypersurface $Sec_1(\nu_4(\mathbb{P}^1)) \subseteq \mathbb{P}^4$.

We have

$$\begin{array}{ccccc} \mathbf{Gor}(\leq T^{(1)}_{1,1}) & \subseteq & \mathbf{Gor}(\leq T^{(1)}_{2,2}) & \subseteq & \mathbf{Gor}(\leq T^{(1)}_{3,3}) \\ \| & & \| & & \| \\ \nu_r(\mathbb{P}^1) & \subseteq & Sec_1(\nu_4(\mathbb{P}^1)) & \subseteq & \mathbb{P}^4 \end{array}$$

The situation changes dramatically when $n \geq 2$.

n=2:

Again there are only a few possibilities for $T^{(2)}_{a,b}$, where $1 \leq a \leq 3$ and $1 \leq b \leq 6$.

We only have the following:

$$\begin{array}{ll}
(1,1,1,1,1) & (1,2,2,2,1) \\
(1,2,3,2,1) & (1,3,3,3,1) \\
(1,3,4,3,1) & (1,3,5,3,1) \\
(1,3,6,3,1) &
\end{array}$$

(while the following are impossible (Exc.) $(1,2,1,2,1)$, $(1,3,1,3,1)$, $(1,3,2,3,1)$.)

Already the situation is much more delicate. Even finding the possible T's has now become a more subtle task (Although for $n=2$ and any j, this problem was solved by R. Stanley.)

There were conjectures which sought to describe the possible $T^{(n)}_{a,b}$ for $n \geq 3$ but these have all been disposed of by examples of Stanley, Bernstein-Iarrobino, and Boij-Laksov. E.g. Stanley has found an example of a Gorenstein artinian algebra which gives $T = (1,13,12,13,1)$, but no example (where the initial part decreases) can exist for $j=3$ and $a \leq 8$ (I was informed of the existence of this latter result by an e-mail of Iarrobino who attributes it to Peskine. I don't know where the proof has appeared, or if it has appeared.) Needless to say, the absence of even a good conjecture for the possible T's which can describe the Hilbert function of a Gorenstein artin algebra points out a part of the subtlety of the problem for $n \geq 3$.

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Unfortunately, I have no more time this term to talk about the many more interesting things that are known. There are, e.g., many interesting results in the paper of Iarrobino and Kanev (that I have continually refereed to) but I think that it is fair to say that our understanding of the structure of these varieties is just beginning. I hope to continue these discussions next year ...