CHAPTER II

Schemes

This chapter and the next form the technical heart of this book. In this chapter we develop the basic theory of schemes, following Grothendieck [EGA]. Sections 1 to 5 are fundamental. They contain a review of sheaf theory (necessary even to define a scheme), then the basic definitions of schemes, morphisms, and coherent sheaves. This is the language that we use for the rest of the book.

Then in Sections 6, 7, 8, we treat some topics which could have been done in the language of varieties, but which are already more convenient to discuss using schemes. For example, the notion of Cartier divisor, and of an invertible sheaf, which belong to the new language, greatly clarify the discussion of Weil divisors and linear systems, which belong to the old language. Then in §8, the systematic use of nonclosed scheme points gives much more flexibility in the discussion of sheaves of differentials and nonsingular varieties, improving the treatment of (I, §5).

In §9 we give the definition of a formal scheme, which did not have an analogue in the theory of varieties. It was invented by Grothendieck as a good way of dealing with Zariski's theory of "holomorphic functions," which Zariski regarded as an analogue in abstract algebraic geometry of the holomorphic functions in a neighborhood of a subvariety in the classical case.

1 Sheaves

The concept of a sheaf provides a systematic way of keeping track of local algebraic data on a topological space. For example, the regular functions on open subsets of a variety, introduced in Chapter I, form a sheaf, as we will see shortly. Sheaves are essential in the study of schemes. In fact, we cannot even define a scheme without using sheaves. So we begin this chapter with sheaves. For additional information, see the book of Godement [1].

Definition. Let X be a topological space. A *presheaf* $\mathscr{F}$ of abelian groups on X consists of the data

(a) for every open subset $U \subseteq X$, an abelian group $\mathscr{F}(U)$, and
(b) for every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV}:\mathscr{F}(U) \to \mathscr{F}(V)$,

subject to the conditions

(0) $\mathscr{F}(\varnothing) = 0$, where $\varnothing$ is the empty set,
(1) ρ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$, and
(2) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

The reader who likes the language of categories may rephrase this definition as follows. For any topological space X, we define a category $\mathfrak{Top}(X)$, whose objects are the open subsets of X, and where the only morphisms are the inclusion maps. Thus $\operatorname{Hom}(V,U)$ is empty if $V \nsubseteq U$, and $\operatorname{Hom}(V,U)$ has just one element if $V \subseteq U$. Now a presheaf is just a contravariant functor from the category $\mathfrak{Top}(X)$ to the category $\mathfrak{Ab}$ of abelian groups.

We define a presheaf of rings, a presheaf of sets, or a presheaf with values in any fixed category $\mathfrak{C}$, by replacing the words "abelian group" in the definition by "ring", "set", or "object of $\mathfrak{C}$" respectively. We will stick to the case of abelian groups in this section, and let the reader make the necessary modifications for the case of rings, sets, etc.

As a matter of terminology, if $\mathscr{F}$ is a presheaf on X, we refer to $\mathscr{F}(U)$ as the *sections* of the presheaf $\mathscr{F}$ over the open set U, and we sometimes use the notation $\Gamma(U,\mathscr{F})$ to denote the group $\mathscr{F}(U)$. We call the maps ρ_{UV} *restriction* maps, and we sometimes write $s|_V$ instead of $\rho_{UV}(s)$, if $s \in \mathscr{F}(U)$.

A sheaf is roughly speaking a presheaf whose sections are determined by local data. To be precise, we give the following definition.

Definition. A presheaf $\mathscr{F}$ on a topological space X is a *sheaf* if it satisfies the following supplementary conditions:

(3) if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$;
(4) if U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each i, with the property that for each i,j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each i. (Note condition (3) implies that s is unique.)

Note. According to our definition, a sheaf is a presheaf satisfying certain extra conditions. This is equivalent to the definition found in some other books, of a sheaf as a topological space over X with certain properties (Ex. 1.13).

Example 1.0.1. Let X be a variety over the field k. For each open set $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions from U to k, and for each $V \subseteq U$, let $\rho_{UV}:\mathcal{O}(U) \to \mathcal{O}(V)$ be the restriction map (in the usual sense). Then $\mathcal{O}$ is a sheaf of rings on X. It is clear that it is a presheaf of rings. To verify the conditions (3) and (4), we note that a function which is 0 locally is 0, and a function which is regular locally is regular, because of the definition of regular function (I, §3). We call $\mathcal{O}$ the *sheaf of regular functions* on X.

Example 1.0.2. In the same way, one can define the sheaf of continuous real-valued functions on any topological space, or the sheaf of differentiable functions on a differentiable manifold, or the sheaf of holomorphic functions on a complex manifold.

Example 1.0.3. Let X be a topological space, and A an abelian group. We define the *constant sheaf* $\mathscr{A}$ on X determined by A as follows. Give A the discrete topology, and for any open set $U \subseteq X$, let $\mathscr{A}(U)$ be the group of all continuous maps of U into A. Then with the usual restriction maps, we obtain a sheaf $\mathscr{A}$. Note that for every connected open set U, $\mathscr{A}(U) \cong A$, whence the name "constant sheaf." If U is an open set whose connected components are open (which is always true on a locally connected topological space), then $\mathscr{A}(U)$ is a direct product of copies of A, one for each connected component of U.

Definition. If $\mathscr{F}$ is a presheaf on X, and if P is a point of X, we define the *stalk* $\mathscr{F}_P$ of $\mathscr{F}$ at P to be the direct limit of the groups $\mathscr{F}(U)$ for all open sets U containing P, via the restriction maps ρ.

Thus an element of $\mathscr{F}_P$ is represented by a pair $\langle U,s\rangle$, where U is an open neighborhood of P, and s is an element of $\mathscr{F}(U)$. Two such pairs $\langle U,s\rangle$ and $\langle V,t\rangle$ define the same element of $\mathscr{F}_P$ if and only if there is an open neighborhood W of P with $W \subseteq U \cap V$, such that $s|_W = t|_W$. Thus we may speak of elements of the stalk $\mathscr{F}_P$ as *germs* of sections of $\mathscr{F}$ at the point P. In the case of a variety X and its sheaf of regular functions $\mathcal{O}$, the stalk $\mathcal{O}_P$ at a point P is just the local ring of P on X, which was defined in (I, §3).

Definition. If $\mathscr{F}$ and $\mathscr{G}$ are presheaves on X, a *morphism* $\varphi:\mathscr{F} \to \mathscr{G}$ consists of a morphism of abelian groups $\varphi(U): \mathscr{F}(U) \to \mathscr{G}(U)$ for each open set U, such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \big\downarrow{\rho_{UV}} & & \big\downarrow{\rho'_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

is commutative, where ρ and ρ' are the restriction maps in $\mathscr{F}$ and $\mathscr{G}$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on X, we use the same definition for a morphism of sheaves. An *isomorphism* is a morphism which has a two-sided inverse.

Note that a morphism $\varphi:\mathscr{F} \to \mathscr{G}$ of presheaves on X induces a morphism $\varphi_P:\mathscr{F}_P \to \mathscr{G}_P$ on the stalks, for any point $P \in X$. The following proposition (which would be false for presheaves) illustrates the local nature of a sheaf.

Proposition 1.1. *Let $\varphi:\mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_P:\mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism for every $P \in X$.*

PROOF. If φ is an isomorphism it is clear that each φ_P is an isomorphism. Conversely, assume φ_P is an isomorphism for all $P \in X$. To show that φ is an isomorphism, it will be sufficient to show that $\varphi(U):\mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism for all U, because then we can define an inverse morphism ψ by $\psi(U) = \varphi(U)^{-1}$ for each U. First we show $\varphi(U)$ is injective. Let $s \in \mathscr{F}(U)$, and suppose $\varphi(s) \in \mathscr{G}(U)$ is 0. Then for every point $P \in U$, the image $\varphi(s)_P$ of $\varphi(s)$ in the stalk $\mathscr{G}_P$ is 0. Since φ_P is injective for each P, we deduce that $s_P = 0$ in $\mathscr{F}_P$ for each $P \in U$. To say that $s_P = 0$ means that s and 0 have the same image in $\mathscr{F}_P$, which means that there is an open neighborhood W_P of P, with $W_P \subseteq U$, such that $s|_{W_P} = 0$. Now U is covered by the neighborhoods W_P of all its points, so by the sheaf property (3), s is 0 on U. Thus $\varphi(U)$ is injective.

Next, we show that $\varphi(U)$ is surjective. Suppose we have a section $t \in \mathscr{G}(U)$. For each $P \in U$, let $t_P \in \mathscr{G}_P$ be its germ at P. Since φ_P is surjective, we can find $s_P \in \mathscr{F}_P$ such that $\varphi_P(s_P) = t_P$. Let s_P be represented by a section $s(P)$ on a neighborhood V_P of P. Then $\varphi(s(P))$ and $t|_{V_P}$ are two elements of $\mathscr{G}(V_P)$, whose germs at P are the same. Hence, replacing V_P by a smaller neighborhood of P if necessary, we may assume that $\varphi(s(P)) = t|_{V_P}$ in $\mathscr{G}(V_P)$. Now U is covered by the open sets V_P, and on each V_P we have a section $s(P) \in \mathscr{F}(V_P)$. If P,Q are two points, then $s(P)|_{V_P \cap V_Q}$ and $s(Q)|_{V_P \cap V_Q}$ are two sections of $\mathscr{F}(V_P \cap V_Q)$, which are both sent by φ to $t|_{V_P \cap V_Q}$. Hence by the injectivity of φ proved above, they are equal. Then by the sheaf property (4), there is a section $s \in \mathscr{F}(U)$ such that $s|_{V_P} = s(P)$ for each P. Finally, we have to check that $\varphi(s) = t$. Indeed, $\varphi(s)$, t are two sections of $\mathscr{G}(U)$, and for each P, $\varphi(s)|_{V_P} = t|_{V_P}$, hence by the sheaf property (3) applied to $\varphi(s) - t$, we conclude that $\varphi(s) = t$.

Our next task is to define kernels, cokernels and images of morphisms of sheaves.

Definition. Let $\varphi:\mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. We define the *presheaf kernel* of φ, *presheaf cokernel* of φ, and *presheaf image* of φ to be the presheaves given by $U \mapsto \ker(\varphi(U))$, $U \mapsto \operatorname{coker}(\varphi(U))$, and $U \mapsto \operatorname{im}(\varphi(U))$ respectively.

Note that if $\varphi:\mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then the presheaf kernel of φ is a sheaf, but the presheaf cokernel and presheaf image of φ are in general not sheaves. This leads us to the notion of a sheaf associated to a presheaf.

Proposition-Definition 1.2. *Given a presheaf $\mathscr{F}$, there is a sheaf $\mathscr{F}^+$ and a morphism $\theta:\mathscr{F} \to \mathscr{F}^+$, with the property that for any sheaf $\mathscr{G}$, and any morphism $\varphi:\mathscr{F} \to \mathscr{G}$, there is a unique morphism $\psi:\mathscr{F}^+ \to \mathscr{G}$ such that $\varphi = \psi \circ \theta$. Furthermore the pair $(\mathscr{F}^+,\theta)$ is unique up to unique isomorphism. $\mathscr{F}^+$ is called the* sheaf associated *to the presheaf $\mathscr{F}$.*

PROOF. We construct the sheaf $\mathscr{F}^+$ as follows. For any open set U, let $\mathscr{F}^+(U)$ be the set of functions s from U to the union $\bigcup_{P \in U} \mathscr{F}_P$ of the stalks of $\mathscr{F}$ over points of U, such that

(1) for each $P \in U$, $s(P) \in \mathscr{F}_P$, and
(2) for each $P \in U$, there is a neighborhood V of P, contained in U, and an element $t \in \mathscr{F}(V)$, such that for all $Q \in V$, the germ t_Q of t at Q is equal to $s(Q)$.

Now one can verify immediately (!) that $\mathscr{F}^+$ with the natural restriction maps is a sheaf, that there is a natural morphism $\theta:\mathscr{F} \to \mathscr{F}^+$, and that it has the universal property described. The uniqueness of $\mathscr{F}^+$ is a formal consequence of the universal property. Note that for any point P, $\mathscr{F}_P = \mathscr{F}^+_P$. Note also that if $\mathscr{F}$ itself was a sheaf, then $\mathscr{F}^+$ is isomorphic to $\mathscr{F}$ via θ.

Definition. A *subsheaf* of a sheaf $\mathscr{F}$ is a sheaf $\mathscr{F}'$ such that for every open set $U \subseteq X$, $\mathscr{F}'(U)$ is a subgroup of $\mathscr{F}(U)$, and the restriction maps of the sheaf $\mathscr{F}'$ are induced by those of $\mathscr{F}$. It follows that for any point P, the stalk $\mathscr{F}'_P$ is a subgroup of $\mathscr{F}_P$.

If $\varphi:\mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the *kernel* of φ, denoted $\ker \varphi$, to be the presheaf kernel of φ (which is a sheaf). Thus $\ker \varphi$ is a subsheaf of $\mathscr{F}$.

We say that a morphism of sheaves $\varphi:\mathscr{F} \to \mathscr{G}$ is *injective* if $\ker \varphi = 0$. Thus φ is injective if and only if the induced map $\varphi(U):\mathscr{F}(U) \to \mathscr{G}(U)$ is injective for every open set of X.

If $\varphi:\mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the *image* of φ, denoted $\operatorname{im} \varphi$, to be the sheaf associated to the presheaf image of φ. By the universal property of the sheaf associated to a presheaf, there is a natural map $\operatorname{im} \varphi \to \mathscr{G}$. In fact this map is injective (see Ex. 1.4), and thus $\operatorname{im} \varphi$ can be identified with a subsheaf of $\mathscr{G}$.

We say that a morphism $\varphi:\mathscr{F} \to \mathscr{G}$ of sheaves is *surjective* if $\operatorname{im} \varphi = \mathscr{G}$.

We say that a sequence $\ldots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is *exact* if at each stage $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$. Thus a sequence

$0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ is exact if and only if φ is injective, and $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \to 0$ is exact if and only if φ is surjective.

Now let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. We define the *quotient sheaf* $\mathscr{F}/\mathscr{F}'$ to be the sheaf associated to the presheaf $U \to \mathscr{F}(U)/\mathscr{F}'(U)$. It follows that for any point P, the stalk $(\mathscr{F}/\mathscr{F}')_P$ is the quotient $\mathscr{F}_P/\mathscr{F}'_P$.

If $\varphi:\mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the *cokernel* of φ, denoted coker φ, to be the sheaf associated to the presheaf cokernel of φ.

Caution 1.2.1. We saw that a morphism $\varphi:\mathscr{F} \to \mathscr{G}$ of sheaves is injective if and only if the map on sections $\varphi(U):\mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. The corresponding statement for surjective morphisms is not true: if $\varphi:\mathscr{F} \to \mathscr{G}$ is surjective, the maps $\varphi(U):\mathscr{F}(U) \to \mathscr{G}(U)$ on sections need not be surjective. However, we can say that φ is surjective if and only if the maps $\varphi_P:\mathscr{F}_P \to \mathscr{G}_P$ on stalks are surjective for each P. More generally, a sequence of sheaves and morphisms is exact if and only if it is exact on stalks (Ex. 1.2). This again illustrates the local nature of sheaves.

So far we have talked only about sheaves on a single topological space. Now we define some operations on sheaves, associated with a continuous map from one topological space to another.

Definition. Let $f:X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathscr{F}$ on X, we define the *direct image* sheaf $f_*\mathscr{F}$ on Y by $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ for any open set $V \subseteq Y$. For any sheaf $\mathscr{G}$ on Y, we define the *inverse image* sheaf $f^{-1}\mathscr{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathscr{G}(V)$, where U is any open set in X, and the limit is taken over all open sets V of Y containing $f(U)$. Do not confuse $f^{-1}\mathscr{G}$ with the sheaf $f^*\mathscr{G}$ which will be defined later for a morphism of ringed spaces (§5).

Note that f_* is a functor from the category $\mathfrak{Ab}(X)$ of sheaves on X to the category $\mathfrak{Ab}(Y)$ of sheaves on Y. Similarly, f^{-1} is a functor from $\mathfrak{Ab}(Y)$ to $\mathfrak{Ab}(X)$.

Definition. If Z is a subset of X, regarded as a topological subspace with the induced topology, if $i:Z \to X$ is the inclusion map, and if $\mathscr{F}$ is a sheaf on X, then we call $i^{-1}\mathscr{F}$ the *restriction* of $\mathscr{F}$ to Z, and we often denote it by $\mathscr{F}|_Z$. Note that the stalk of $\mathscr{F}|_Z$ at any point $P \in Z$ is just $\mathscr{F}_P$.

Exercises

1.1. Let A be an abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.