Equivariant de Rham cohomology and gauged field theories

August 15, 2013

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1 Introduction

1.1 Differential forms and de Rham cohomology

Let X be a smooth manifold of dimension n. We recall that

$$\Omega^k(X) := C^\infty(X, \Lambda^k T^*X)$$

is the vector space of *differential k-forms on X*. What is the available structure on differential forms?

1. There is the *wedge product*

$$\Omega^k(M) \otimes \Omega^\ell(M) \xrightarrow{\wedge} \Omega^{k+\ell}(X)$$

induced by a vector bundle homomorphism

$$\Lambda^k T^*X \otimes \Lambda^\ell T^*X \longrightarrow \Lambda^{k+\ell} T^*X$$

This gives $\Omega^*(X) := \bigoplus_{k=0}^n \Omega^k(X)$ the structure of a $\mathbb{Z}$-graded algebra. This algebra is *graded commutative* in the sense that

$$\omega \wedge \eta = (-1)^{|\omega||\eta|} \eta \wedge \omega \tag{1.1}$$

for homogeneous elements $\omega, \eta \in \Omega^*(X)$ of degree $|\omega| \in \mathbb{Z}$ resp. $|\eta| \in \mathbb{Z}$ (every element $\omega \in \Omega^*(X)$ can be written in the form $\omega = \sum \omega^i$ with $\omega^i \in \Omega^i(X)$; ω is called *homogeneous of degree k* if $\omega^i = 0$ for $i \neq k$.

2. There is the *de Rham differential*

$$d \colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

It has the following properties:

(a) $d \colon \Omega^0(X) = C^\infty(X) \to \Omega^1(X) = C^\infty(X, T^*X)$ sends a function f to its differential df;

(b) d is a *graded derivation*, i.e., for $\omega, \eta \in \Omega^*(X)$ we have

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge \eta$$

(c) $d^2 = 0$

Conversely, d is determined by these three properties.

Definition 1.2. Let X be a manifold. Then the *k-th de Rham cohomology group* $H^k_{dR}(X)$ is defined by

$$H^k_{dR}(X) := \frac{\ker\big(d\colon \Omega^k(X) \to \Omega^{k+1}(X)\big)}{\operatorname{im}\left(d\colon \Omega^{k-1}(X) \to \Omega^k(X)\right)} = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}$$

These cohomology groups are a measure for the topological complexity of the manifold X. Very roughly, the dimension of $H^k_{dR}(X)$ can be thought of the the "number of k-dimensional holes" in X. According to the De Rham Theorem the de Rham cohomology group $H^k_{dR}(X)$ is isomorphic to the singular cohomology group $H^k(X;\mathbb{R})$ with coefficients in $\mathbb{R}$. Under this isomorphism the cup-product on singular cohomology corresponds to the multiplication on de Rham cohomology induced by the wedge product of differential forms.

Question. Is there a more geometric way to understand $\Omega^*(X)$ and its structure?

Answer. Yes, we can interpret $\Omega^*(X)$ as *smooth functions* on a supermanifold associated to X as will be explained in the next section.

1.2 A commercial for supermanifolds

As the name suggests, a *supermanifold* is a more general object than a smooth manifold. For the purposes of this introduction, we don't need to know what exactly a supermanifold is (see Definition 4.3 for a precise definition). Rather it will suffice that associated to any supermanifold M there is an associated *algebra of functions* $C^\infty(M)$ which is a *commutative super algebra*, i.e., $\mathbb{Z}/2$-graded algebra $A = A^{ev} \oplus A^{odd}$ which is graded commutative in the sense of equation (1.1). We note that a commutative algebra A can be thought of as a commutative superalgebra which is *even* in the sense that $A^{odd} = 0$. In the same way an ordinary smooth manifold M can be thought of as a supermanifold whose ring of functions $C^\infty(M)$ is even.

We note that if M is a smooth manifold then the commutative algebra of smooth functions $C^\infty(M)$ captures everything we want to know about M in the sense that the map

$$\mathsf{Man}(M,N) \longrightarrow \mathsf{Alg}(C^\infty(N), C^\infty(M)) \qquad f \mapsto f^*$$

is a bijection [GoSa, Thm. 2.3]. Here $\mathsf{Man}(M,N)$ is the set of smooth maps from M to N and $\mathsf{Alg}(C^\infty(N), C^\infty(M))$ is the set of algebra homomorphisms from $C^\infty(N)$ to $C^\infty(M)$. In particular, two smooth manifolds M, N are diffeomorphic if and only if the commutative algebras $C^\infty(M)$ and $C^\infty(N)$ are isomorphic.

The above observation suggests a definition of the morphisms of the category SM of supermanifolds: If M, N are supermanifolds, we define the set $\mathsf{SM}(M,N)$ of morphisms from M to N by

$$\mathsf{SM}(M,N) := \mathsf{SAlg}(C^\infty(N), C^\infty(M))$$

Here SAlg is the category of super algebras whose morphisms are grading preserving algebra homomorphisms.

Theorem 1.3. *Let X be a manifold. Then*

1. *there is a supermanifold of maps $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ with $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)) \cong \Omega^*(X)$;*
2. *There is a super Lie group $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ of diffeomorphisms of $\mathbb{R}^{0|1}$ which acts on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ by precomposition.*
3. *The super Lie algebra of $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ has a basis consisting of one even generator N and one odd generator Q; the non-trivial graded commutators are $[N,Q] = Q$.*
4. *The induced action of this Lie algebra on $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)) = \Omega^*(X)$ is given by letting Q act by $\omega \mapsto d\omega$ and N by $\omega \mapsto k\omega$ for $\omega \in \Omega^k(X)$.*

Explanations.

1. Given supermanifolds M, N, we have the *set* $\mathsf{SM}(M,N)$ of morphisms from M to N in the category of supermanifolds and the group $\mathrm{Diff}(M)$ of isomorphisms $M \to M$ which acts on $\mathsf{SM}(M,N)$ by precomposition. There is a more sophisticated construction which gives a *supermanifold* of maps $\underline{\mathsf{SM}}(M,N)$ and a super Lie group $\underline{\mathrm{Diff}}(M)$ (see below) which acts on $\underline{\mathsf{SM}}(M,N)$ (for general M, these are strictly speaking not supermanifolds, but more general objects, namely presheaves on SM as we will discuss. However, for $M = \mathbb{R}^{0|1}$ these presheaves are representable and hence we in fact obtain supermanifolds). A simpler case is the category SVect of super vector spaces (= $\mathbb{Z}/2$-graded vector spaces): if V, W are super vector spaces, we can consider the set $\mathsf{SVect}(V,W)$ of morphisms from V to W which are the grading preserving linear maps $V \to W$. This is quite different from the super vector space $\underline{\mathsf{SVect}}(V,W) = \underline{\mathsf{SVect}}^{ev}(V,W) \oplus \underline{\mathsf{SVect}}^{odd}(V,W)$ of *all* linear maps. Here $\underline{\mathsf{SVect}}^{ev}(V,W)$ (resp. $\underline{\mathsf{SVect}}^{odd}(V,W)$) consists of maps which preserve (resp. reverse) parity.
2. We recall that a Lie group G is a smooth manifold together with smooth maps

$$m\colon G \times G \longrightarrow G \text{ (multiplication)} \qquad \text{and} \qquad u\colon \mathrm{pt} \to G \text{ (unit)}$$

making the usual diagrams commutative which express associativity and the unit property (here pt is the 0-dimensional manifold consisting of one point; in particular, the map u amounts just to a point in G). It is customary to require an additional smooth map $i\colon G \to G$ that sends a group element to its inverse, and to require the usual relationships between i, m and u, again expressed by the commutativity of certain diagrams. However, the map i is uniquely determined by m and u, and its existence is guaranteed by requiring that the map

$$G \times G \xrightarrow{m\times p_2} G \times G$$

is an isomorphism (here $m \times p_2$ is the map whose first component is the multiplication map $m\colon G\times G \to G$, and whose second component is the projection map $p_2\colon G\times G \to G$ onto the second factor).

A *super Lie group* is defined exactly the same way as a Lie group, just replacing manifolds by supermanifolds, and morphisms in the category Man of manifolds by morphisms in the category SM of supermanifolds. An action of a super Lie group G on a super manifold M is a morphism $G \times M \to M$ of super manifolds which makes the usual associativity diagram commutative.

3. Associated to any super Lie group G is a super Lie algebra $\mathfrak{g}$ consisting of all left-invariant vector fields on G. As for ordinary manifolds, vector fields on a supermanifold M can be defined as (graded) derivations of the superalgebra of functions $C^\infty(M)$. The graded commutator of derivations gives $\mathfrak{g}$ the structure of a super Lie algebra (which is a super vector space with a graded skew-symmetric bracket which satisfies a graded version of the Jacobi identity, see **??**).

   If G acts on a supermanifold M, this action can be thought of as Lie group homomorphism from $G \to \underline{\mathrm{Diff}}(M)$. Differentiating, we obtain the corresponding Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Der}(C^\infty(M))$ (we recall that the Lie algebra of the diffeomorphism group of a (super) manifold M is the Lie algebra of vectorfields on M also known as $\mathrm{Der}(C^\infty(M))$ = derivations of $C^\infty(M)$).

1.3 Topological field theories

Definition 1.4. (Preliminary!) A d-dimensional topological field theory is a functor

$$E\colon d\text{-}\mathsf{Bord} \longrightarrow \mathsf{TV}$$

Here TV is the category of (locally convex, complete) topological vector spaces and d-Bord is the d-dimensional bordism category (named after its morphisms, rather than its objects):

- objects of d-Bord are closed $(d-1)$-manifolds Y;

- d-$\mathsf{Bord}(Y_0, Y_1) := \{d\text{-bordisms } \Sigma \text{ from } Y_0 \text{ to } Y_1\}/\text{diffeomorphism}$.

In addition, it is required that E sends the disjoint union of closed $(d-1)$-manifolds (resp. d-bordisms) to the tensor product of the corresponding vectorspaces (resp. operators). In technical terms, d-Bord (resp. TV) is a symmetric monoidal category (see 3.6), where the monoidal structure on d-Bord (resp. TV) is given by disjoint union (resp. tensor product). The functor E is required to be a symmetric monoidal functor, which is a compatibility condition for E and the two monoidal structures.

Example 1.5. The choice of a non-zero complex number λ determines a field theory $E_\lambda \in$ d-TFT for any even d, which is called the *Euler characteristic theory* since it is constructed using the Euler characteristic $\chi(\Sigma)$ of the bordisms Σ:

$$E(Y) := \mathbb{C} \qquad E(\Sigma) := \lambda^{\chi(\Sigma)} \in \mathbb{C} = \mathrm{Hom}(E(Y_0), E(Y_1))$$

To check that E_λ is in fact a functor, let Σ_1 be a bordism from Y_0 to Y_1 and let Σ_2 be a bordism from Y_1 to Y_2. Then the composition $\Sigma_2 \circ \Sigma_1$ of these morphisms in d-Bord is given by $\Sigma_2 \cup_{Y_1} \Sigma_1$, and

$$E(\Sigma_2 \circ \Sigma_1) = E(\Sigma_2 \cup_{Y_1} \Sigma_1) = \lambda^{\chi(\Sigma_2 \cup_{Y_1} \Sigma_1)} = \lambda^{\chi(\Sigma_2)+\chi(\Sigma_1)} = \lambda^{\chi(\Sigma_2)}\lambda^{\chi(\Sigma_1)} = E(\Sigma_2) \circ E(\Sigma_1)$$

Here we've used the assumption that d is even, since in general

$$\chi(\Sigma_2 \cup_{Y_1} \Sigma_1) = \chi(\Sigma_2) + \chi(\Sigma_1) - \chi(Y_1),$$

but the Euler characteristic of the closed manifold Y_1 vanishes its dimension is odd. Next we check that E is a monoidal functor. This is clear on the level of objects, since $E(Y_1 \amalg Y_2) = \mathbb{C} \cong \mathbb{C} \otimes \mathbb{C} = E(Y_1) \otimes E(Y_2)$. If Σ is a bordism from Y_0 to Y_1 and Σ' is a bordism from Y_0' to Y_1', then

$$E(\Sigma \amalg \Sigma') = \lambda^{\chi(\Sigma \amalg \Sigma')} = \lambda^{\chi(\Sigma)+\chi(\Sigma')} = \lambda^{\chi(\Sigma)}\lambda^{\chi(\Sigma')} = E(\Sigma) \cdot E(\Sigma')$$

There are a number of variants of the definition of field theory that will be important to us.

Definition 1.6. (Field theories over a manifold.) A *d-dimensional field theory over a manifold X* is a functor

$$E\colon d\text{-}\mathsf{Bord}(X) \longrightarrow \mathsf{TV}$$

where the objects of d-$\mathsf{Bord}(X)$ are pairs $(Y, f\colon Y \to X)$ where Y is as above and f is a smooth map. A morphism from (Y_0, f_0) to (Y_1, f_1) consists of isomorphism classes of pairs $(\Sigma, f\colon \Sigma \to X))$ with Σ as above and f a smooth map which restricts to f_0 resp. f_1 on $\partial\Sigma = Y_1 \amalg Y_0$.

Example 1.7. A function $f\colon X \to \mathbb{C}$ determines a 0-dimensional topological field theory $E_f\colon 0\text{-}\mathsf{Bord}(X) \to \mathsf{TV}$ as follows. The only object in $0\text{-}\mathsf{Bord}(X)$ is the empty set (the only manifold of dimension -1), and we set $E_f(\emptyset) := \mathbb{C}$. Each point $x \in X$ can be interpreted as a map $x\colon \text{pt} \to X$ and we set

$$E_f(x) := f(x) \in \mathbb{C} = \mathsf{TV}(\mathbb{C},\mathbb{C}) = \mathsf{TV}(E_f(\emptyset), E_f(\emptyset)).$$

In fact, every morphism of $\emptyset$ is a isomorphic to a finite disjoint union $x_1 \amalg \cdots \amalg x_k$ of such morphisms. Their image under E_f is then determined by the monoidal property of E to be

$$E_f(x_1 \amalg \cdots \amalg x_k) = \prod_{i=1}^{k} E_f(x_i) = \prod_{i=1}^{k} f(x_i)$$

Conversely, if $E \in 0\text{-TFT}(X)$, then E determines a function $f\colon X \to \mathbb{C}$ by $x \mapsto E(x\colon \text{pt} \to X)$.

Caveat I. With our preliminary Definition 1.4 the above construction gives a bijection between (isomorphism classes of) $0\text{-TFT}(X)$ and the set of functions $f\colon X \to \mathbb{C}$ *without any smoothness or continuity assumptions on* f. This reveals a weakness of the definition above, since we would like to obtain a bijection with the set $C^\infty(X)$ of *smooth* functions on X.

To fix this, we need to build in "smoothness" for the functors $E\colon d\text{-TFT}(X) \to \mathsf{TV}$. This is done by replacing domain- and range categories by their "family versions" whose objects are smooth families of the original objects. For example, a morphism in the family version of d-Bord is a submersion $\Sigma \to S$ of smooth manifolds whose fibers Σ_s for $s \in S$ are d-dimensional bordisms. We think of $\{\Sigma_s\}_{s\in S}$ as a smooth family of bordisms parametrized by the manifold S. Thus enhanced, isomorphism classes of 0-TFT's over X correspond indeed to smooth functions on X.

Example 1.8. A vector bundle $V \to X$ with connection ∇ determines a field theory $E_{V,\nabla}$ over X. To describe it, we note that a point $x \in X$ can be interpreted as an object of $1\text{-}\mathsf{Bord}(X)$, namely as map $x\colon \text{pt} \to X$ from the one-point manifold pt to X which sends pt to x. Similarly, a path $\gamma\colon [a,b] \to X$ can be thought of as a morphism in $1\text{-}\mathsf{Bord}(X)$ from $\gamma(a)$ to $\gamma(b)$. We define

$$E_{V,\nabla}(x) := V_x \qquad \text{and} \qquad E_{V,\nabla}(\gamma) := ||(\gamma)\colon V_{\gamma(a)} \to V_{\gamma(b)}$$

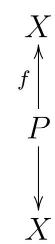
for any point $x \in X$ and path $\gamma\colon [a,b] \to X$. Here V_x is the fiber of the vector bundle over x, and $||(\gamma)\colon V_{\gamma(a)} \to V_{\gamma(b)}$ is the parallel translation along the path gamma determined by the connection ∇. It can be shown that this construction gives in fact an equivalence between the category of vector bundles with connections over X and the category $1\text{-TFT}(X)$ (the objects of this category are functors, the morphisms are invertible natural transformations).

Caveat II. This example shows that we've been careless in our definition of the bordism category $d\text{-}\mathsf{Bord}(X)$: if a point $x \in X$ gives an object in $1\text{-}\mathsf{Bord}(X)$ as claimed above, and a smooth path γ from x to y can be interpreted as a morphism from x to y, then how do we compose γ with a path δ from y to z? The problem is that the concatenation $\delta * \gamma$ of γ and δ won't be smooth in general and hence cannot be interpreted as a morphism in $1\text{-}\mathsf{Bord}(X)$. The solution is to provide a more careful definition of the objects of $d\text{-}\mathsf{Bord}(X)$: the closed $(d-1)$-manifolds Y will come equipped with a (germ of a) d-dimensional bicollar and an extension of the map $f\colon Y \to X$ to that bicollar. If (Σ, F) is a morphism with domain or range (Y, f), the restriction of this pair to a neighborhood of Y is required to agree with the one given by the object.

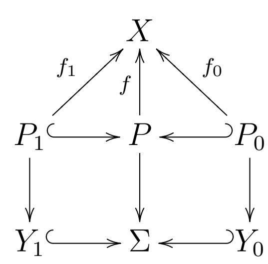
Definition 1.9. (Gauged field theories.) If X is a manifold equipped with a smooth action of a Lie group G, then a *G-gauged field theory over X* is a functor

$$E\colon d\text{-}\mathsf{Bord}_G(X) \longrightarrow \mathsf{TV}$$

where objects of $d\text{-}\mathsf{Bord}_G(X)$ are quadruples (Y, P, f, ∇^P), where Y as before is a closed $(d-1)$-manifold, $P \to Y$ is a principal G-bundle, $f\colon P \to X$ is a G-equivariant map, and ∇^P is a connection on P. The first three data we typically write as a diagram



A morphism from $(Y_0, P_0, \nabla_0, f_0)$ to $(Y_1, P_1, \nabla_1, f_1)$ consists of the additional data displayed in the following diagram:



Here Σ is a bordism from Y_0 to Y_1, $P \to \Sigma$ is a principal G-bundle whose restriction to Y_i is equal to $P_i \to Y_i$, $f\colon P \to X$ is a G-equivariant map, and the unlabeled maps in the above diagram are the bundle projection maps and the obvious inclusion maps.

Example 1.10. Example (1.8) can be generalized to the equivariant setting. If $V \to X$ is an equivariant vector bundle equipped with a G-equivariant connection ∇, a variant of the above construction gives a G-gauged 1-dimensional topological field theory $E_{V,\nabla}$ over X. Again, it can be shown that this construction gives an equivalence between the category of equivariant vector bundles with connections over X and the category $1\text{-TFT}_G(X)$ of G-gauged topological field theories of dimension 1 over X.

Definition 1.11. (Supersymmetric field theories.) A *(supersymmetric) topological field theory of dimension* $d|\delta$ is a symmetric monoidal functor

$$E\colon d|\delta\text{-}\mathsf{Bord} \longrightarrow \mathsf{TV}$$

Here the domain category is defined completely analogous to d-Bord: the objects are closed supermanifolds Y of dimension $d-1|\delta$, the morphisms are superbordisms of dimension $d|\delta$. We note that the category $d|0$-Bord agrees with the previously defined category d-Bord. Abusing notation, the target category here also has been slightly enlarged: from now on TV stands for the symmetric monoidal category of topological super vector spaces (see Definition 2.1). Also, the parameter spaces for the family versions of these categories (see Caveat I) are now allowed to be supermanifolds.

We will write $d|\delta$-TFT for the category of topological field theories of dimension $d|\delta$. If X is a manifold, we define as in Definition 1.6 the category $d|\delta\text{-TFT}(X)$ of topological field theories of dimension $d|\delta$ over X. If X is equipped with a smooth action of a Lie group G, we define the category $d|\delta\text{-TFT}_G(X)$ of G-gauged topological field theories of dimension $d|\delta$ over X as in Definition 1.9.

Let Σ be a closed d-manifold, and $E \in d\text{-TFT}(X)$. Given a smooth map $f\colon \Sigma \to X$, we can view (Σ, f) as a morphism in d-$\mathsf{Bord}(X)$ (as endomorphism of the object $\emptyset \in d\text{-}\mathsf{Bord}(X)$). We will call the function

$$Z_{\Sigma,E}\colon \operatorname{map}(\Sigma, X) \longrightarrow \mathbb{C} \qquad f \mapsto E(\Sigma, f) \in \mathsf{TV}(E(\emptyset), E(\emptyset)) \cong \mathbb{C}$$

the Σ*-partition function of* E. We note that if $g\colon \Sigma \to \Sigma$ is a diffeomorphism and $f\colon \Sigma \to X$ is a smooth map, then the pairs (Σ, f) and $(\Sigma, f \circ g)$ represent the *same* morphism in d-$\mathsf{Bord}(X)$, and hence $E(\Sigma, f) = E(\Sigma, f \circ g)$. In other words, $Z_{\Sigma,E}$ belongs to the fixed point set $\operatorname{map}(\Sigma, X)^{\mathrm{Diff}(\Sigma)}$ for the obvious $\mathrm{Diff}(\Sigma)$-action on $\operatorname{map}(\Sigma, X)$. With the proper definition of field theories as "smooth" functors (as explained in Caveat I), the function $Z_{\Sigma,E}$ will be *smooth*. It is not clear what we mean by a 'smooth' function here, since $\operatorname{map}(\Sigma, X)$ is not a finite dimensional smooth manifold. However, we will define a gadget denoted $\underline{\mathsf{Man}}(\Sigma, X)$ (to be thought of as the 'smooth manifold' of smooth maps from Σ to X, see section **??**) for which it makes sense to talk about the algebra $C^\infty(\underline{\mathsf{Man}}(\Sigma, X))$ of smooth functions on $\underline{\mathsf{Man}}(\Sigma, X)$. So in particular we obtain a functor

$$Z_\Sigma\colon d\text{-TFT} \longrightarrow \underline{\mathsf{Man}}(\Sigma, X)^{\underline{\mathrm{Diff}(\Sigma)}} \qquad E \mapsto Z_{\Sigma,E}$$

Here we view the set $\underline{\mathsf{Man}}(\Sigma, X)^{\underline{\mathrm{Diff}}(\Sigma)}$ as a discrete category (i.e., a category whose only morphisms are the identity morphisms). Analogously, if Σ is a closed supermanifold of dimension $d|\delta$, we have a functor

$$Z_\Sigma\colon d|\delta\text{-TFT} \longrightarrow \underline{\mathsf{SM}}(\Sigma, X)^{\underline{\mathrm{Diff}}(\Sigma)}$$

If $d = 0$, then $\emptyset$ is the only object of d-$\mathsf{Bord}(X)$ resp. $d|\delta$-$\mathsf{Bord}(X)$ and it is an easy consequence of the definition of field theories that the functors Z_{pt} resp. $Z_{\mathbb{R}^{0|1}}$ are equivalence of categories. It follows that we have equivalences of categories

$$0\text{-TFT}(X) \cong C^\infty(\underline{\mathsf{Man}}(\mathrm{pt}, X)) = C^\infty(X)$$
$$0\text{-TFT}(X) \cong C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X))^{\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})}$$

By Theorem 1.3 the algebra $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X))$ can be identified with the algebra $\Omega^*(X)$ of differential forms on X. It turns out that a form $\omega \in \Omega^k(X)$ is invariant under the $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$-action if and only if it is annihilated by the action of the Lie algebra $\mathrm{Lie}(\underline{\mathrm{Diff}}(\mathbb{R}^{0|1}))$. This is the case if and only if $Q\omega = 0$ and $N\omega = 0$ since Q and N form a basis of $\mathrm{Lie}(\underline{\mathrm{Diff}}(\mathbb{R}^{0|1}))$. By part (4) of Theorem 1.3, the condition $Q\omega = 0$ is equivalent to ω being a *closed form*, and $N\omega = 0$ is equivalent to $\omega \in \Omega^0(X)$.

More generally, for $\delta = 1$ and $d = 0, 1, 2$ we can define categories $d|\delta\text{-TFT}^n(X)$ of topological field theories over a manifold X of dimension $d|\delta$ and *degree* $n \in \mathbb{Z}$ in such a way that degree 0 field theories correspond to the field theories considered so far. Generalizing what we did for degree $n = 0$ above, we can construct the Σ-partition function

$$Z_{\Sigma,E}\colon \underline{\mathsf{SM}}(\Sigma, X) \to \mathbb{C}$$

for a field theory E of arbitrary degree $n \in \mathbb{Z}$. This function, instead of being *invariant* under the $\underline{\mathrm{Diff}}(\Sigma)$-action, is *equivariant*, where $g \in \underline{\mathrm{Diff}}(\Sigma)$ acts on $\mathbb{C}$ via multiplication by $\rho(g)^n$, where $\rho\colon \underline{\mathrm{Diff}}(\Sigma) \to \mathbb{C}^\times$ is a certain group homomorphism. Then Theorem 1.3 has the following consequence.

Corollary 1.12. *The category* $0|1\text{-TFT}^n(X)$ *of* $0|1$*-dimensional topological field theories of degree* n *over* X *is equivalent to the discrete category* $\Omega^n_{cl}(X;\mathbb{C})$ *of closed complex* n*-forms on* X*.*

We think of field theories over a manifold X as *geometric* objects, which for small $d|\delta$ amount to very classical objects, namely smooth functions on X (for $d|\delta = 0|0$, see Example 1.7), vector bundles with connections (for $d|\delta = 1|0$, see Example 1.8) , and closed differential forms (for $d|\delta = 0|1$, by the above corollary). These objects are *contravariant* objects in the sense that a smooth map $f\colon X \to Y$ induces a functor

$$f^*\colon d|\delta\text{-TFT}(Y) \longrightarrow d|\delta\text{-TFT}(X)$$

There is a systematic way to forget geometric information, but to retain topological information contained in these objects by passing to equivalence classes for the following equivalence relation.

Definition 1.13. Two topological field theories $E_\pm \in d|\delta\text{-TFT}(X)$ are *concordant* if and only if there is a field theory $E \in d|\delta\text{-TFT}(X \times \mathbb{R})$ such that E restricted to $X \times (-\infty, -1)$ is isomorphic to $p_2^* E_-$, and E restricted to $X \times (1, \infty)$ is isomorphic to $p_2^* E_+$.

We remark that the notion of concordance is a very general one, not restricted to field theories, but applicable to 'contravariant objects over manifolds', for example, functions, closed differential forms or vector bundles with connections. We note that any two functions are concordant, that two vector bundles with connections are concordant if and only if these vector bundles are isomorphic (this isomorphism *doesn't* need to be compatible with the connections), and two closed forms are concordant if and only if they represent the same de Rham cohomology class. These examples hopefully illustrate well what is meant by the slogan that passing to concordance classes *forgets the geometry, but retains the topology.*

As a consequence of Corollary 1.12 we obtain the following result.

Corollary 1.14. *For a manifold X there is a bijection*

$$0|1\text{-TFT}^n(X)/\textit{concordance} \longleftrightarrow H^n_{dR}(X) \otimes \mathbb{C}$$

One of the goals of this course is to provide a proof of this result as well as its equivariant analog:

Theorem 1.15. *For a manifold X equipped with an action of a compact Lie group G, there is a bijection*

$$0|1\text{-TFT}^n_G(X)/\textit{concordance} \longleftrightarrow H^n_{G,dR}(X) \otimes \mathbb{C}$$

where $0|1\text{-TFT}^n_G(X)$ is the category of G-gauged topological field theories over X of dimension $0|1$ and degree n the equivariant de Rham cohomology of X.

2 Super vector spaces, super algebras and super Lie algebras

In an algebraic context, the adjective "super" usually just refers to a $\mathbb{Z}/2$-grading, for example as in the following definition.

Definition 2.1. A *super vector space* (over the field $k = \mathbb{R}$ or $\mathbb{C}$) is a vector space V equipped with a $\mathbb{Z}/2$-grading, that is, a decomposition $V = V^{ev} \oplus V^{odd}$. The elements of $V^{ev} \subset V$ are called *even* elements of V, the elements of $V^{odd} \subset V$ are *odd* elements; elements which are either even or odd are the *homogeneous elements* of V. Equivalently, a $\mathbb{Z}/2$-grading is an involution $\epsilon \in \operatorname{End}(V)$. This determines $V^{ev/odd}$ by setting

$$V^{ev} := (+1)\text{-eigenspace} \qquad \text{and} \qquad V^{odd} := (-1)\text{-eigenspace}$$

Super vector spaces are the objects in a category SVect. For $V, W \in \mathsf{SVect}$,

$$\begin{aligned} \mathsf{SVect}(V,W) &= \{f\colon V \to W \mid f \text{ is linear and grading preserving}\} \\ &= \{f\colon V \to W \mid f \text{ is linear and } f \circ \epsilon_V = \epsilon_W \circ f\} \end{aligned}$$

Our next goal is to define superalgebras also known as $\mathbb{Z}/2$-graded algebras. It is straightforward to define this, but we would like to think of the passage from algebras to superalgebras as obtained by replacing "vector spaces" by "super vector spaces".

2.1 Super algebras

We recall that a unital algebra over a field k is a vector space A together with a multiplication map $m\colon A \otimes A \longrightarrow A$ and a unit $u \in A$ which satisfy the usual conditions: m should be associative, and u should be a left- and right-unit for m. We note that an element of A can be interpreted as a linear map from k to A, and thinking of the unit u as a map, the unital algebra structure on the vector space A is given by the linear maps

$$m\colon A \otimes A \longrightarrow A \text{ (multiplication)} \qquad u\colon k \longrightarrow A \text{ (unit)}$$

making the following diagrams commutative:

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{m\otimes 1} & A\otimes A \\ {\scriptstyle 1\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A\otimes A & \xrightarrow{m} & A \end{array} \qquad \begin{array}{ccccc} k\otimes A & \xrightarrow{u\otimes 1} & A\otimes A & \xleftarrow{1\otimes u} & A\otimes k \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle\cong} & \\ & & A & & \end{array} \tag{2.2}$$

We note that the commutativity of the first diagram expresses associativity of the multiplication m, while the commutativity of the second expresses that u is a left- and right-unit for m.

This definition of algebra generalizes directly as follows.

Definition 2.3. A *super algebra* is a super vector space A together with morphisms

$$m\colon A \otimes A \to A \qquad \text{and} \qquad u\colon k \to A$$

making the diagrams (2.2) commutative. Here the field k is regarded as a super vector space by declaring it to be purely even (i.e., the odd part of this super vector space is trivial), and by defining the *tensor product of super vector spaces* $V \otimes W$ to be the usual tensor product of the underlying vector spaces, equipped with the $\mathbb{Z}/2$-grading given by the grading involution

$$\epsilon := \epsilon_V \otimes \epsilon_W \colon V \otimes W \longrightarrow V \otimes W$$

We note that this implies that the tenor product $v \otimes w \in V \otimes W$ of homogeneous elements $v \in V$, $w \in W$ is even if and only if v and w have the same parity.

Example 2.4. (Examples of super algebras.)

1. Every algebra A is an example of a super algebra by declaring A to be purely even. Algebras we will most care about is the symmetric algebra $\mathbb{R}[x^1, \dots, x^p]$ generated by elements x^i, and the algebra of smooth functions $C^\infty(\mathbb{R}^p)$. Identifying the x^i with the coordinate functions on $\mathbb{R}^p$, we will think of $\mathbb{R}[x^1, \dots, x^p]$ as the subalgebra of $C^\infty(\mathbb{R}^p)$ consisting of polynomial functions.

2. The exterior algebra $\Lambda[\theta^1, \dots, \theta^q]$ generated by odd elements $\theta^1, \dots, \theta^q$.

3. $\Omega^*(X) = \Omega^{ev}(X) \oplus \Omega^{odd}(X)$ equipped with the wedge-product, and the induced super algebra structure on the deRham cohomology groups $H^*_{dR}(X)$. We note that the de Rham cohomology super algebra of the q-dimensional torus T^q is isomorphic to the exterior algebra $\Lambda[\theta^1, \dots, \theta^q]$, where $\{\theta_i\}$ is a basis of $H^1_{dR}(T^q)$.

4. $\mathbb{R}[x^1, \dots, x^q] \otimes \Lambda[\theta^1, \dots, \theta^q] \subset C^\infty(\mathbb{R}^p) \otimes \Lambda[\theta^1, \dots, \theta^q]$

To make sense of the last example we need to define the multiplication map for these tensor products. This can be done quite generally: If A, B are super algebras, we define the *tensor product super algebra* to be $A \otimes B$ equipped with the multiplication map

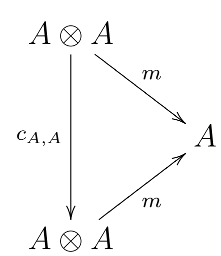
$$A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes c_{B,A} \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

and the monoidal unit $k \cong k \otimes k \overset{u_A \otimes u_B}{\longrightarrow} A \otimes B$. Here $c_{A,B}$ is the isomorphism defined for any pair of super vector spaces A, B by

$$c_{A,B} \colon A \otimes B \longrightarrow B \otimes A \qquad a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$$

All of the above superalgebras are *commutative* in the following sense.

Definition 2.5. A superalgebra (A, m, u) is *commutative* if the diagram



commutes. More explicitly, this means that for homogenous elements $a, b \in A$ we have

$$a \cdot b = (-1)^{|a||b|} b \cdot a$$

2.2 Super Lie algebra

Definition 2.6. A *super Lie algebra* is a $\mathbb{Z}/2$-graded vector space $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ equipped with a bilinear map

$$[\quad , \quad] \colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

with the following properties:

1. $[\,,\,]$ is skew-symmetric in the graded sense, that is $[a, b] = -(-1)^{|a||b|}[b, a]$ for homogeneous elements $a, b \in \mathfrak{g}$ of degree $|a|, |b| \in \mathbb{Z}/2$.

2. $[\,,\,]$ satisfies the graded version of the Jacobi identity.

We recall that the Jacobi identity for a Lie algebra $\mathfrak{g}$ has the following form:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

Letting $\alpha = [a, \,] \colon \mathfrak{g} \to \mathfrak{g}$ and using the skew-symmetry, this can be rewritten as

$$\alpha([b, c]) = -[c, [a, b]] - [b, [c, a]] = [\alpha(b), c] + [b, \alpha(c)].$$

In other words, the Jacobi identity says that for each $a \in \mathfrak{g}$ the map $[a, \,] \colon \mathfrak{g} \to \mathfrak{g}$ is a derivation w.r.t. the bracket. The graded version of the Jacobi identity is saying that for each homogeneous element $a \in \mathfrak{g}$ the endomorphism $[a, \,]$ is a *graded derivation* of $(\mathfrak{g}, [\,,\,])$, that is

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$$

Definition 2.7. If A is a super algebra, a linear map $D \colon A \to A$ (not necessarily grading preserving) is an *derivation of parity* $|D| \in \mathbb{Z}/2$ if

$$D \circ \epsilon_A = (-1)^{|D|} \epsilon_A \circ D \qquad \text{and} \qquad D(ab) = (Da)b + (-1)^{|D||a|}$$

for all homogeneous elements $a, b \in A$. Derivations of parity 0 (resp. 1) will be called *even* (resp. *odd*). We will write $\mathrm{Der}(A)$ for the $\mathbb{Z}/2$-graded vector space $\mathrm{Der}(A) = \mathrm{Der}^{ev}(A) \oplus \mathrm{Der}^{odd}(A)$.

Homework 2.8. Let A be a super algebra. Show that the bracket

$$[\quad , \quad] \colon \mathrm{Der}(A) \otimes \mathrm{Der}(A) \longrightarrow \mathrm{Der}(A) \qquad \text{defined by} \qquad [D, E] := D \circ E - (-1)^{|D||E|} D \circ E$$

gives $\mathrm{Der}(A)$ the structure of a super Lie algebra in the sense of the Definition (2.6). This generalizes the well-known case for algebras.

3 A categorical Digression

The goal of this section is threefold. One is to define symmetric monoidal categories and functors; as explained in the introduction this will be needed for our definition of field theories as functors. Secondly, we want to provide a categorical point of view on (commutative) super algebras as (commutative) monoids in the (symmetric) monoidal category of super vector spaces. Finally, we want to define the notion of a group object in a monoidal category. This will allow us in the next section to define super Lie groups as group objects in the category of supermanifolds.

This section is quite independent; the reader could easily skip it at first, and come back to it for the specific notions needed later.

3.1 Monoidal categories

We recall that a *monoid with unit* is a set A together with a map $m\colon A \times A \to A$ (multiplication) and an element $u \in A$ (unit) such that the multiplication is associative and u is a left- and right-unit for m. If we think of u as a map $u\colon \mathrm{pt} \to A$ from the one-point-set pt to A, the associativity of m and the unit property of u is given by the commutativity of two diagrams which are exactly the two diagrams (2.2), except that

- these are diagrams in the category Set of sets rather than the category Vect of vector spaces,
- the tensor product of vector spaces is replaced by the Cartesian product of sets, and
- the ground field $k \in \mathsf{Vect}$ is replaced by the one-point-set $\mathrm{pt} \in \mathsf{Set}$.

This suggests that the notions "monoid", "algebra" and "super algebra" are special cases of a more general notion, where A is an object of some category C. Of course to make sense of $m\colon A \otimes A \to A$ and the unit u as morphisms in C we need:

- There should be a functor
$$\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C},$$
allowing us to form for any objects $V, W \in \mathsf{C}$ an object $V \otimes W \in \mathsf{C}$. Moreover, if $f\colon V_1 \to V_2$ and $g\colon W_1 \to W_2$ are morphisms, we have a morphism
$$f \otimes f\colon V_1 \otimes W_1 \longrightarrow V_2 \otimes W_2$$
Here $\mathsf{C} \times \mathsf{C}$ is the category whose objects are pairs (V, W) of objects in C; a morphism from (V_1, W_1) to (V_2, W_2) is a pair of morphisms $(f\colon V_1 \to W_1, g\colon V_2 \to W_2)$.

- There should be an object $I \in \mathsf{C}$ (playing the role of $k \in \mathsf{Vect}$ and $\mathrm{pt} \in \mathsf{Set}$ in these special cases) and isomorphisms

$$I \otimes V \cong V \cong V \otimes I$$

These structures are encoded in the following notion of a monoidal category, see e.g., §1, Ch. VII of Mac Lane's book *Categories for the working mathematician* (second edition) [McL].

Definition 3.1. A *monoidal category* is a category C equipped with the following data:

1. A functor $\otimes \colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$;
2. an object $I \in \mathsf{C}$ called *monoidal unit*;
3. isomorphisms $\alpha_{U,V,W} \colon U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ for $U, V, W \in \mathsf{C}$ called *associators*;
4. isomorphisms $\lambda_V \colon I \otimes V \cong V$ and $\rho_V \colon V \otimes I \cong V$.

These isomorphisms are required to satisfy some conditions that are spelled out for example in Mac Lane's book. In particular, it is required that the family of isomorphisms $\alpha_{U,V,W}$ is *natural* in U, V and W. Naturality in U means that if $f \colon U_1 \to U_2$ is a morphism, then the following diagram commutes:

$$\begin{array}{ccc} U_1 \otimes (V \otimes W) & \xrightarrow{f \otimes (1 \otimes 1)} & U_2 \otimes (V \otimes W) \\ \downarrow{\scriptstyle \alpha_{U_1,V,W}} & & \downarrow{\scriptstyle \alpha_{U_2,V,W}} \\ (U_1 \otimes V) \otimes W & \xrightarrow{(f \otimes 1) \otimes 1} & (U_2 \otimes V) \otimes W \end{array}$$

Similarly, it is required that the family of isomorphisms λ_V (resp. ρ_V) is natural in V.

There are two further requirements.

- There are five ways to put parentheses in a tensor product of four factors $U, V, W, X \in \mathsf{C}$. Each of these five ways leads to a priori different objects of C, but the associators provide us with five isomorphisms between these objects. This gives a pentagon shaped diagram, and the requirement is that this diagram commutes.

- There is another diagram involving both, the associator α as well as the unit isomorphisms λ, ρ. This diagram is also required to commute.

In the notation for monoidal categories we tend to suppress the data (α, λ, ρ) and just write $(\mathsf{C}, \otimes, I)$ for a monoidal category.

Example 3.2. There are plenty of examples of monoidal categories:

1. $(\mathsf{Set}, \times, \mathrm{pt})$ and $(\mathsf{Set}, \amalg, \emptyset)$, where $\times$ stands for Cartesian product, and $\amalg$ for disjoint union. Similarly, Set can be replaced by the category Top of topological spaces or the category Man of smooth manifolds.

2. Our motivating examples: $(\mathsf{Vect}, \otimes, k)$ and $(\mathsf{SVect}, \otimes, k)$.

3. $(\mathsf{Vect}, \oplus, \{0\})$ and $(\mathsf{SVect}, \oplus, \{0\})$, where $\{0\}$ denotes the trivial (0-dimensional) vector space.

4. More generally in the last two examples, vector spaces could be replaced by cochain complexes.

Definition 3.3. Given a monoidal category $(\mathsf{C}, \otimes, I)$, a *monoid in* C is an object $A \in \mathsf{C}$ together with morphisms $m\colon A \otimes A \to A$ and $u\colon I \to A$ making the diagrams (2.2) commutative.

Example 3.4. (Examples of monoids.)

1. A monoid in $(\mathsf{Set}, \times)$ is what is usually called a monoid.

2. A monoid in $(\mathsf{Vect}, \otimes)$ is an algebra.

3. A monoid in $(\mathsf{SVect}, \otimes)$ is a superalgebra.

3.2 Symmetric monoidal categories

The family of isomorphisms $c_{A,B}$ is natural in A, B and has the property that the composition

$$A \otimes B \xrightarrow{c_{A,B}} B \otimes A \xrightarrow{c_{B,A}} A \otimes B \qquad \text{is the identity on } A \otimes B. \tag{3.5}$$

Hence equipped with the braiding isomorphisms $c_{A,B}$, the category $(\mathsf{SVect}, \otimes, I)$ is a symmetric monoidal category which is defined a follows.

Definition 3.6. A *symmetric monoidal category* is a monoidal category $(\mathsf{C}, \otimes, I)$ equipped with a family of isomorphisms $c_{A,B}\colon A \otimes B \cong B \otimes A$ for $A, B \in \mathsf{C}$ called *braiding isomorphisms* which are natural in A, B and satisfy condition (3.5).

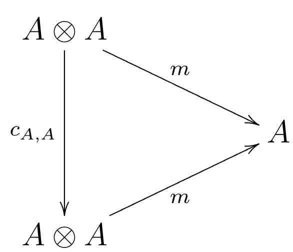
Example 3.7. Further examples of symmetric monoidal categories.

1. Let $\mathbb{Z}/2$-Vect be the *symmetric monoidal category of $\mathbb{Z}/2$-graded vector spaces*. The underlying monoidal category agrees with SVect, but the braiding isomorphisms $c_{A,B}$ are different: they are given by $a \otimes b \mapsto b \otimes a$. We note that the definition of the tensor product of superalgebras above generalizes to define the tensor product of monoids in any symmetric monoidal category. It should be emphasized that this tensor product *depends on the braiding isomorphism*. For example, let $C\ell(V)$ be the Clifford algebra generated by an inner product space V (see [LM, Ch. I, §1]). This is a $\mathbb{Z}/2$-graded algebra, or equivalently, a super algebra. If W is another inner product space, the Clifford algebra $C\ell(V \oplus W)$ is isomorphic to the tensor product $C\ell(V) \otimes C\ell(W)$ as *super algebras*, but not as $\mathbb{Z}/2$-*graded algebras*.

2. The category TV of topological super vector spaces equipped with the projective tensor product (see Definition **??**) and the usual braiding isomorphism for super vector spaces. This is the target category for field theories as in Definition 1.11. It contains as a subcategory the symmetric monoidal category of (ungraded) topological vector spaces, which was the target category for $d|0$-field theories as in Definition 1.4.

3. The category $d|\delta$-Bord (or its variants $d|\delta$-$\mathsf{Bord}(X)$, $d|\delta$-$\mathsf{Bord}_G(X)$), the domain category of our field theories, is a symmetric monoidal category: the monoidal structure is given by disjoint union, and the braiding isomorphism is the obvious one.

Definition 3.8. Let $(\mathsf{C}, \otimes, I, c)$ be a symmetric monoidal category. A monoid $A \in \mathsf{C}$ is *commutative* if the diagram



is commutative. A *commutative super algebra* is a commutative monoid in the symmetric monoidal category SVect of super vector spaces. In more down to earth terms, a superalgebra A is commutative if

$$a \cdot b = (-1)^{|a||b|} b \cdot a \tag{3.9}$$

for all homogeneous elements $a, b \in A$.

We want to emphasize that the commutativity of a monoid in a symmetric monoidal category depends on the braiding isomorphisms. For example, the symmetric monoidal categories SVect and $\mathbb{Z}/2$-Vect agree except for their braiding isomorphisms. A commutative $\mathbb{Z}/2$-graded algebra (aka monoidal object in $\mathbb{Z}/2$-Vect) is a $\mathbb{Z}/2$-graded algebra which is commutative in the sense that $ab = ba$ for all $a, b \in A$, while a commutative super algebra (aka commutative monoid in SVect) is a $\mathbb{Z}/2$-graded algebra which is graded commutative in the sense that equation (3.9) holds.

4 Supermanifolds

As mentioned in the introduction, supermanifolds are objects which are more general than manifolds: the algebra $C^\infty(M)$ of smooth functions on a smooth manifold M is a commutative algebra, while a supermanifold M has an associated algebra of functions $C^\infty(M)$ which is a commutative superalgebra in the sense of Definition 2.5. Moreover, the supermanifold M is just an ordinary smooth manifold precisely if the odd part of the superalgebra $C^\infty(M)$ is zero (which in particular makes $C^\infty(M)$ an ordinary commutative algebra). This motivates our approach to defining supermanifolds as well as notions like vector fields and vector bundles on supermanifolds:

Leitmotif 4.1. To generalize notions from the world of manifolds to the the larger worlds of supermanifolds, we first express these well-known notions for ordinary manifolds *in terms of the commutative algebra of smooth functions*, and then generalize to the 'super' version by replacing commutative algebras by commutative superalgebras.

A good reference for supermanifolds is the paper [DM].

4.1 Definition and examples of supermanifolds

Let M be a manifold. Then the smooth functions on open subsets of M define a sheaf $\mathcal{C}^\infty_M$ of commutative algebras on M which is called the *structure sheaf*. More precisely, for each open subset $U \subset M$ we define

$$\mathcal{C}^\infty_M(U) := C^\infty(U), \text{ the algebra of smooth functions on } U$$

In particular, the algebra of global sections of $\mathcal{C}^\infty_M$ is the algebra $C^\infty(M)$ of smooth functions on M. A diffeomorphism $f\colon M \to N$ induces an isomorphism of pairs $(M, \mathcal{C}^\infty_M) \cong (N, \mathcal{C}^\infty_N)$. In particular, using the diffeomorphisms provided by the charts of a p-dimensional manifold M, we see that $(M, \mathcal{C}^\infty_M)$ is *locally isomorphic* to $(\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p})$, that is, for each point $x \in M$ there is a neighborhood U such that $(U, (\mathcal{C}^\infty_M)_{|U})$ is isomorphic to $(V, \mathcal{C}^\infty_V)$, where V is an open subset of $\mathbb{R}^p$. It turns out that the converse of this statement holds as well.

Proposition 4.2. *Let M be a topological space and let $\mathcal{O}_M$ be a sheaf of commutative algebras on M such that $(M, \mathcal{O}_M)$ is locally isomorphic to $(\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p})$. Then M is a smooth manifold of dimension p.*

Idea of proof. The key is to embed the abstract sheaf $\mathcal{O}_M$ as a subsheaf of the sheaf $\mathcal{C}^0_M$ of continuous functions on M. If we think of this subsheaf as the sheaf of *smooth functions* on M, smooth functions on open subsets $U \subset M$ correspond to smooth functions on their images $V \subset \mathbb{R}^p$ via the local isomorphisms relating $(M, \mathcal{O}_M)$ and $(\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p})$. In particular, these local isomorphisms provide smooth charts for M, since the transition maps between two such charts is smooth, since pulling back any smooth function results in a smooth function.

To embed $\mathcal{O}_M$ as a subsheaf of $\mathcal{C}^0_M$, we declare the value of $s \in \mathcal{O}_M(U)$ at a point $x \in U$ to be the unique $\lambda \in \mathbb{C}$ such that the restriction of $s - \lambda \in \mathcal{O}_M(U)$ to $\mathcal{O}_M(V)$ is not invertible for any $V \subset U$ which contains x (the assumption that $\mathcal{O}_M$ locally looks like the sheaf $\mathcal{C}^\infty_{\mathbb{R}^p}$ of smooth functions on $\mathbb{R}^p$ implies that there is exactly one such $\lambda \in \mathbb{C}$). □

Our discussion of manifolds above suggests the following definition.

Definition 4.3. A *supermanifold* of dimension $p|q$ is a pair $M = (M_{\text{red}}, \mathcal{O}_M)$, where

- M_{red} is a topological space called the *reduced manifold* and
- $\mathcal{O}_M$ is a sheaf of commutative superalgebras on M_{red} such that $(M_{\text{red}}, \mathcal{O}_M)$ is locally isomorphic to $\mathbb{R}^{p|q} := (\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p} \otimes \Lambda[\theta^1, \dots, \theta^q])$

The algebra of global sections of $\mathcal{O}_M$ is denoted $C^\infty(M)$; we think of $C^\infty(M)$ as the commutative superalgebra of functions on M.

Given an open subset $U \subset M_{\text{red}}$ let $\text{Nil}(U) \subset \mathcal{O}_M(U)$ be the nilpotent ideal generated by the odd elements of $\mathcal{O}_M(U)$. We note that

$$\mathcal{O}_M / \text{Nil} \text{ is locally isomorphic to } \mathcal{O}_{\mathbb{R}^p} \otimes (\Lambda[\theta^1, \dots, \theta^q] / \text{Nil}) \cong \mathcal{O}_{\mathbb{R}^p}$$

This implies in particular that M_{red} is a smooth manifold of dimension p.

Example 4.4. Let $E \to N$ be a real vector bundle of rank q over a p-manifold. Then define

$$\Pi E := (N, \mathcal{C}^\infty(N, \Lambda(E^\vee)))$$

where $\mathcal{C}^\infty(N, \Lambda(E^\vee))$ is the sheaf of sections of the exterior power $\Lambda(E^\vee) := \bigoplus_{k=0}^q \Lambda^k(E^\vee)$ of the dual vector bundle $E^\vee$. In particular,

$$C^\infty(\Pi E) = C^\infty(N, \Lambda(E^\vee))$$

Special cases:

1. $E = \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$; then

   $$C^\infty(\Pi E) = C^\infty(\mathbb{R}^p, \Lambda((\mathbb{R}^q)^\vee)) = C^\infty(\mathbb{R}^p) \otimes \Lambda((\mathbb{R}^q)^\vee) = C^\infty(\mathbb{R}^p) \otimes \Lambda[\theta^1, \dots, \theta^q]$$

   where $\theta^i \in (\mathbb{R}^q)^\vee$ are the coordinate functions on $\mathbb{R}^q$. We note that any vector bundle $E \to N$ is locally isomorphic to the trivial vector bundle and hence ΠE is in fact a supermanifold.

2. $C^\infty(\Pi TN) = C^\infty(N, \Lambda(T^*N)) = \Omega^*(N)$

There is an obvious notion of morphisms between supermanifolds in terms of maps of their structure sheaves as explained in [DM, p. 65]. Alternatively, thanks to the existence of partitions of unity for smooth manifolds, it is possible to work with the algebra of global sections instead of the sheaves. The following result (see e.g. [GoSa, Thm. 2.3]) shows that smooth maps between manifolds can be understood in terms of the induced map of functions.

Theorem 4.5. *If M, N are manifolds (Hausdorff and countable basis) then the map*

$$\mathsf{Man}(M,N) \longrightarrow \mathsf{Alg}(C^\infty(N), C^\infty(M)) \qquad f \mapsto f^*$$

is a bijection.

This motivates the following definition.

Definition 4.6. Supermanifolds are the objects of the *category of supermanifolds* SM. Given supermanifolds M and N, the set of morphisms $\mathsf{SM}(M,N)$ is defined by

$$\mathsf{SM}(M,N) := \mathsf{SAlg}(C^\infty(N), C^\infty(M))$$

where $C^\infty(M)$ is the commutative superalgebra of global sections of the sheaf $\mathcal{O}_M$, and $\mathsf{SAlg}(C^\infty(N), C^\infty(M))$ is the set of grading preserving algebra homomorphisms Notationally, it will be convenient to distinguish between a morphism $f\colon M \to N$ and the induced map $f^*\colon C^\infty(N) \to C^\infty(M)$, since given morphisms $f \in \mathsf{SM}(M,N)$ and $g\colon N \to P$ in SM, their composition $g \circ f\colon M \to P$ corresponds by definition to the algebra map $(g \circ f)^* := f^* \circ g^*\colon C^\infty(P) \to C^\infty(M)$. From an abstract point of view, the category of supermanifolds is the full subcategory of SAlg^{op} consisting of those super algebras which are algebras of functions on supermanifolds.

Warning 4.7. For an ordinary manifold M the set $\mathsf{Man}(M,\mathbb{R})$ of smooth maps from M to $\mathbb{R}$ is equal to the set $C^\infty(M)$ of global sections of the sheaf $\mathcal{C}^\infty(M)$ of smooth functions on M. In general, *this is not the case* if M is a supermanifold. Rather, the image of the map

$$\mathsf{SM}(M,\mathbb{R}) = \mathsf{SAlg}(C^\infty(\mathbb{R}), C^\infty(M)) \longrightarrow C^\infty(M)$$

which sends a map $f\colon M \to \mathbb{R}$ to the pullback $f^*(t)$ of the coordinate function $t \in C^\infty(\mathbb{R})$ is contained in the *even* part $C^\infty(M)^{\mathrm{ev}}$. In fact, as a special case of Theorem 4.18 we will see that this map is a bijection between $\mathsf{SM}(M,\mathbb{R})$ and $C^\infty(M)^{\mathrm{ev}}$. Another special case worth pointing out is that we'll obtain a bijection between $C^\infty(M)$ and $\mathsf{SM}(M,\mathbb{R}^{1|1})$.

Theorem 4.8 (Batchelor's Theorem). *Every supermanifold M of dimension $p|q$ is isomorphic to ΠE for some rank q vector bundle $E \to N$ over a p-manifold (everything smooth).*

Idea of proof. Let $M = (M_{\mathrm{red}}, \mathcal{O}_M)$ be a supermanifold of dimension $p|q$. Our goal is to construct a rank q vector bundle $E \to M_{\mathrm{red}}$ and to show that M is isomorphic to ΠE. Let $\mathrm{Nil} \subset C^\infty(M)$ be the ideal generated by the odd elements. Then $\mathrm{Nil} / \mathrm{Nil}^2$ is a module over $C^\infty(M)/\mathrm{Nil} \cong C^\infty(M_{\mathrm{red}})$. For sufficiently small open subsets $U \subset M_{\mathrm{red}}$, $\mathcal{O}_M(U)$ is isomorphic to $C^\infty(U) \otimes \Lambda[V]$, where $\Lambda[V]$ is the exterior algebra generated by a q-dimensional vector space V (the elements of which are odd). Hence we have the following isomorphisms of $\mathcal{O}_M(U)$-modules

$$\mathrm{Nil}(U) \cong C^\infty(U) \otimes \bigoplus_{k=1}^{q} \Lambda^k(V) \qquad \mathrm{Nil}^2(U) \cong C^\infty(U) \otimes \bigoplus_{k=2}^{q} \Lambda^k(V)$$

This implies an isomorphism

$$\mathrm{Nil}(U) / \mathrm{Nil}^2(U) \cong C^\infty(U) \otimes V$$

of modules over $\mathcal{O}_M(U)/\mathrm{Nil}(U) \cong C^\infty(U)$. In particular, $U \mapsto \mathrm{Nil}(U)/\mathrm{Nil}^2(U)$ is a locally free sheaf of modules of rank q over the sheaf $\mathcal{C}^\infty(M_{\mathrm{red}})$ of smooth functions on M_{red}. It is well-known that locally free sheaves of modules over the sheaf of smooth functions are given as sheaves of sections of smooth vector bundles. In particular, there is a rank q vector bundle $E \to M_{\mathrm{red}}$ with $C^\infty(E^\vee) \cong \mathrm{Nil} / \mathrm{Nil}^2$.

To finish the proof we need to show that the superalgebra $C^\infty(M)$ is isomorphic to the superalgebra $C^\infty(\Pi E)$. We note that the powers $\mathrm{Nil}^k \subset C^\infty(M)$ give $C^\infty(M)$ the structure of a *filtered superalgebra* (a filtered superalgebra is a superalgebra A equipped with a sequence of decreasing subspaces $A = F_0 \supset F_1 \supset F_2 \dots$ such that $F_m \cdot F_n \subset F_{m+n}$). The superalgebra

$$C^\infty(\Pi E) = C^\infty(M_{\mathrm{red}}; \Lambda(E^\vee)) = \bigoplus_{k=0}^{q} C^\infty(M_{\mathrm{red}}; \Lambda^k(E^\vee))$$

is a *graded superalgebra* (which is a superalgebra of the form $A = \bigoplus_{k=0}^\infty G_k$ with $G_m \cdot G_n \subset G_{m+n}$). In fact, the isomorphism $\mathrm{Nil} / \mathrm{Nil}^2 \cong C^\infty(M_{\mathrm{red}}; E^\vee)$ induces isomorphisms

$$\mathrm{Nil}^k / \mathrm{Nil}^{k+1} \cong C^\infty(\Lambda^k(E^\vee))$$

which are compatible with multiplication and hence the graded superalgebra $C^\infty(\Pi E)$ is the *graded superalgebras associated to the filtered superalgebra* $C^\infty(M)$ (if $A = F_0 \supset F_1 \supset F_2 \dots$ is a filtered superalgebra, the associated graded superalgebra is $\bigoplus_k G_k$ with $G_k = F_k/F_{k+1}$).

We note that locally, the superalgebra $C^\infty(M)$ is of the form $C^\infty(U) \otimes \Lambda[V]$, and it is easy to check that this filtered algebra is isomorphic to the associated graded algebra. Using a partition of unity on M_{red} these local isomorphisms can be glued to provide an isomorphism between the filtered algebra $C^\infty(M)$ and the associated graded algebra. $\square$

Warning 4.9. Let $\mathsf{VectBun}$ be a category of smooth vectorbundles. The functor

$$\mathsf{VectBun} \longrightarrow \mathsf{SM} \qquad \text{given by} \qquad E \mapsto \Pi E$$

is *essentially surjective*, i.e., every object of SM is isomorphic to one in the image of this functor by Batchelor's Theorem. However, this is *not an equivalence of categories* since there are more morphisms in the category of supermanifolds than in the category of vector bundles in the sense that for $E, F \in \mathsf{VectBun}$ the map of morphisms

$$\mathsf{VectBun}(E, F) \longrightarrow \mathsf{SM}(\Pi E, \Pi F)$$

given by the functor is injective, but in general *not surjective*.

Example 4.10. Here is such an example: take E be a q-dimensional vector space, considered as a vector bundle over the point pt, and let F be the trival line bundle over pt. Then

$$\mathsf{VectBun}(E, F) = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R}) = E^\vee$$

while

$$\mathsf{SM}(\Pi E, \Pi F) = \mathsf{SAlg}(C^\infty(\Pi F), C^\infty(\Pi E)) = \mathsf{SAlg}(\Lambda[\theta], \Lambda^*(E^\vee)) \longleftrightarrow \Lambda^*(E^\vee)^{odd}$$

The bijection is given by sending $f^* \in \mathsf{SAlg}(\Lambda[\theta], \Lambda^*(E^\vee))$ to $f^*(\theta)$ which is an odd element of the exterior algebra $\Lambda^*(E^\vee)$ generated by the dual vector space $E^\vee$. We note that $\Lambda^*(E^\vee)^{odd}$ contains the generating vector space $E^\vee$, but it is strictly larger than $E^\vee$ if $\dim E \geq 3$ (if $\theta^1, \theta^2, \theta^3$ are linearly independent elements of $E^\vee$, then their product is a non-trivial element in $\Lambda^*(E^\vee)^{odd}$ not belonging to $E^\vee$.

4.2 Super Lie groups and their Lie algebras

Definition 4.11. An *even (resp. odd) vector field* on a supermanifold M is a even/odd graded derivation $D\colon C^\infty(M) \to C^\infty(M)$ (see Definition 2.7). We will write $\mathrm{Der}^{ev}(C^\infty(M))$ (resp. $\mathrm{Der}^{odd}(C^\infty(M))$ for the vector space of even (resp. odd) derivations of $C^\infty(M)$, and

$$\mathrm{Der}(C^\infty(M)) := \mathrm{Der}^{ev}(C^\infty(M)) \oplus \mathrm{Der}^{odd}(C^\infty(M))$$

for the super vector space of derivations of $C^\infty(M)$. This is a super Lie algebra with respect to the graded commutator $[\ ,\ ]$ of derivations defined by

$$[V, W] := V \circ W - (-1)^{|V||W|} W \circ V$$

for homogeneous vectorfields V, W.

We note that if V is an odd vectorfield, then $[V,V] = 2V^2$. In particular, the square of an odd vectorfield is again a vectorfield (unlike the situation for vectorfields on ordinary manifolds where the square of a vector is not a vector field, but rather a second order differential operator).

Example 4.12. $C^\infty(\mathbb{R}^{1|1}) = C^\infty(\mathbb{R}) \otimes \Lambda[\theta]$, and hence every function $f \in C^\infty(\mathbb{R}^{1|1})$ can uniquely be written as $f = f_0 + \theta f_1$ for $f_0, f_1 \in C^\infty(\mathbb{R})$. Examples of vectorfields aka derivations:

$$\begin{aligned}
C^\infty(\mathbb{R}^{1|1}) \quad &\longrightarrow \quad C^\infty(\mathbb{R}^{1|1})\\
\frac{\partial}{\partial t}\colon f_0 + \theta f_1 &\mapsto f_0' + \theta f_1'\\
\theta\frac{\partial}{\partial t}\colon f_0 + \theta f_1 &\mapsto \theta(f_0' + \theta f_1') = \theta f_0'\\
\frac{\partial}{\partial \theta}\colon f_0 + \theta f_1 &\mapsto f_1\\
D := \frac{\partial}{\partial \theta} - \theta\frac{\partial}{\partial t}\colon f_0 + \theta f_1 &\mapsto f_1 - \theta f_0'\\
D^2\colon f_0 + \theta f_1 &\mapsto D(f_1 - \theta f_0') = -f_0' - \theta f_1' = -\frac{\partial}{\partial t}(f_0 + \theta f_1)
\end{aligned}$$

This shows that $D^2 = -\frac{\partial}{\partial t}$.

Definition 4.13. A *super Lie group* G is a group object in the category SM of supermanifolds, i.e., G is a supermanifold equipped with morphisms in SM

$$m\colon G \times G \longrightarrow G \text{ (multiplication) } u\colon \text{ pt} \to G \text{ (unit)}$$

such that $G \times G \xrightarrow{\mu \times p_2} G \times G$ is an isomorphism and the usual diagrams, expressing associativity of m and the unit property of u, are commutative.

Example 4.14. $G = \mathbb{R}^{1|1}$ with multiplication $m \in \mathsf{SM}(G \times G, G) = \mathsf{SAlg}(C^\infty(G), C^\infty(G \times G)$ given by

$$\begin{aligned}
m^*\colon \mathbb{C}^\infty(G) = C^\infty(\mathbb{R})[\theta] &\longrightarrow C^\infty(G \times G) = C^\infty(G) \otimes C^\infty(G)\\
t &\mapsto t \otimes 1 + 1 \otimes t + \theta \otimes \theta\\
\theta &\mapsto \theta \otimes 1 + 1 \otimes \theta
\end{aligned}$$

Homework 4.15. Check the properties required for a super Lie group.

Definition 4.16. Let G be a super Lie group. Then we define its *super Lie algebra* $\mathrm{Lie}(G)$ to be the $\mathbb{Z}/2$-graded vector space

$$\mathrm{Lie}(G) := \{\text{left-invariant vector fields on } G\}$$

equipped with Lie bracket given by the graded commutator of vectorfields. A vector field D on a supermanifold M equipped with a left G-action $\mu\colon G \times M \to M$ is *(left) invariant* if it makes the following diagram commutative:

$$\begin{array}{ccccc}
C^\infty(M) & \xrightarrow{m^*} & C^\infty(G \times M) & = & C^\infty(G) \otimes_\pi C^\infty(M) \\
\downarrow{\scriptstyle D} & & & & \downarrow{\scriptstyle 1\otimes D} \\
C^\infty(M) & \xrightarrow{m^*} & C^\infty(G \times M) & = & C^\infty(G) \otimes_\pi C^\infty(M)
\end{array}$$

If G is a super Lie group of dimension $p|q$, then $\mathrm{Lie}(G)$ is a super vector space of dimension $p|q$, i.e., $\dim \mathrm{Lie}(G)^{ev} = p$, $\dim \mathrm{Lie}(G)^{odd} = q$.

Example 4.17. $\mathrm{Lie}(\mathbb{R}^{1|1})$ has dimension $1|1$ with basis $\{\frac{\partial}{\partial t}, D\}$. To prove this, it suffices to show that the vectorfields $\frac{\partial}{\partial t}$ and D are indeed left-invariant.

$$\begin{array}{ccc}
\theta & \stackrel{m^*}{\longmapsto} & \theta \otimes 1 + 1 \otimes \theta \\
 & & \downarrow{\scriptstyle 1\otimes D} \\
\downarrow{\scriptstyle D} & & 1 \otimes D\theta = 1 \\
 & & \\
1 & \stackrel{m^*}{\longmapsto} & 1
\end{array}
\qquad
\begin{array}{ccc}
t & \stackrel{m^*}{\longmapsto} & t \otimes 1 + 1 \otimes t + \theta \otimes \theta \\
 & & \downarrow{\scriptstyle 1\otimes D} \\
\downarrow{\scriptstyle D} & & 1 \otimes Dt - \theta \otimes D\theta = -1 \otimes \theta - \theta \otimes 1 \\
 & & \\
-\theta & \stackrel{m^*}{\longmapsto} & -\theta \otimes 1 - 1 \otimes \theta
\end{array}$$

This shows that the vectorfield D is left-invariant. A similar calculation shows that $\frac{\partial}{\partial t}$ is left-invariant. Alternatively, we note

$$\frac{\partial}{\partial t} = -D^2 = -\frac{1}{2}[D, D]$$

and that the graded bracket of left-invariant vectorfields is invariant. We remark that this shows that the Lie algebra $\mathrm{Lie}(\mathbb{R}^{1|1})$ is the *free Lie algebra with one odd generator D* (this means that for any super Lie algebra algebra $\mathfrak{g}$ the map

$$\{\text{Lie algebra homomorphisms } f\colon \mathrm{Lie}(\mathbb{R}^{1|1}) \to \mathfrak{g}\} \longrightarrow \mathfrak{g}^{odd} \qquad f \mapsto f(D)$$

is a bijection).

4.3 Determining maps from a supermanifold to $\mathbb{R}^{p|q}$

Theorem 4.18. *If S is a supermanifold, then the map*

$$\mathsf{SM}(S, \mathbb{R}^{p|q}) \longrightarrow \underbrace{C^\infty(S)^{ev} \times \cdots \times C^\infty(S)^{ev}}_{p} \times \underbrace{C^\infty(S)^{odd} \times \cdots \times C^\infty(S)^{odd}}_{q}$$
$$f \mapsto (f^*x^1, \dots, f^*x^p, f^*\theta^1, \dots, f^*\theta^q)$$

is a bijection. Here $x^i, \theta^j \in C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \Lambda[\theta^1, \dots, \theta^q]$ are the coordinate functions.

We note that this result implies in particular that $\mathbb{R}^{p|q}$ is the *categorical product* of p copies of $\mathbb{R}$ and q copies of $\mathbb{R}^{0|1}$ in the category SM. Conversely, to prove it, we first construct the *Cartesian product* $M \times N$ of supermanifolds M, N, and show that it is the categorical product in SM. It is well-known of course that this is the case if M, N are ordinary manifolds. In order to extend the construction of the Cartesian product from manifolds to supermanifolds, we should follow our Leitmotiv **??** and describe the algebra of functions $C^\infty(M \times N)$ on the Cartesian product of manifolds M, N in terms of the algebras $C^\infty(M)$ and $C^\infty(N)$. There is the bilinear

$$C^\infty(M) \times C^\infty(N) \longrightarrow C^\infty(M \times N) \qquad \text{given by} \qquad (f, g) \mapsto p_1^* f \cdot p_2^* g, \tag{4.19}$$

where $p_1 \colon M \times N \to M$, $p_2 \colon M \times N \to M$ are the projection maps. A first guess might be that the corresponding linear map

$$C^\infty(M) \otimes C^\infty(N) \longrightarrow C^\infty(M \times N)$$

is an isomorphism. Alas, the answer is more complicated and given by the following theorem which can be found for example in

Theorem 4.20. *The bilinear map* (4.19) *induces an isomorphism*

$$C^\infty(M) \otimes_\pi C^\infty(N) \xrightarrow{\cong} C^\infty(M \times N).$$

Here $C^\infty(M) \otimes_\pi C^\infty(N)$ is the completed projective tensor product of the topological vector spaces $C^\infty(M)$ and $C^\infty(N)$ equipped with their standard Frechét topology.

Definition 4.21. We recall that a *topological vector space* is a vector space V equipped with a topology such that addition and scalar multiplication are continuous. Note that in particular translations of open subsets are open and hence a topology is determined by a neighborhood base of $0 \in V$. This allows us to define: a sequence $v_i \in V$ is *Cauchy* if for all open neighborhoods U of $0 \in V$ there is some N such that $v_i - v_j \in U$ for all $i, j \geq N$. In particular, we can define V to be *complete* if every Cauchy sequence converges. A topological vector space is *locally convex* if there is a neighborhood basis of 0 consisting of convex subsets. In particular, a family of semi-norms $|| \quad ||_{i\in I}$ on V determines a locally convex topology on V whose neighborhood basis of 0 consists of all the balls

$$B_r^i := \{x \in V \mid ||x||_i < r\}$$

If the index set I is countable, the topological vector space V is called a *Frechét space*

Example 4.22. (The Frechét topology on $C^\infty(M)$.) Let M be a manifold (we will always assume that our manifolds are Hausdorff with a countable basis). Let K_i, $i \in \mathbb{N}$ be a family of compact subsets whose interiors cover M, each of which is contained in some coordinate chart. The *Frechet topology* on $C^\infty(M)$ is the topology defined by the semi-norms

$$||f||_{i,j} := \max \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right|$$

where the maximum is taken over all points $x \in K_i$ and all $\alpha = (\alpha_1, \dots, \alpha_p)$ with $|\alpha| := \alpha_1 + \cdots + \alpha_p \le j$. While these semi-norms depend on the choices (of the local coordinates and the compact subsets K_i), the topology is independent of them. More generally, if $V \to M$ is a smooth vector bundle, then using coordinate charts on M and local trivializations for V, we can similarly define semi-norms for the vector space $C^\infty(V) = C^\infty(M, V)$ of smooth sections of V. These define a Frechét topology on $C^\infty(V)$.

Definition 4.23. (The projective tensor product). (see [Sch, §6.1]) If V, W are locally convex topological vector spaces, the *projective topology* on the algebraic tensor product $V \otimes W$ is the finest topology (that is, the topology with the most open subsets) such that the tautological bilinear map

$$V \otimes W \longrightarrow V \otimes W$$

is continuous. The *(completed) projective tensor product* $V \otimes_\pi W$ is complete locally convex topological Hausdorff vector space obtained as the completion of the algebraic tensor product $V \otimes W$ equipped with the projective topology.

We note that if $f\colon V \otimes_\pi W \to U$ is a continuous linear map, then the composition

$$V \times W \longrightarrow V \otimes W \hookrightarrow V \otimes_\pi W \xrightarrow{f} U$$

is a continuous bilinear map. It is easy to check that this construction gives a bijection between $\mathsf{TV}(V \otimes_\pi W, U)$ and the set of continuous bilinear maps $V \times W \to U$ (see e.g. [Sch, §6.2]). In particular, the bilinear map (4.19) is continuous, and hence induces a continuous linear map $C^\infty(M) \otimes_\pi C^\infty(N) \to C^\infty(M \times N)$ (which according to Theorem 4.20 is an isomorphism).

Similarly, the map $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $(f,g) \mapsto f \cdot g$ is a continuous bilinear map which leads to a continuous linear map

$$C^\infty(M) \otimes_\pi C^\infty(M) \longrightarrow C^\infty(M)$$

This gives $C^\infty(M)$ the structure of a *topological algebra*, i.e., a monoid in the symmetric monoidal category $(\mathsf{TV}, \otimes_\pi)$.

Remark 4.24. Let A, B be monoids in a symmetric monoidal category $(\mathsf{C}, \otimes)$. Then $A \otimes B$ is a again a monoid with multiplication given by

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{1 \otimes c_{B,A} \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B$$

Given another monoid $C \in \mathsf{C}$ and monoidal maps $f\colon A \to C$, $g\colon B \to C$ the composition

$$A \otimes B \xrightarrow{f \otimes g} C \otimes C \xrightarrow{\mu_C} C$$

is again a monoidal map. In fact, this construction provides a bijection between pairs $(f\colon A \to C, g\colon B \to C)$ of monoidal maps and monoidal maps from $A \otimes B$ to C. In other words, $A \otimes B$ is the *coproduct* of A and B in the category of monoidal objects and monoidal maps in C. In particular, the projective tensor product $A \otimes_\pi B$ of topological algebras A, B is their coproduct in the category TopAlg of topological (super) algebras.

Definition 4.25. Let $M = (M_{\mathrm{red}}, \mathcal{O}_M)$ and $N = (N_{\mathrm{red}}, \mathcal{O}_N)$ be supermanifolds. Then their *Cartesian product* $M \times N$ is defined to be the supermanifold whose reduced manifold is $M_{\mathrm{red}} \times N_{\mathrm{red}}$ and whose structure sheaf $\mathcal{O}_{M \times N}$ is determined by

$$\mathcal{O}_{M \times N}(U \times V) := \mathcal{O}_M(U) \otimes_\pi \mathcal{O}_N(V)$$

for open subsets $U \subset M$, $V \subset N$. In particular, $C^\infty(M \times N) = C^\infty(M) \otimes_\pi C^\infty(N)$.

Lemma 4.26. *If M, N are supermanifolds, then the Cartesian product $M \times N$ is the* categorical product*; i.e., for any supermanifold S the map*

$$\mathsf{SM}(S, M \times N) \longrightarrow \mathsf{SM}(S, M) \times \mathsf{SM}(S, N) \qquad f \mapsto (p_1 \circ f, p_2 \circ f)$$

is a bijection.

Proof. If $f\colon M \to N$ is a smooth map between manifolds, the induced map $f^*\colon C^\infty(N) \to C^\infty(M)$ is a *continuous* algebra homomorphism, i.e., a morphisms in the TopAlg, the category of topological algebras. The composition

$$\mathsf{Man}(M,N) \longrightarrow \mathrm{TopAlg}(C^\infty(N), C^\infty(M)) \hookrightarrow \mathsf{Alg}(C^\infty(N), C^\infty(M))$$

is a bijection by Theorem 4.1. This implies that *both* maps above are bijections. In particular, the functor $\mathsf{Man}^{op} \to \mathrm{TopAlg}$ given by $M \mapsto C^\infty(M)$ is full and faithful (i.e., it induces bijections on morphism sets). More generally, the statements above hold for *supermanifolds* M, N leading to a full and faithful embedding $\mathsf{SM}^{op} \to \mathrm{TopAlg}$. This implies that $M \times N$ is the product of M and N in the category SM, since $C^\infty(M \times N) = C^\infty(M) \otimes_\pi C^\infty(N)$ is the coproduct of $C^\infty(M)$ and $C^\infty(N)$ in $\mathsf{SM}^{op} \subset \mathrm{TopAlg}$ by Remark 4.24. □

Proof of Theorem 4.18. Since $\mathbb{R}^{p|q}$ can be identified with the Cartesian (= categorical) product

$$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{p} \times \underbrace{\mathbb{R}^{0|1} \times \cdots \times \mathbb{R}^{0|1}}_{q}$$

it suffices to prove the theorem for $p|q = 1|0$ and $p|q = 0|1$. In the latter case we have bijections

$$\begin{array}{ccccc} \mathsf{SM}(S, \mathbb{R}^{0|1}) & \longleftrightarrow & \mathsf{Alg}(C^\infty(\mathbb{R}^{0|1}), C^\infty(S)) & \longleftrightarrow & C^\infty(S)^{odd} \\ f & \longmapsto & f^* & \longmapsto & f^*(\theta) \end{array}$$

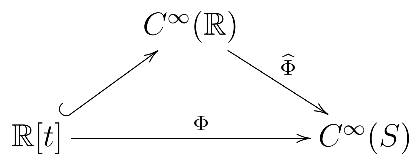
The first map is a bijection by definition of the morphisms in the category SM (see Definition 4.6). The second map is a bijection since a grading preserving algebra homomorphism Φ from $C^\infty(\mathbb{R}^{0|1}) = \Lambda[\theta]$ to any superalgebra A is completely determined by the image $\Phi(\theta) \in A^{odd}$; conversely, given any $a \in A^{odd}$, there is a unique grading preserving homomorphism Φ with $\Phi(\theta) = a$.

To prove the theorem for $p|q = 1|0$, we consider the composition

$$\begin{array}{ccccccc} \mathsf{SM}(S, \mathbb{R}) & \longleftrightarrow & \mathsf{Alg}(C^\infty(\mathbb{R}), C^\infty(S)) & \xrightarrow{res} & \mathsf{Alg}(\mathbb{R}[t], C^\infty(S)) & \longleftrightarrow & C^\infty(S)^{odd} \\ f & \longmapsto & f^* & \longmapsto & f^*_{|\mathbb{R}[t]} & \longmapsto & f^*(t) \end{array}$$

where res sends f^* to the restriction of f^* to the polynomial algebra $\mathbb{R}[t] \subset C^\infty(\mathbb{R})$. The first and third of these maps are evidently bijections, and so i suffices to show that the restriction map res is a bijection as well.

To show that res is surjective, we need to prove that every parity preserving algebra homomorphism $\Phi\colon \mathbb{R}[t] \to C^\infty(S)$ can be extended to a parity preserving algebra homomorphism $\widehat{\Phi}$ making the diagram



commutative. This is easy if S is an ordinary manifold, since in that case $\phi := \Phi(t) \in C^\infty(S)$ is a smooth map $\phi\colon S \to \mathbb{R}$ and we can define

$$\widehat{\Phi}(h) := h \circ \phi,$$

where $\phi := \Phi(t) \in C^\infty(S)$.

For the general case, we note that if h is a polynomial $h = a_n t^n + \cdots + a_1 t + a_0 \in \mathbb{R}[t]$, then

$$\begin{aligned}\widehat{\Phi}(h) &= \Phi(h) = \Phi(a_n t^n + \cdots + a_1 t + a_0)\\ &= a_n \Phi(t^n) + \cdots + a_1 \Phi(t) + a_0\\ &= a_n \phi^n + \cdots + a_1 \phi + a_0 = h(\phi).\end{aligned}$$

If h is a smooth function, the idea is to use the Taylor expansion of h to make sense of $h(\phi) = \widehat{\Phi}(h) \in C^\infty(S)$. To do this, we use Batchelor's Theorem to identify the supermanifold S with ΠE for some rank s vector bundle $E \to M_{red}$ and hence

$$C^\infty(S) \qquad \text{with} \qquad \bigoplus_{k=0}^{q} C^\infty(M_{red}; \Lambda^k(E^\vee))$$

This allows us to write ϕ in the form

$$\phi = \phi_b + \phi_s \qquad \text{with} \qquad \phi_b \in C^\infty(M_{\mathrm{red}}),\ \ \phi_s \in \bigoplus_{k=1}^{q} C^\infty(M_{\mathrm{red}}; \Lambda^k(E^\vee))$$

We note that ϕ_s is nilpotent (in the colorful language often used, ϕ_s is the "soul" and ϕ_b is the "body" of ϕ). This allows us to define

$$h(\phi) = h(\phi_b + \phi_s) := \sum_{k=0}^{\infty} h^{(k)}(\phi_b) \frac{(\phi_s)^k}{k!} \ \in C^\infty(S).$$

Here $h^{(k)} \in C^\infty(\mathbb{R})$ is the k-th derivative of h, and the term $h^{(k)}(\phi_b) := h^{(k)} \circ \phi_b \in C^\infty(M_{\mathrm{red}}) \subset C^\infty(S)$ is defined since M_{red} is an ordinary manifold. The sum above is finite since $\phi_s \in C^\infty(S)$ is nilpotent. $\square$

Remark 4.27. The above construction defines the composition

$$S \xrightarrow{\phi_s} \mathbb{R} \xrightarrow{h} \mathbb{R}$$

of $\phi_s \in C^\infty(S)^{\mathrm{ev}}$ and a smooth function $h \in C^\infty(\mathbb{R})$. If we only assume that h is of class C^k, but $h^{(k+1)}$ doesn't exist, then the above construction does not work in general, since it's easy to come up with supermanifolds S and nilpotent elements $\phi_s \in C^\infty(S)^{\mathrm{ev}}$ with $(\phi_s)^{k+1} \neq 0$. This is not just a defect of this construction, but rather the composition $h \circ \phi_s$ *can't be defined*. An interesting consequence of this observation is that the category of C^k-manifolds and C^k-maps between them cannot be extended to a category of C^k-supermanifolds.

5 Mapping supermanifolds and the functor of points formalism

The goal of this section is to make sense of the *supermanifold of maps* or *mapping supermanifold* $\underline{\mathsf{SM}}(Y, Z)$ for supermanifolds Y, Z. As explained in the introduction, we will be particularly interested in the mapping supermanifold $\mathsf{SM}(\mathbb{R}^{0|1}, Z)$. We want to emphasize that despite the notational similarities, there is a big difference between $\mathsf{SM}(Y, Z)$ and $\underline{\mathsf{SM}}(Y, Z)$:

1. $\mathsf{SM}(Y, Z)$ is the *set* of morphisms from Y to Z in the category SM of supermanifolds, while

2. $\underline{\mathsf{SM}}(Y, Z)$ is a supermanifold, an *object in the category* SM. For this reason $\underline{\mathsf{SM}}(Y, Z)$ is referred to as an *internal hom object*.

Warning 5.1. The above is an oversimplification: in general $\underline{\mathsf{SM}}(Y, Z)$ is *not* a supermanifold, and so the word "internal" in the sentence above should be taken with a grain of salt. This is already evident when discussing ordinary manifolds, and trying to construct the mapping manifold $\underline{\mathsf{Man}}(Y, Z)$ for manifolds Y, Z. The space of smooth maps $Y \to Z$ can be viewed as a manifold locally modeled on certain topological vector spaces [Mi]. However, these vector spaces are *infinite dimensional* unless $\dim Y = 0$, and hence these mapping spaces aren't objects of the category Man of finite dimensional manifolds.

As we'll explain below, in general $\underline{\mathsf{SM}}(Y, Z)$ is an object of the functor category

$$\mathsf{Fun}(\mathsf{SM}^{\mathrm{op}}, \mathsf{Set}),$$

whose objects are functors from the opposite category $\mathsf{SM}^{\mathrm{op}}$ to the category Set of sets, and whose morphisms are natural transformations. This functor category contains SM as a full subcategory via the Yoneda embedding. We'll show that for $Y = \mathbb{R}^{0|1}$ the mapping supermanifold $\underline{\mathsf{SM}}(Y, Z)$ belongs to the image of the Yoneda embedding (see Proposition 5.24).

5.1 Internal hom objects

We begin by discussing the meaning of "internal hom object". If X, Y, Z are sets, there is a bijection of sets

$$\mathsf{Set}(X \times Y, Z) \leftrightarrow \mathsf{Set}(X, \mathsf{Set}(Y, Z)) \tag{5.2}$$

given by sending a map $f\colon X \times Y \to Z$ to the map $X \to \mathsf{Set}(Y, Z)$, $x \mapsto f_x$, where $f_x\colon Y \to Z$ is the restriction of f to $\{x\} \times Y \subset X \times Y$. This bijection is natural in X, Y and Z.

We would like to generalize this bijection from the category Set to other categories C. To make sense of the left hand side for $X, Y, Z \in \mathsf{C}$ we will require that C is a category equipped with a monoidal structure $\otimes$, and we replace $X \times Y$ by $X \otimes Y$ (if C has products, the monoidal structure could be given by the categorical product, but we wish to consider more general examples). Concerning the right hand side, we observe that just blindly replacing Set by C doesn't work: $\mathsf{C}(X, \mathsf{C}(Y, Z))$ does not make sense, since $\mathsf{C}(Y, Z)$ is a *set*, not an *object of* C. To fix this, we postulate that for every pair of objects $Y, Z \in \mathsf{C}$ there is an object $\underline{\mathsf{C}}(Y, Z) \in \mathsf{C}$, depending functorially on Y, Z; in other words, we require that we have a functor

$$\underline{\mathsf{C}}(\ ,\ )\colon \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}.$$

Moreover, we require that there is a family of bijections

$$\mathsf{C}(X \otimes Y, Z) \xleftrightarrow{b_{X,Y,Z}} \mathsf{C}(X, \underline{\mathsf{C}}(Y, Z)) \tag{5.3}$$

for $X, Y, Z \in \mathsf{C}$, which is natural in X, Y, Z. In categorical lingo, this is expressed by saying that for each $Y \in \mathsf{C}$ the functor

$$- \otimes Y\colon \mathsf{C} \to \mathsf{C} \qquad \textit{has a right adjoint} \qquad \underline{\mathsf{C}}(Y, -)\colon \mathsf{C} \to \mathsf{C}$$

A monoidal category $(\mathsf{C}, \otimes)$ equipped with this additional structure is called a *closed monoidal category* and the object $\underline{\mathsf{C}}(Y, Z) \in \mathsf{C}$ is referred to as an *internal hom object* [McL, ch. VII, §7]. Unfortunately, there is no uniform notation for internal hom objects: competing notations include Y^X and $[X, Y]$, but we prefer a notation indicating the category.

Example 5.4. (Examples of closed monoidal categories.)

1. The monodial category $(\mathsf{Ab}, \times)$ of abelian groups equipped with the Cartesian product is closed with $\underline{\mathsf{Ab}}(X, Y)$ defined to be the abelian group of homomorphisms from X to Y.

2. The monoidal category $(\mathsf{Vect}, \otimes)$ of vector spaces is closed, where $\underline{\mathsf{Vect}}(X, Y)$ is the vector space of linear maps.

3. The monoidal category $(\mathsf{SVect}, \otimes)$ of super vector spaces is closed, where $\underline{\mathsf{SVect}}(X, Y)$ is the vector space of all linear maps equipped with the grading involution ϵ that sends a linear map $f\colon X \to Y$ to $X \xrightarrow{\epsilon_X} X \xrightarrow{f} Y \xrightarrow{\epsilon_Y} Y$, where ϵ_X, ϵ_Y are the grading involutions on X resp. Y. In particular, $\underline{\mathsf{SVect}}(X, Y)^{ev}$ is the vector space of grading preserving linear maps. We note that for all the other examples listed here the internal hom object $\underline{\mathsf{C}}(X, Y)$ is given by additional structure (abelian group structure, vector space structure of topology) on the morphism set $\mathsf{C}(X, Y)$, while this is not the case in this example.

4. The category $(\mathsf{Top}, \times)$ of topological spaces equipped with the categorical product is closed, provided this category is defined with great care as explained in detail in [McL, ch. VII, §8]: the objects of Top are those topological spaces which are Hausdorff and compactly generated (i.e., a subset C is closed if and only if its intersection with every compact subset is closed). For compactly generated Hausdorff spaces X, Y, the product $X \times Y$ and the internal hom object $\underline{\mathsf{Top}}(X, Y)$ is defined by modifying the standard product topology on the Cartesian product resp. the compact-open topology on the space of continuous maps $X \to Y$ in a functorial way to turn them into compactly generated spaces (this procedure turns every Hausdorff space into a compactly generated Hausdorff space with the same underlying set).

Even if a given monoidal category is not closed, one might ask whether for given $Y, Z \in \mathsf{C}$ there is an object $\underline{\mathsf{C}}(Y, Z) \in \mathsf{C}$ such that we have a family of bijections $b_{X,Y,Z}$ as in (5.3) above, which is natural in X. This question can be rephrased as follows. The left hand side defines a functor

$$\mathsf{C}(- \otimes Y, Z)\colon \mathsf{C}^{op} \longrightarrow \mathsf{Set} \qquad \text{given by} \qquad X \mapsto \mathsf{C}(X \otimes Y, Z) \tag{5.5}$$

where C^{op} is the *opposite category* of C which consists of the same objects as C, and for $X, Y \in \mathsf{C}$, we have $\mathsf{C}^{op}(X, Y) = \mathsf{C}(Y, X)$. In other words, a functor $\mathsf{C}^{op} \to \mathsf{D}$ is the same thing as a contravariant functor from C to D. Let us denote by $\mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set})$ the *functor category* whose objects are functors $F\colon \mathsf{C}^{op} \to \mathsf{Set}$ and whose morphisms are natural transformations between these functors. This category is also known as the category of *presheaves on* C. We recall that according to Yoneda's Lemma the functor

$$\mathsf{Y}\colon \mathsf{C} \longrightarrow \mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set}) \qquad \text{given by} \qquad Y \mapsto (X \mapsto \mathsf{C}(X, Y))$$

is fully faithful, which means that the map

$$\mathsf{C}(Y, Z) \longrightarrow \mathsf{Fun}(\mathsf{Y}(Y), \mathsf{Y}(Z))$$

given by applying the functor Y to morphisms is a bijection for every pair of objects $Y, Z \in \mathsf{C}$. In other words, we can think of C as a full subcategory of the generally much larger category

$\mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set})$ (a subcategory $\mathsf{C} \subset \mathsf{D}$ is full if for any $Y, Z \in \mathsf{C}$ the inclusion map $\mathsf{C}(Y, Z) \to \mathsf{D}(Y, Z)$ is a bijection). We note that $\mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set})$ can be thought of as the "cocompletion" of C in the sense that every object of $\mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set})$ is the colimit of objects in the image of the Yoneda embedding. Moreover, $\mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set})$ is cocomplete (i.e., all colimits exist) and complete (i.e., all limits exist).

Definition 5.6. An object of $\widehat{\mathsf{C}} := \mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set})$ is called a *presheaf on* C or a *generalized object of* C (we will be interested in *generalized supermanifolds*, i.e., objects of $\widehat{\mathsf{SM}}$). A generalized object is called *representable* if it is isomorphic to an object in the image of the Yoneda embedding Y.

Passing from a category C to the larger category $\widehat{\mathsf{C}}$ is useful for the same reasons it is useful to pass from real numbers to complex numbers or from functions to distributions: it might happen that an equation has a solution only in the "enlargement", like the root of a polynomial equation with coefficients in $\mathbb{R}$ might be in $\mathbb{C} \setminus \mathbb{R}$, or a partial differential equation might only have solutions which are distributions rather than functions. The same can be said about our quest to find an internal hom object $\underline{\mathsf{C}}(Y, Z) \in \mathsf{C}$: there is always the generalized object

$$\underline{\mathsf{C}}(Y, Z) := \mathsf{C}(- \otimes Y, Z) \in \widehat{\mathsf{C}} = \mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set}) \tag{5.7}$$

defined in (5.5). Abusing terminology, we will call this an *internal hom object*, although it is an object of the larger category $\widehat{\mathsf{C}}$. If we insist on an object of C, we need to study whether $\underline{\mathsf{C}}(Y, Z) \in \widehat{\mathsf{C}}$ is representable (which is analogous to deciding whether the root of a polynomial is a real number, or whether a weak solution to a PDE belongs to a suitable function space).

Remark 5.8. A monoidal structure on a category C extends to a monoidal structure on its cocompletion $\widehat{\mathsf{C}} = \mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Set})$ (called the *convolution tensor product* [Day]). This is a closed monoidal category, and for representable objects $Y, Z \in \mathsf{C} \subset \widehat{\mathsf{C}}$ the internal hom object is given by (5.7).

5.2 The functor of points formalism

Via the Yoneda embedding

$$\mathsf{SM} \hookrightarrow \mathsf{Fun}(\mathsf{SM}^{\mathrm{op}}, \mathsf{Set})$$

we will identify a supermanifold X with the corresponding functor $X \colon \mathsf{SM}^{\mathrm{op}} \to \mathsf{Set}$. On objects, the functor X sends a supermanifold S to the set $X(S) = \mathsf{SM}(S, X)$ of morphisms from S to X. The $X(S)$ is called the set of *S-points of X*. This terminology is motivated by the special case where X is an ordinary manifold and S is the 0-manifold pt consisting of one point. Then $\mathsf{SM}(S, X) = \mathsf{Man}(\mathrm{pt}, X)$ is in fact the set of points of the manifold X.

More generally, if S is an ordinary manifold, a smooth map $S \to X$ can be thought of as a "smooth family of points of X parametrized by S". This is also the mental picture of an S-point of a supermanifold X: a family of points in X parametrized by the supermanifold S.

Via the Yoneda embedding, the morphism set $\mathsf{SM}(X,Y)$ for supermanifolds X is in bijective correspondence to natural transformations

$$\mathsf{SM} \underset{Y}{\overset{X}{\rightrightarrows}} \mathsf{Set}, \qquad \Phi\colon X \Rightarrow Y \tag{5.9}$$

between the corresponding functors. We recall that such a natural transformation f gives for each object S of the domain category SM a morphism

$$\Phi_S\colon X(S) \longrightarrow Y(S)$$

in the range category Set subject to compatibility conditions. So a natural transformation $\Phi\colon X \to Y$ amounts to a set map $\Phi_S\colon X(S) \to Y(S)$ between the S-points of X and Y for each supermanifold S. These are required to be *natural in* S in the sense that for any morphism $g\colon S' \to S$ of supermanifolds, the diagram

$$\begin{array}{ccc} X(S) & \xrightarrow{X(g)} & X(S') \\ {\scriptstyle\Phi_S}\downarrow & & \downarrow{\scriptstyle\Phi_{S'}} \\ Y(S) & \xrightarrow{Y(g)} & Y(S') \end{array} \tag{5.10}$$

commutes.

Using the terminology introduced above, Theorem 4.18 can be reinterpreted as calculating the S-points of $\mathbb{R}^{p|q}$: there are bijections

$$\begin{aligned} \mathbb{R}^{p|q}(S) &\leftrightarrow (C^\infty(S)^{\mathrm{ev}})^p \times (C^\infty(S)^{\mathrm{odd}})^q \\ f &\mapsto (f^*t^1, \dots, f^*t^p, f^*\theta^1, \dots, f^*\theta^q) \end{aligned}$$

which are natural in S.

Determining the S-points of certain (generalized) supermanifolds explicitly, and constructing maps Φ_S between them satisfying the naturality condition (5.10) is a main technical point of this paper. In particular we will show that certain generalized supermanifolds $X\colon \mathsf{SM}^{\mathrm{op}} \to \mathsf{Set}$ are representable by giving a supermanifold Y and a family of bijections $\Phi_S\colon X(S) \to Y(S)$ satisfying the naturality condition.

5.3 The supermanifold of endomorphisms of $\mathbb{R}^{0|1}$

In this section we show that the internal hom object $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \in \widehat{\mathsf{SM}}$ is representable (see Prop. 5.11), and we describe the monoidal structure on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ given by composition explicitly in terms of the S-point formalism (see Prop. 5.13).

Proposition 5.11. *The generalized supermanifold* $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ *is isomorphic to the supermanifold* $\mathbb{R}^{1|1}$*.*

Proof. We need to show that for any supermanifold S there is a bijection of S-points

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(S) \longleftrightarrow \mathbb{R}^{1|1}(S)$$

which is natural in S. We recall that

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(S) = \underline{\mathsf{SM}}(S \times \mathbb{R}^{0|1}, \mathbb{R}^{0|1}) = \mathsf{SAlg}(C^\infty(\mathbb{R}^{0|1}), C^\infty(S \times \mathbb{R}^{0|1}))$$

In particular, any map $g\colon S \times \mathbb{R}^{0|1} \to \mathbb{R}^{0|1}$ is determined by the induced map

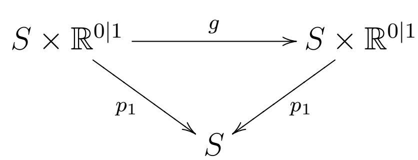
$$g^*\colon C^\infty(\mathbb{R}^{0|1}) = \Lambda[\theta] \longrightarrow C^\infty(S \times \mathbb{R}^{0|1}) = C^\infty(S)[\theta]$$

which in turn is determined by $g^*(\theta) \in (C^\infty(S)[\theta])^{odd}$. This shows that we have a bijection

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(S) \longleftrightarrow C^\infty(S)^{odd} \times C^\infty(S)^{ev} = \mathbb{R}^{1|1}(S) \tag{5.12}$$

given by $g \mapsto (g_1, g_0)$, where the g_i's are determined by $g^*(\theta) = g_1 + g_0\theta$. □

To show that composition gives a monoidal structure on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ it is useful to think of an S-point $g \in \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(S)$ as a map $g\colon S \times \mathbb{R}^{0|1} \to S \times \mathbb{R}^{0|1}$ making the diagram



commutative. We write $\mathsf{SM}_S(S \times \mathbb{R}^{0|1}, S \times \mathbb{R}^{0|1})$ for this set of *bundle endomorphism* of $S \times \mathbb{R}^{0|1} \to S$ over the identity on the base space S. Composition of these endomorphisms gives a family of maps

$$m_S\colon \mathsf{SM}_S(S \times \mathbb{R}^{0|1}, S \times \mathbb{R}^{0|1}) \times \mathsf{SM}_S(S \times \mathbb{R}^{0|1}, S \times \mathbb{R}^{0|1}) \longrightarrow \mathsf{SM}_S(S \times \mathbb{R}^{0|1}, S \times \mathbb{R}^{0|1})$$

which is natural in S and hence provides a morphism

$$m\colon \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \times \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \longrightarrow \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$$

of generalized supermanifolds. Associativity of the multiplications m_S implies that m is associative, thus making $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ a monoidal object in $\widehat{\mathsf{SM}}$.

Proposition 5.13. *The monoid* $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ *is isomorphic to the semi-direct product* $\mathbb{R}^{0|1} \rtimes \mathbb{R}$ *where the monoidal structure on* $\mathbb{R}^{0|1}$ *(resp.* $\mathbb{R}$*) is given by addition (resp. multiplication) and* $\mathbb{R}$ *acts on* $\mathbb{R}^{0|1}$ *by multiplication (see* (5.14)*).*

It is usual to define the semi-direct product for groups, but the same definition applies to monoids: let H, G be monoids and let $G \times H \to H$, $(g, h) \mapsto g(h)$ be an action of G on H compatible with the monoidal structure on H (i.e., for each $g \in G$ the map $h \mapsto g(h)$ is a monoidal map). Then the *semi-direct product* $H \rtimes G$ is the set $H \times G$ equipped with the following monoidal structure (aka associative multiplication):

$$(H \rtimes G) \times (H \rtimes G) \longrightarrow H \rtimes G \qquad (h, g), (h', g') \mapsto (h \cdot g(h'), g \cdot g')$$

The monoidal structure on $\mathbb{R}^{0|1} \rtimes \mathbb{R}$ is defined by a family of monoidal structures on

$$(\mathbb{R}^{0|1} \rtimes \mathbb{R})(S) = \mathbb{R}^{0|1}(S) \times \mathbb{R}(S) = C^\infty(S)^{odd} \times C^\infty(S)^{ev}$$

given by

$$\begin{aligned} (\mathbb{R}^{0|1} \rtimes \mathbb{R})(S) \times (\mathbb{R}^{0|1} \rtimes \mathbb{R})(S) \longrightarrow (\mathbb{R}^{0|1} \rtimes \mathbb{R})(S) \\ (g_1, g_0), (g_1', g_0') \mapsto (g_1 + g_0 g_1', g_0 g_0') \end{aligned} \tag{5.14}$$

Proof. We need to show that the bijection (5.12) is compatible with the monoidal structure on both sides. So let $g, g' \in \mathsf{SM}_S(S \times \mathbb{R}^{0|1}, S \times \mathbb{R}^{0|1})$ and let $g_0, g_0' \in C^\infty(S)^{ev}$, $g_1, g_1' \in C^\infty(S)$ be determined by $g^*(\theta) = g_1 + g_0\theta$ and $(g')^*(\theta) = g_1' + g_0'\theta$. Then

$$\begin{aligned} (g \circ g')^* =& (g')^*(g^*\theta) = (g')^*(g_1 + g_0\theta) = g_1 + g_0(g')^*(\theta) \\ =& g_1 + g_0(g_1' + g_0'\theta) = g_1 + g_0 g_1' + g_0 g_0'\theta \end{aligned}$$

Here the third equality comes from the fact that g' is a bundle endomorphism over the identity of S, which implies that the induced map $(g')^*$ is $C^\infty(S)$-linear. This shows that via the bijection (5.12) the composition $g \circ g'$ corresponds to the pair

$$(g_1 + g_0 g_1', g_0 g_0') := (g_1, g_0)(g_1', g_0') \in (\mathbb{R}^{0|1} \rtimes \mathbb{R})(S)$$

□

5.4 Vector bundles on supermanifolds

Our next goal is to greatly generalize Proposition 5.11 by showing that for any supermanifold X the generalized supermanifold $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ is representable. More precisely we will show that the representing supermanifold is the totalspace of the parity-reversed tangent bundle of X. In preparation for that proof in the next section, we introduce in this section the notion vector bundles over supermanifolds.

To do that, we follow our Leitmotif 4.1 according to which we should generalize a notion from manifolds to supermanifolds by expressing the classical definition *in terms of the sheaf of smooth functions*. We note that if $E \to X$ is a smooth vector bundle of rank r over a manifold X, then the sheaf $\mathcal{C}^\infty(E)$ of smooth sections of E is a sheaf of modules over the sheaf $\mathcal{C}^\infty(X)$ of smooth functions. Moreover, if E is the trivial bundle of rank r, then $\mathcal{C}^\infty(E)$ can be identified with the sum of r copies of $\mathcal{C}^\infty(X)$ so that $\mathcal{C}^\infty(E)$ is a sheaf of free modules of rank r. For a general vector bundle E, its local triviality implies that $\mathcal{C}^\infty(E)$ is a sheaf of *locally* free modules over $\mathcal{C}^\infty(X)$, which means that each point $x \in X$ has an open neighborhood U such that $\mathcal{C}^\infty(E)(U)$ is a free $\mathcal{C}^\infty(X)(U)$-module of rank r.

Definition 5.15. A *vector bundle E of rank $r|s$ over a supermanifold X* consists of a sheaf $\mathcal{C}^\infty(X, E)$ over X_{red} of modules over the structure sheaf $\mathcal{O}_X$ which is locally free of rank $r|s$ (i.e., it has a basis consisting of r even and s odd elements). The *space of sections of E* is the space $C^\infty(X, E) := \mathcal{C}^\infty(X, E)(X_{\text{red}})$ of global sections of the sheaf $\mathcal{C}^\infty(X, E)$.

Example 5.16. Let X be a supermanifold of dimension $p|q$. Then its *tangent bundle* TX is a vector bundle defined by

$$\mathcal{C}^\infty(TX)(U) := \operatorname{Der}(\mathcal{O}_X(U)) \qquad \text{for open subsets } U \subset X_{\text{red}},$$

where $\operatorname{Der}(\mathcal{O}_X(U))$ is the $\mathbb{Z}/2$-graded vector space of derivations of the superalgebra $\mathcal{O}_X(U)$. The sheaf $\mathcal{C}^\infty(TX)$ is a sheaf of modules over $\mathcal{O}_X$, since if $D\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is a graded derivation, and $f \in \mathcal{O}_X(U)$, then $fD\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is again a derivation, due to the graded commutativity of $\mathcal{O}_X(U)$. For sufficiently small U, $\mathcal{O}_X(U) \cong C^\infty(V)[\theta^1, \dots, \theta^q]$, where V is an open subset of $\mathbb{R}^p$, and hence

$$\operatorname{Der}(C^\infty(V)[\theta^1, \dots, \theta^q]) \text{ is a free } C^\infty(V)[\theta^1, \dots, \theta^q]\text{-module}$$

with basis

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}, \frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^q}.$$

Since the $\frac{\partial}{\partial x^i}$'s are even derivations, and the $\frac{\partial}{\partial \theta^j}$'s are odd derivations, this shows that $\mathcal{C}^\infty(TX)$ is a locally free sheaf of $\mathcal{O}_X$-modules of rank $p|q$.

If $f\colon X \to Y$ is a smooth map between manifolds and E is a vector bundle over Y, we can pull back E via f to obtain the vector bundle f^*E over X. It is not hard to show that for any open set $U \subset X$ the map

$$C^\infty(U) \otimes_{C^\infty(Y)} C^\infty(E) \longrightarrow C^\infty(f^*E_{|U}) \qquad g \otimes s \mapsto g \cdot f^*s$$

is an isomorphism (the tensor product here is again the projective tensor product with respect to the standard Frechét topology on these vector spaces). The vector space $C^\infty(U)$ is regarded as a module over $C^\infty(Y)$ via the algebra homomorphism $(f_{|U})^*\colon C^\infty(Y) \to C^\infty(U)$. In other words, pulling back vector bundles corresponds to extension of scalars for their spaces of sections. This motivates the following definition.

Definition 5.17. Let $f\colon X \to Y$ be a map of supermanifolds, and let E be a vector bundle over Y. Then the *pull-back vector bundle* f^*E over X is defined by

$$\mathcal{C}^\infty(f^*E)(U) := \mathcal{O}_X(U) \otimes_{C^\infty(Y)} C^\infty(E)$$

where $\mathcal{O}_X(U)$ is a module over $C^\infty(Y)$ via the algebra homomorphism

$$C^\infty(Y) \xrightarrow{f^*} C^\infty(X) = \mathcal{O}_X(X) \xrightarrow{res} \mathcal{O}_X(U)$$

If E is a vector bundle of rank r over a manifold X of dimension p, then its total space can be equipped with the structure of a smooth manifold of dimension $p+r$. Abusing notation, this manifold is again denoted E. This works also in the world of supermanifolds: if X is a supermanifold of dimension $p|q$ and E is a vector bundle of rank $r|s$ over X, we can construct a supermanifold of dimension $p+r|q+s$. Abusing notation we write E for this total space supermanifold.

It is easiest to first construct E as a generalized supermanifold, i.e., as a functor $\mathsf{SM}^{op} \to \mathsf{Set}$. To motivate this construction, we first consider the case $q = s = 0$; in other words, we assume that $\pi\colon E \to X$ is just a vector bundle of rank r over an ordinary manifold of dimension p. Then a point e in the total space E is a pair (x, v), where x is a point of X and v is a vector in the fiber $E_x = \pi^*(x)$ of E.

The same holds for S-points of E. Let e be an S-point of E, where S is some manifold; in other words, $e\colon S \to E$ is a smooth map. Let $x := \pi \circ e \in X(S)$ be the corresponding S-point of X, and let v be the section of the pullback bundle $x^*E \to S$ uniquely determined by the commutative diagram

$$\begin{array}{ccc} x^*E & \longrightarrow & E \\ \uparrow v \ \downarrow & \overset{e}{\nearrow} & \downarrow \pi \\ S & \xrightarrow{x} & X. \end{array}$$

This construction gives a bijection

$$E(S) \ \longleftrightarrow \ \{(x,v) \mid x \in X(S), v \in E_x := C^\infty(x^*E)\}.$$

We will refer to the vector space E_x as the *fiber of E over the S-point x*; we note that for $S = \mathrm{pt}$, this agrees with the usual meaning of fiber.

Warning 5.18. Suppose E is the trivial bundle of rank $r|s$ over a supermanifold X. Then the space $C^\infty(X, E)$ of sections of E is the free module of rank $r|s$ over $C^\infty(X)$. On the other hand, the total space E should be the product $X \times \mathbb{R}^{p|q}$, and hence sections of the bundle

$$E = X \times \mathbb{R}^{r|s} \to X$$

should just correspond to maps $X \to \mathbb{R}^{r|s}$, which by Theorem 4.18 are in bijective correspondence to

$$\underbrace{C^\infty(X)^{ev} \times \cdots \times C^\infty(X)^{ev}}_{r} \times \underbrace{C^\infty(X)^{odd} \times \cdots \times C^\infty(X)^{odd}}_{s},$$

which is just the *even* part of the free $C^\infty(X)$-module of rank $r|s$.

The apparent contradiction is resolved by contemplating Warning 4.7: the set of morphisms $\mathsf{SM}(X, \mathbb{R})$ is the *even part* $C^\infty(X)^{\text{ev}}$ of the vector space $C^\infty(X)$ of smooth functions on X. Similarly, $\mathsf{SM}(X, \mathbb{R}^{r|s})$ can be identified with the even part of the vector space of sections $C^\infty(X, E)$ of the trivial bundle of rank $r|s$ over X.

Definition 5.19. Let E be a vector bundle of rank $r|s$ over a supermanifold X. Then the *total space* is the generalized supermanifold with S-points given by

$$E(S) := \{(x, v) \mid x \in X(S), v \in E_x^{ev}\}. \tag{5.20}$$

Here E_x is the *fiber of E over the S-point* $x \in X(S)$, defined by $E_x := C^\infty(S, x^*E)$.

Homework 5.21. Show that the generalized supermanifold E is representable. Hint: What is the candidate for the reduced manifold E_{red}?

If E is a rank $r|s$ vector bundle over a supermanifold X, we denote by ΠE the *parity reversed* vector bundle of rank $s|r$; in terms of the sheaves of sections, $\mathcal{C}^\infty(X, \Pi E)$ is obtained from $\mathcal{C}^\infty(X, E)$ by reversing the parity, i.e., by declaring

$$\mathcal{C}^\infty(X, \Pi E)^{\text{ev}} := \mathcal{C}^\infty(X, E)^{\text{odd}} \qquad \mathcal{C}^\infty(X, \Pi E)^{\text{odd}} := \mathcal{C}^\infty(X, E)^{\text{ev}}.$$

In particular, the total space supermanifold ΠE has dimension $p + s|q + r$.

Homework 5.22. Let E be a vector bundle of rank r over a manifold X of dimension p. Then as observed above, the total space supermanifold ΠE has dimension $p|r$. Show that this generalized supermanifold is isomorphic to the supermanifold we called ΠE in Example 4.4.

We will be particularly interested in the S-points of ΠTX for a supermanifold X, which according to (5.20) are pairs (x, v) with $x \in X(S)$ and $v \in T_x X^{\text{odd}}$. It will be convenient to describe the tangent space $T_x X$ of X at an S-point $x \in X(S)$ more directly in terms of derivations.

Given a super algebra homomorphism $e\colon A \to B$, we define $\text{Der}_f(A, B)$ to be the vectorspace of *f-derivations*, which are linear maps $D\colon A \to B$ with the property that

$$D(fg) = D(f)e(g) + (-1)^{|D||f|}e(f)D(g) \qquad \text{for all } f, g \in A.$$

For example, it is well-known that the tangent space T_xX of a manifold X at a point x can be described algebraically as $\mathrm{Der}_{\mathrm{ev}_x}(C^\infty(X), \mathbb{R})$, where $\mathrm{ev}_x \colon C^\infty(X) \to \mathbb{R}$ is the evaluation map which sends a function $f \in C^\infty(X)$ to $f(x) \in \mathbb{R}$. This holds more generally: if X is a supermanifold, x is an S-point of X, and $x^* \colon C^\infty(X) \to C^\infty(S)$ is the induced map on functions, then the map

$$T_xX = C^\infty(S, x^*TX) = C^\infty(S) \otimes_{C^\infty(X)} \mathrm{Der}(C^\infty(X)) \longrightarrow \mathrm{Der}_{x^*}(C^\infty(X), C^\infty(S)) \quad (5.23)$$

which sends $a \otimes D$ to $a \cdot (x^* \circ D)$ is well-defined. In fact, it is a bijection, as is not hard to show.

5.5 The supermanifold of maps $\mathbb{R}^{0|1} \to X$

The goal of this section is to show that for any supermanifold X the generalized supermanifold $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ is representable (Prop. 5.24). It follows that $\underline{\mathsf{SM}}(\mathbb{R}^{0|q}, X)$ is representable for every q (Cor. 5.25). Then we describe the action of $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ via precomposition explicitly in the S-point formalism (Prop. 5.27).

Proposition 5.24. *For any supermanifold X the generalized supermanifold $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ is isomorphic to the supermanifold ΠTX. In particular, it is representable.*

Corollary 5.25. *For any supermanifold X the generalized supermanifold $\underline{\mathsf{SM}}(\mathbb{R}^{0|q}, X)$ is representable.*

The case $q = 1$ is an obvious consequence of the proposition. The case $q = 2$ follows from

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X) = \underline{\mathsf{SM}}(\mathbb{R}^{0|1} \times \mathbb{R}^{0|1}, X) = \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)) \cong \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \Pi TX) \cong \Pi T(\Pi TX),$$

where the last isomorphism follows from applying the proposition to the supermanifold ΠTX. The general case follows by induction on q.

Proof of Proposition 5.24. We need to show that for any supermanifold S there is a bijection of S-points

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X) \longleftrightarrow (\Pi TX)(S)$$

which is natural in S. We recall that by Definition 5.19

$$(\Pi TX)(S) = \{(x, v) \mid x \in X(S), v \in T_xX^{\mathrm{odd}}\}$$

where $T_xX = C^\infty(S, x^*TX)$.

We recall that

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)(S) = \mathsf{SM}(S \times \mathbb{R}^{0|1}, X) = \mathsf{SAlg}(C^\infty(X), C^\infty(S \times \mathbb{R}^{0|1}))$$

In particular, any map $f\colon S\times\mathbb{R}^{0|1}\to X$ is determined by the induced map

$$f^*\colon C^\infty(X)\longrightarrow C^\infty(S\times\mathbb{R}^{0|1})=C^\infty(S)[\theta]$$

Given $a\in C^\infty(X)$, the element $f^*(a)\in C^\infty(S)[\theta]$ can be written uniquely in the form

$$f^*(a)=f_0(a)+\theta f_1(a)\qquad\text{with } f_0(a),f_1(a)\in C^\infty(S)$$

Since f^* is parity preserving, this implies for the linear maps $f_i\colon C^\infty(X)\to C^\infty(S)$ that f_0 is even and f_1 is odd. Next we determine how to express that f^* is an algebra homomorphism in terms of f_0 and f_1. For $a,b\in C^\infty(S)$ we have

$$\begin{aligned}f^*(a)f^*(b)&=(f_0(a)+\theta f_1(a))(f_0(b)+\theta f_1(b))\\&=f_0(a)f_0(b)+\theta(f_1(a)f_0(b)+(-1)^{|a|}f_0(a)f_1(b))\end{aligned}$$

Here the last equality holds since θ is odd and $|f_0(a)|=|a|$. Comparing with

$$f^*(a)f^*(b)=f^*(ab)=f_0(ab)+\theta f_1(ab)$$

we see that f_0 is an element of $\mathsf{SAlg}(C^\infty(X),C^\infty(S))=\mathsf{SM}(S,X)$, and $f_1\colon C^\infty(X)\to C^\infty(S)$ is an odd graded derivation w.r.t. f_0. So geometrically speaking, f_0 is an S-point of X, and the derivation $f_1\in\mathrm{Der}_{f_0}(C^\infty(X),C^\infty(S))$ via the bijection (5.23) can be interpreted as an odd tangent vector $f_1\in T_{f_0}X^{odd}$ at f_0. Summarizing, we have shown that we have a bijection

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|1},X)(S)\ \longleftrightarrow\ \{(x,v)\mid x\in X(S),v\in T_xX^{odd}\}=(\Pi TX)(S)\tag{5.26}$$

given by $f\mapsto(f_0,f_1)$, where the f_i's are determined by $f^*(a)=f_0(a)+\theta f_1(a)$. □

The monoid $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})$ acts on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},X)$ by precomposition. More explicitly, on S-points this action is given by

$$\begin{aligned}\mathsf{SM}_S(S\times\mathbb{R}^{0|1},S\times X)\times\mathsf{SM}_S(S\times\mathbb{R}^{0|1},S\times\mathbb{R}^{0|1})&\longrightarrow\mathsf{SM}_S(S\times\mathbb{R}^{0|1},S\times X)\\(f,g)&\mapsto f\circ g\end{aligned}$$

where as before, we make the identifications $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},X)(S)=\mathsf{SM}_S(S\times\mathbb{R}^{0|1},S\times X)$ and $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})(S)=\mathsf{SM}_S(S\times\mathbb{R}^{0|1},S\times\mathbb{R}^{0|1})$.

Proposition 5.27. *Identifying $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},X)(S)$ with $(\Pi TX)(S)$ via bijection (5.26) and $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})(S)$ with $(\mathbb{R}^{0|1}\rtimes\mathbb{R})(S)$ via bijection (5.12), the action map*

$$\mu\colon\underline{\mathsf{SM}}(\mathbb{R}^{0|1},X)\times\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})\longrightarrow\underline{\mathsf{SM}}(\mathbb{R}^{0|1},X)\tag{5.28}$$

is given on S-points by

$$(x,v),(g_1,g_0)\mapsto(x+g_1v,g_0v)\tag{5.29}$$

Remark 5.30. Thinking of $C^\infty(S)$ as "scalars" (which they are if $S = \mathrm{pt}$), the scalar g_0 acts as a dilation on the tangent vector $v \in T_xX$ while g_1 acts via $x \mapsto x + g_1 v$ which can be interpreted as "moving the point x in the direction of the tangent vector $w = g_1 v \in T_x X^{ev}$". The classical analog is the image of the tangent vector $w \in T_xX$ under the exponential map $\exp\colon T_xX \to X$. The reader is probably aware that the construction of the exponential map does require additional data on X, e.g., a Riemannian metric, and might wonder why it is possible to describe the "exponential map" by the simple-minded formula $w \mapsto x + w$. This in fact *does not work in general*, i.e., if $x \in \mathsf{SAlg}(C^\infty(X), C^\infty(S))$, and $w \in T_xX$, then $x + w$ is in general not an algebra homomorphism:

$$(x+w)(ab) = x(ab) + w(ab) = x(a)x(b) + w(a)x(b) + x(a)w(b)$$
$$(x+w)(a)(x+w)(b) = x(a)x(b) + w(a)x(b) + x(a)w(b) + w(a)w(b)$$

However, if $w = g_1 v$, then the term $w(a)w(b)$ vanishes, and so $x + w$ is again an S-point of X.

Proof of Proposition 5.27. Let $f \in \underline{\mathsf{SM}}_S(S \times \mathbb{R}^{0|1}, S \times X)$, $g \in \underline{\mathsf{SM}}_S(S \times \mathbb{R}^{0|1}, S \times \mathbb{R}^{0|1})$ be S-points that correspond to (x, v) resp. (g_1, g_0) via the bijections (5.26) resp. (5.12), i.e.,

$$f^*(a) = x(a) + \theta v(a) \text{ for } a \in C^\infty(X) \qquad \text{and} \qquad g^*(\theta) = g_1 + g_0\theta$$

Then

$$\begin{aligned}(f \circ g)^*(a) &= g^*(f^*(a)) = g^*(x(a) + \theta v(a)) = x(a) + g^*(\theta)v(a)\\ &= x(a) + (g_1 + g_0\theta)v(a) = x(a) + g_1 v(a) + \theta g_0 v(a)\end{aligned}$$

Here the third equation holds since g^* is $C^\infty(S)$-linear. This shows that the composition $f \circ g$ corresponds to the pair $(x + g_1 v, g_0 v)$. □

5.6 The algebra of functions on a generalized supermanifold

In this subsection we first define the algebra of functions $C^\infty(X)$ for a generalized supermanifold X, extending our previous notion for supermanifolds. More importantly, we will explain how functions on a generalized supermanifold are related to its S-points. This will prepare the ground for calculating the map

$$\mu^*\colon C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)) \longrightarrow C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X) \times \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}))$$

induced by the action map μ (see (5.28)).

We recall from Theorem 4.18 that for any supermanifold X the map

$$\mathsf{SM}(X, \mathbb{R}^{1|1}) \longrightarrow C^\infty(X)^{ev} \times C^\infty(X)^{odd} = C^\infty(X) \qquad f \mapsto (f^*(t), f^*(\theta))$$

is a bijection. Here $t,\theta \in C^\infty(\mathbb{R}^{1|1}) = C^\infty(\mathbb{R})[\theta]$ is the even (resp. odd) coordinate function. The addition (resp. multiplication) on $C^\infty(X)$ is induced by maps $a, m\colon \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$. On S-points, these maps are given by

$$a((t_1,\theta_1),(t_2,\theta_2)) = (t_1+t_2, \theta_1+\theta_2) \qquad m((t_1,\theta_1),(t_2,\theta_2)) = (t_1t_2+\theta_1\theta_2, t_1\theta_2+\theta_1 t_2)$$

for $(t_1,\theta_1),(t_2,\theta_2) \in \mathbb{R}^{1|1}_S$. Moreover, the map $C^\infty(X) \to C^\infty(X)$ given by multiplication by $r \in \mathbb{R}$ is induced by the map

$$m_r\colon \mathbb{R}^{1|1} \to \mathbb{R}^{1|1} \qquad \text{given on } S\text{-points by} \qquad (t_1,\theta_1) \mapsto (rt_1, r\theta_1)$$

Homework 5.31. Prove the claims that the maps a, m, m_r in fact induce the desired maps on $C^\infty(X)$.

Definition 5.32. Let X be a generalized supermanifold; that is, $X \in \widehat{\mathsf{SM}} = \mathsf{Fun}(\mathsf{SM}^{op}, \mathsf{Set})$. The *algebra of functions on* X, denoted by $C^\infty(X)$ is defined by $C^\infty(X) := \widehat{\mathsf{SM}}(X, \mathbb{R}^{1|1})$ with algebra structure induced by the maps a, m, m_r (as before, we identify SM with its image under the Yoneda embedding $\mathsf{SM} \hookrightarrow \widehat{\mathsf{SM}}$).

Let us unpack this abstract definition. Given a generalized supermanifold X, how do we write down a function $f \in C^\infty(X)$? According to the definition, f is a morphism in $\widehat{\mathsf{SM}} = \mathsf{Fun}(\mathsf{SM}, \mathsf{Set})$ from X to $\mathbb{R}^{1|1}$, that is, a natural transformation. In other words, f consists of a natural family of maps

$$f_S\colon X(S) \longrightarrow \mathbb{R}^{1|1}(S) = C^\infty(S) \qquad \text{for } S \in \mathsf{SM} \tag{5.33}$$

We want to emphasize that while this might sound exotic at first glance, this is actually very close to our usual way to think about functions: a function on a manifold X is something we can evaluate at a point $x \in X$ to get a number $f(x) \in \mathbb{R}$. We note that this is exactly what f_S does for $S = \mathrm{pt}$ and $X \in \mathsf{Man} \subset \mathsf{SM} \subset \widehat{\mathsf{SM}}$, since then $X(S)$ is just X considered as a set, and $C^\infty(S) = \mathbb{R}$. So one way to think about a function $f \in C^\infty(X)$ for a generalized supermanifold X is that *we can evaluate f on any S-point $x \in X(S)$ to obtain $f(x) := f_S(x) \in C^\infty(S)$.* We note that f is an even (resp. odd) function if $f(x)$ is an even (resp. odd) function on S for every S and every S-point $x \in X(S)$.

Let us illustrate the evaluation of functions on supermanifolds on S-points in some examples. The easy proofs of these statements are left to the reader.

Example 5.34. 1. We recall that $C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \Lambda[\theta^1, \dots, \theta^q]$. Here θ^i are the odd coordinate functions, and we will write $t^1, \dots, t^p \in C^\infty(\mathbb{R}^p) \subset C^\infty(\mathbb{R}^{p|q})$ for the even coordinate functions. We recall that the set of S-points $\mathbb{R}^{p|q}(S)$ can be identified with tupels $(t_1, \dots, t_p, \theta_1, \dots, \theta_q)$ with $t_i \in C^\infty(S)^{ev}$, $\theta_j \in C^\infty(S)^{odd}$ (see Theorem 4.18). We want to emphasize that despite their typographical similarities, the t^i's and θ^j's are very different mathematical objects than the t_i's and θ_j's: the former are functions on $\mathbb{R}^{p|q}$, while the ladder are functions on S. While this might be confusing, we have long been accustomed to this: we write $t^1, \dots, t^p \in C^\infty(\mathbb{R}^p)$ for the coordinate functions on $\mathbb{R}^p$, but $(t_1, \dots, t_p)$ for a point in $\mathbb{R}^p$ (so that $t_i \in \mathbb{R} = C^\infty(\mathrm{pt})$). Let us write $x^1, \dots, x^{p+q}$ for the coordinate functions $t^1, \dots, t^q, \theta^1, \dots, \theta^q \in C^\infty(\mathbb{R}^{p|q})$ and $(x_1, \dots, x_{p+q})$ for the S-point $(t_1, \dots, t_q, \theta_1, \dots, \theta_q) \in \mathbb{R}^{p|q}(S)$. To illustrate the evaluation of functions on an S-point of $\mathbb{R}^{p|q}$, we evaluate the functions x^i and $x^i x^j$ on the S-point $(x_1, \dots, x_{p+q})$ to obtain:

$$\begin{aligned} x^i(x_1, \dots, x_{p+q}) &= x_i \\ x^i x^j(x_1, \dots, x_{p+q}) &= x^i(x_1, \dots, x_{p+q}) x^j(x_1, \dots, x_{p+q}) = x_i x_j \end{aligned}$$

2. Let E be a rank r vector bundle over a p-manifold X, and let ΠE be the associated supermanifold of dimension $p|r$. We recall that

$$\begin{aligned} C^\infty(\Pi E) = C^\infty(X; \Lambda^*(E^\vee)) &= \bigoplus_{k=1}^{q} C^\infty(X; \Lambda^k(E^\vee)) \\ &= C^\infty(X) \oplus C^\infty(X, E^\vee) \oplus C^\infty(X, \Lambda^2(E^\vee)) \oplus \dots \end{aligned}$$

and

$$(\Pi E)(S) = \{(x, v) \mid \ x \in X(S), \ v \in C^\infty(x^* E)^{odd}\}$$

Then

$$f(x, v) = \begin{cases} x^* f \in C^\infty(S)^{ev} & f \in C^\infty(X) \\ \langle x^* f, v \rangle \in C^\infty(S)^{odd} & f \in C^\infty(X, E^\vee) \end{cases}$$

Here $x^* \colon C^\infty(X) \to C^\infty(S)$ is the algebra homomorphism induced by $x \in \mathsf{SM}(S, X)$, and

$$\langle \ , \ \rangle \colon C^\infty(S, x^* E^\vee) \otimes C^\infty(S, x^* E) \longrightarrow C^\infty(S)$$

is the pairing induced by the fiberwise evaluation pairing. We note that the algebra $C^\infty(\Pi E)$ is generated by $C^\infty(X)$ and $C^\infty(E^\vee)$, and hence the formula above allows us to calculate $f(x, v)$ for *every* function f.

3. Specializing the above example to $E = TX$ and $f = dg \in \Omega^1(X)$, and

$$v = a \otimes D \in C^\infty(x^* TX) = C^\infty(S) \otimes_{C^\infty(X)} C^\infty(TX) = C^\infty(S) \otimes_{C^\infty(X)} \mathrm{Der}(C^\infty(X))$$

we obtain

$$(dg)(x, v) = \langle x^*(dg), a \otimes D \rangle = a x^* \langle dg, D \rangle = a x^*(Dg)$$

Here Dg stands for applying the derivation $D \colon C^\infty(X) \to C^\infty(X)$ to the function g (note that there is no sign involved when switching the symbols D and g since X is a manifold here).

Proposition 5.35. *Let G be the monoid $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}$ in $\widehat{\mathsf{SM}}$. Let m be the multiplication map. Then*

$$m^*(t) = t \otimes t \qquad m^*(\theta) = \theta \otimes 1 + t \otimes \theta$$

Proof. Let $(g_1, g_0), (g_1', g_0') \in (\mathbb{R}^{0|1} \rtimes \mathbb{R})(S) = \mathbb{R}^{0|1}(S) \times \mathbb{R}(S)$. We recall that

$$m((g_1, g_0), (g_1', g_0')) = (g_1 + g_0 g_1', g_0 g_0')$$

and hence for any $f \in C^\infty(\mathbb{R}^{0|1} \rtimes \mathbb{R})$ we have

$$(m^*(f))((g_1, g_0), (g_1', g_0')) = f(m((g_1, g_0), (g_1', g_0'))) = f(g_1 + g_0 g_1', g_0 g_0')$$

Evaluating the coordinate functions $t, \theta \in C^\infty(\mathbb{R}^{0|1} \rtimes \mathbb{R}) = C^\infty(\mathbb{R})[\theta]$ on an S-point (g_1, g_0) we get

$$t(g_1, g_0) = g_0 \qquad \theta(g_1, g_0) = g_1$$

Hence

$$\begin{aligned}(m^*(t))((g_1, g_0), (g_1', g_0')) &= t(g_1 + g_0 g_1', g_0 g_0') = g_0 g_0' = t(g_1, g_0) t(g_1', g_0') \\ &= (t \otimes t)((g_1, g_0), (g_1', g_0'))\end{aligned}$$

and

$$\begin{aligned}(m^*(\theta))((g_1, g_0), (g_1', g_0')) &= \theta(g_1 + g_0 g_1', g_0 g_0') = g_1 + g_0 g_1' \\ &= \theta(g_1, g_0) + t(g_1 + g_0)\theta(g_1'(g_1', g_0') \\ &= (\theta \otimes 1 + t \otimes \theta)((g_1, g_0), (g_1', g_0'))\end{aligned}$$

□

Proposition 5.36. *Let $\mu \colon Y \times G \to Y$ be the right action of the monoid $G = \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ on the generalized supermanifold $Y = \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$. Let $\omega \in \Omega^*(X) \cong C^\infty(Y)$ be a differential form of degree k. Then*

$$\mu^*(\omega) = \omega \otimes t^k - (-1)^k d\omega \otimes \theta t^k$$

Proof. We recall that on S-points the action map μ has the explicit description

$$\mu((x, v), (g_1, g_0)) = (x + g_1 v, g_0 v)$$

where as in Example 5.34

- x is an S-point of the manifold X, and
- $v = a \otimes D \in C^\infty(S) \otimes_{C^\infty(X)} C^\infty(TX)$ is a tangent vector at v ($D \in C^\infty(TX)$ is algebraically speaking a derivation of $C^\infty(X)$).

Hence for any $\omega \in C^\infty(\Pi TX) = \Omega^*(X)$ we have

$$(\mu^*(\omega))((x,v),(g_1,g_0)) = \omega(\mu((x,v),(g_1,g_0))) = \omega(x + g_1 v, g_0 v)$$

We first calculate $\mu^*(\omega)$ for $\omega = f \in \Omega^0(X)$ and $\omega = dg \in \Omega^1(X)$.

$$\begin{aligned}(\mu^*(f))((g_1,g_0),(x,v)) &= f(x+g_1v, g_0v) = x^*(f) + g_1v^*(f) = x^*(f) - v^*(f)g_1\\ &= f(x,v)) - df(x,v)\theta(g_1,g_0) = (f\otimes 1 - df\otimes\theta)((g_1,g_0),(x,v))\end{aligned}$$

$$\begin{aligned}(\mu^*(dh))((g_1,g_0),(x,v)) &= dh(x+g_1v, g_0v) = g_0v^*(h) = v^*(h)g_0\\ &= dh(x,v)t(g_1,g_0) = (dh\otimes t)((g_1,g_0),(x,v))\end{aligned}$$

This shows that $\mu^*(f) = f \otimes 1 - df \otimes \theta$ and $\mu^*(dh) = dh \otimes t$. For $\omega = f dh_1 \dots dh_k \in \Omega^k(X)$ we conclude

$$\begin{aligned}\mu^*(\omega) &= \mu^*(f)\mu^*(dh_1)\dots\mu^*(dh_k) = (f\otimes 1 - df\otimes\theta)(dh_1\otimes t)\dots(dh_k\otimes t)\\ &= f dh_1\dots dh_k \otimes t^k - (-1)^k df dh_1\dots dh_k\otimes\theta t^k = \omega\otimes t^k - (-1)^k d\omega\otimes\theta t^k\end{aligned}$$

Here the sign $(-1)^k$ is due to moving the odd element θ past the k odd elements $dh_1, \dots, dh_k$. □

6 Gauged field theories and equivariant de Rham cohomology

Let G be a Lie group that acts on a manifold X. The goal of this section is to show that there is an isomorphism between the equivariant de Rham cohomology group $H^n_{dR,G}(X)$ and concordance classes of G-gauged $0|1$-EFT's over X.

6.1 Equivariant cohomology

Equivariant de Rham cohomology was introduced by Henri Cartan in 1950. It took nine more years before Armand Borel introduced equivariant cohomology for topological spaces equipped with the action of a topological group. Reversing historical order, we first introduce equivariant cohomology in this subsection before discussing equivariant de Rham cohomology in the next subsection.

Definition 6.1. [**Borel cohomology**] Let G be a topological group and let X be a topological space equipped with a left G-action $\mu\colon G \times X \to X$. Let EG be a space equipped with a right G-action such that

- the space EG is contractible, and
- the G-action is free.

We remark that these two properties characterize G up to G-homotopy equivalence. Then

$$H^n_G(X) := H^n(EG \times_G X)$$

is the *n-th G-equivariant cohomology group* of X. Here $EG \times_G X$ is the quotient space of the diagonal G-action on $EG \times X$ (with respect to the left G-action on EG given by $gx := xg^{-1}$ for $g \in G$, $x \in X$).

Remark 6.2. 1. We note that the G-action on $EG \times X$ is *free*. The quotient space $EG \times_G X$ is called the *homotopy quotient space* or *Borel construction*.

2. It is customary to call the quotient space $BG := EG/G$ the *classifying space*, since it classifies principal G-bundles in the sense that for any topological space X there is a bijection

$$\{f\colon X \to BG\}/\text{homotopy} \;\longleftrightarrow\; \{\text{principal } G\text{-bundles over } X\}/\text{isomorphism}$$

This bijection is given by sending a map $f\colon X \to BG$ to the pull-back of the principal G-bundle $EG \to BG$ via f.

Example 6.3. 1. If G acts freely on a space X (via a left action), then the projection map $p\colon EG \times_G X \to G\backslash X$ is a fiber bundle with fiber EG. Since EG is contractible, p is a homotopy equivalence and hence

$$H^*_G(X) = H^*(EG \times_G X) \cong H^*(G\backslash X)$$

2. $H^*_G(\mathrm{pt}) = H^*(EG \times_G \mathrm{pt}) = H^*(BG)$.

3. If G acts trivially on X, then $H^*_G(X) = H^*(EG \times_G X) = H^*(BG \times X)$ which can be calculated via the Künneth formula in terms of $H^*(BG)$ and $H^*(X)$.

4. In general, the projection map $EG \times_G X \to BG$ is a fiber bundle with fiber X. The associated Leray-Hirsch spectral sequence

$$H^p(BG; H^q(X)) \implies H^{p+q}(EG \times_G X) = H^{p+q}_G(X)$$

provides a good tool for the calculation of the equivariant cohomology.

5. If $H \subset G$ is a subgroup of G, then

$$H^*_G(G/H) = H^*(EG \times_G G/H) = H^*(EG \times_H \mathrm{pt}) = H^*(BH),$$

since EG serves also as a model for EH.

Lemma 6.4. *Let the circle group $\mathbb{T}$ act on a manifold X. Then the action is free if and only if $H^i_{\mathbb{T}}(X) = 0$ for all $i \geq \dim X$.*

Proof. If $G = \mathbb{T}$ acts freely, then $H^*_G(X)$ is the cohomology of the quotient manifold $G\backslash X$ of dimension $\dim X - 1$. In particular, $H^q_G(X) = 0$ for $q \geq \dim X$.

To prove the converse, assume that the action is not free, that is, there is a point $x \in X$ whose isotropy subgroup $G_x = \{g \in G \mid gx = x\}$ is non-trivial. Then inclusion of the orbit through x and the projection onto pt are equivariant maps

$$Gx \xrightarrow{i} X \xrightarrow{p} \mathrm{pt}$$

which induce maps

$$BH_x = EG \times_G G/H_x \longrightarrow EG \times_G X \longrightarrow BG$$

on homotopy orbit spaces. In other words, the map of classifying spaces $Bi\colon BH_x \to BG$ induced by the inclusion map $i\colon H_x \hookrightarrow G$ factors through $EG \times_G X$. In particular, the map in cohomology $Bi^*\colon H^*(BG) \to H^*(BH_x)$ factors through $H^*_G(X)$.

If x is a fixed point, then the isotropy subgroup H_x is equal to G, and hence $H^{2q}(BG) = \mathbb{Z}$ implies that $H^{2q}_G(X) \neq 0$ for all q. If x is not a fixed point, but H_x is a cyclic subgroup $\mathbb{Z}/k \subset \mathbb{T} = G$. The inclusion map $i\colon \mathbb{Z}/k \to \mathbb{T}$ induces on even dimensional cohomology a surjective map

$$Bi^*\colon H^{2q}(B\mathbb{T}) \cong \mathbb{Z} \twoheadrightarrow H^{2q}(B\mathbb{Z}/k) = \mathbb{Z}/k$$

which again implies $H^{2q}_G(X) \neq 0$ for all q. □

6.2 Differential forms on G-manifolds.

The goal of this section is the study of the algebraic structure on the algebra $\Omega^\bullet(X)$ of differential forms on a manifold X induced by the action of a Lie group G on X. We will see that $\Omega^\bullet(X)$ is a G^*-algebra in the sense of Definition **??** below.

Digression on vectorfields and differential forms. Associated to a vectorfield v on a manifold X, there are two graded derivations on the algebra $\Omega^\bullet(X)$:

The contraction operator ι_v. This is a graded derivation which has degree -1 in the sense that if ω is a k-form, then $\iota_v\omega$ is a $(k-1)$-form. Explicitly, it is given by

$$\iota_v\omega(v_1,\dots,v_{k-1}) := \omega(v,v_1,\dots,v_{k-1}) \qquad \text{for} \quad v_i \in T_xX$$

The Lie derivative L_v. This is a graded derivation of degree 0, in other words, it is degree preserving. Explicitly, it can be defined by the formula

$$L_v := [d,\iota_v] = d\iota_v + \iota_v d, \tag{6.5}$$

often referred to as the *Cartan formula*. Alternatively, the Lie derivative can be described more geometrically as

$$L_v\omega = \frac{d}{dt}_{|t=0}\,\varphi_t^*\omega,$$

where φ_t is flow generated by the vectorfield v. In other words, $\varphi_t\colon X \to X$ is the 1-parameter group of diffeomorphisms of X determined by v via the differential equation

$$\frac{d}{dt}_{|t=0}\,\varphi_t(x) = v(x) \in T_xX \quad \text{for every } x \in X.$$

Lemma 6.6. *Let v,w be vectorfields on a manifold X. Then the operators ι_v, ι_w, L_v, L_w satisfy the following graded commutation relations*

$$[\iota_v,\iota_w] = 0 \qquad [L_v,\iota_w] = \iota_{[v,w]} \qquad [L_v,L_w] = L_{[v,w]} \tag{6.7}$$

Here $[v,w] = vw - wv$ is the commutator of the vectorfields v,w thought of as derivations acting on $C^\infty(X)$.

Proof. The algebra $\Omega^\bullet(X)$ is generated by functions $f \in C^\infty(X) = \Omega^0(X)$ and their differentials $df \in \Omega^1(X)$. Hence for checking the bracket relations above, it suffices evaluate these graded derivations of $\Omega^\bullet(X)$ on differential forms of this type. For example, the derivation $[\iota_v,\iota_w]$ has degree -2 and hence evaluating it on $\Omega^i(X)$ for $i = 0,1$ gives zero, which proves the first relation. Similarly, both sides of the second relation have degree -1 and hence evaluating them on functions gives zero.

The key for proving the second relation is that for a vectorfield v and a function f we have $L_vf = \iota_v df = vf$ (here vf mean applying the derivation v to the function f). Evaluating $[L_v,\iota_w]$ on df we have

$$[L_v,\iota_w]df = L_v\iota_w df - \iota_w L_v df = L_v\iota_w df - \iota_w dL_v f = vwf - wvf = [v,w]f = L_{[v,w]}f$$

Here the second equation holds since $L_v = d\iota_v + \iota_v d$ and d commute.

To prove the last relation, we use the fact that the graded commutator is a derivation w.r.t. the product given by the graded commutator (writing out the graded commutators is another possibility). This gives the second of the following equalities.

$$[L_v, L_w] = [L_v, [d, \iota_w]] = [[L_v, d], \iota_w] + [d, [L_v, \iota_w] = [d, \iota_{[v,w]}] = L_{[v,w]}$$

The third equation follows from the vanishing of $[L_v, d]$ and the second relation. □

Corollary 6.8. *The action of a Lie algebra $\mathfrak{g}$ by derivations on $C^\infty(X)$ induces an action of the super Lie algebra $\mathfrak{sg}$ defined below on $\Omega^\bullet(X)$.*

Definition 6.9. Given a Lie algebra $\mathfrak{g}$, we define the super Lie algebra $\mathfrak{sg}$ as follows. As super vector space $\mathfrak{sg} = \mathfrak{g} \oplus \Pi\mathfrak{g}$ (i.e., the sum of two copies of $\mathfrak{g}$, one of which has even, and one of which has odd parity). The Lie bracket is given by the relations (6.7), where we write L_v (resp. ι_v) for the even (resp. odd) element of $\mathfrak{sg}$ given by $v \in \mathfrak{g}$.

There is a super Lie algebra that acts tautologically on $\Omega^\bullet(X)$ for any manifold X, namely the super Lie algebra $\mathfrak{d}$ spanned by one even element N and one odd element Q equipped with the graded commutator relations

$$[N, Q] = Q \qquad [Q, Q] = 0$$

This acts on $\Omega^\bullet(X)$ by letting Q act as the de Rham differential d, and letting N act as the *grading operator* defined by $N\omega = k\omega$ for $\omega \in \Omega^k(X)$. The first relation holds since the de Rham differential has degree +1, the second holds since $d^2 = 0$.

We note that the Lie algebra actions of $\mathfrak{sg}$ and $\mathfrak{d}$ on $\Omega^\bullet(X)$ don't commute. Rather, we have the commutator relations

$$[Q, \iota_v] = L_v \qquad [Q, L_v] = 0 \qquad [N, \iota_v] = \iota_v \qquad [N, L_v] = 0 \tag{6.10}$$

The next lemma shows that we can interpret these bracket relations as defining an action of the Lie algebra $\mathfrak{d}$ on $\mathfrak{sg}$ by Lie algebra endomorphisms.

Lemma 6.11. *Let $\rho\colon \mathfrak{d} \to \mathrm{End}(\mathfrak{sg})$ be the linear map defined by the commutator relations* (6.10)*. Then*

1. *ρ is a Lie algebra homomorphisms, and*
2. *the image of ρ consists of derivations of the Lie algebra $\mathfrak{sg}$.*

Proof. Explicitly, the linear map ρ is given by

$$\rho(Q)\iota_v = [Q, \iota_v] = L_v \qquad \rho(Q)L_v = [Q, L_v] = 0$$
$$\rho(N)\iota_v = [N, \iota_v] = -\iota_v \qquad \rho(N)L_v = [N, L_v] = 0$$

To prove part (1), we first note that $[\rho(Q), \rho(Q)] = 0 = \rho([Q, Q])$. To check that the bracket relations involving N are preserved by ρ, we note that N and $\rho(N)$ can be interpreted as grading operators for a $\mathbb{Z}$-grading on $\mathfrak{d}$ and $\mathfrak{sg}$, respectively (explicitly, $\deg(Q) = +1$, $\deg(L_v) = 0$, $\deg(\iota_v) = -1$). Then compatibility of ρ with bracket involving N is equivalent to ρ being grading preserving, which it is, since $\deg(Q) = 1$ and the degree of the operator $\rho(Q)\colon \mathfrak{sg} \to \mathfrak{sg}$ is $+1$ as well.

To prove part (2), we need to show that $\rho(N)$ and $\rho(Q)$ are derivations of $\mathfrak{sg}$. For $\rho(N)$, this is equivalent to saying that $\mathfrak{sg}$ is a $\mathbb{Z}$-graded Lie algebra, i.e., that the Lie bracket of elements of degree m resp. n is an element of degree $m + n$. This is the case, as a glance at the defining relations (6.7) shows.

To show that $\rho(Q)$ is a graded derivation of $\mathfrak{sg}$, we need to check for each of the three bracket relations (6.7). Concerning the relation $[\iota_v, \iota_w] = 0$, we calculate:

$$\rho(Q)[\iota_v, \iota_w] = 0$$
$$[\rho(Q)\iota_v, \iota_w] - [\iota_v, \rho(Q)\iota_w] = [L_v, \iota_w] - [\iota_v, L_w] = [L_v, \iota_w] + [L_w, \iota_v] = L_{[v,w]} + L_{[w,v]} = 0$$

For the relation $[L_v, \iota_w] = L_{[v,w]}$, we check

$$\rho(Q)[L_v, \iota_w] = \rho(Q)\iota_{[v,w]} = L_{[v,w]}$$
$$[\rho(Q)L_v, \iota_w] + [L_v, \rho(Q)\iota_w] = [L_v, L_w] = L_{[v,w]}$$

Finally, applying concerning the relation $[L_v, L_w] = L_{[v,w]}$ we have

$$\rho(Q)[L_v, L_w] = \rho(Q)L_{[v,w]} = 0$$
$$[\rho(Q)L_v, L_w] + [L_v, \rho(Q)L_w] = 0$$

which finishes the proof that $\rho(Q)$ is a derivation of $\mathfrak{sg}$. □

Digression on semidirect products of Lie algebras. Let $\mathfrak{g}$, $\mathfrak{h}$ be (super) Lie algebras and assume that $\rho\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{g})$ is a (super) Lie algebra homomorphism. Here $\mathrm{Der}(\mathfrak{g})$ is the Lie algebra of (graded) derivations of $\mathfrak{g}$ consisting of all linear maps $A\colon \mathfrak{g} \to \mathfrak{g}$ such that

$$A([v, w]) = [Av, w] + (-1)^{|A||v|}[v, Aw] \qquad \text{for } v, w \in \mathfrak{g}$$

for homogeneous elements $v, w \in \mathfrak{g}$. Then the *semidirect product* $\mathfrak{h} \ltimes \mathfrak{g}$ is defined to be the vector space $\mathfrak{h} \oplus \mathfrak{g}$ with a (graded) Lie bracket $[\ ,\ ]$ given by

$$[x, y] = [x, y]_{\mathfrak{h}} \qquad [v, w] = [v, w]_{\mathfrak{g}} \qquad [x, v] = \rho(x)v \qquad \text{for } x, y \in \mathfrak{h},\ v, w \in \mathfrak{g}$$

where $[x,y]_{\mathfrak{h}}$ and $[v,w]_{\mathfrak{g}}$ is the Lie bracket of $\mathfrak{h}$ and $\mathfrak{g}$, respectively.

From now we will assume that a compact Lie group G acts on the manifold X via a smooth action map

$$\mu\colon X \times G \longrightarrow X.$$

Given an element v of the Lie algebra $\mathfrak{g}$ of G, we obtain a vectorfield $\mu_*(v)$ on X as the composition

$$C^\infty(X) \xrightarrow{\mu^*} C^\infty(X \times G) \cong C^\infty(X) \otimes C^\infty(G) \xrightarrow{1\otimes v} C^\infty(X) \otimes \mathbb{R} \cong C^\infty(X)$$

Suppressing the action map μ in the notation, we will often write v instead instead of $\mu_*(v)$. In particular, a Lie algebra element $v \in \mathfrak{g}$ will determine derivations ι_v, L_v of $\Omega^\bullet(X)$ of degree -1 resp. 0.

6.3 The Weil algebra

In this section we will construct for every compact Lie group G a G^*-algebra W called the *Weil algebra*. Moreover, we will show that the G^*-algebra W is acylic and that it has a connection θ. In particular, we can use the Weil algebra as an algebraic model $\mathcal{E}$ for the de Rham complex on EG.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}^\vee = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$ be its dual, considered as $\mathbb{Z}$-graded vector spaces concentrated in degree 0. Let $\mathfrak{g}^\vee[-1]$ (resp. $\mathfrak{g}^\vee[-2]$) be shifted versions of $\mathfrak{g}^\vee$ (see Definition **??**); in particular, all elements in $\mathfrak{g}^\vee[-1]$ have degree 1, and all elements of $\mathfrak{g}^\vee[-2]$ have degree 2. The *Weil algebra* is defined to be the tensor product

$$W := \Lambda(\mathfrak{g}^\vee[-1]) \otimes S(\mathfrak{g}^\vee[-2])$$

of the exterior algebra $\Lambda(\mathfrak{g}^\vee[-1])$ generated by $\mathfrak{g}^\vee[-1]$ and the symmetric algebra $S(\mathfrak{g}^\vee[-2])$ generated by $\mathfrak{g}^\vee[-2]$. For $f \in \mathfrak{g}^\vee$ we will use the notation

$$f[-1] \in \mathfrak{g}^\vee[-1] \subset \Lambda(\mathfrak{g}^\vee) \subset W \qquad f[-2] \in \mathfrak{g}^\vee[-2] \subset S(\mathfrak{g}^\vee) \subset W$$

for the corresponding elements of the Weil algebra.

The Weil algebra has the structure of a G^*-algebra, consisting of a $\mathbb{Z}$-grading, a G-action and an action of the super Lie algebra $\widetilde{\mathfrak{g}}$ as follows:

$\mathbb{Z}$-grading The $\mathbb{Z}$-grading on W is the implicit in the construction: $\mathfrak{g}^\vee[-1]$ (resp. $\mathfrak{g}^\vee[-2]$) has degree 1 (resp. 2), and this determines a $\mathbb{Z}$-grading on the algebra W.

G-action G acts on $\mathfrak{g}^\vee$ by the *coadjoint action* (the dual of the adjoint action on $\mathfrak{g}$).

$\widetilde{\mathfrak{g}}$-action We recall that $\widetilde{\mathfrak{g}} = \mathbb{R}N \oplus \mathbb{R}Q \oplus \mathfrak{g} \oplus \mathfrak{g}[1]$, and that for $v \in \mathfrak{g}$ we write $L_v \in \widetilde{\mathfrak{g}}$ (resp. ι_v) for v considered as an element of $\mathfrak{g} \subset \widetilde{\mathfrak{g}}$ (resp. $\mathfrak{g}[1] \subset \widetilde{\mathfrak{g}}$). The action of these generators on W is the following.

1. N acts on W as the grading operator, i.e., N is a derivation with $Nw = \deg(w)w$ for any element $w \in W$. This is the required compatibility condition between the $\mathbb{Z}$-grading and the $\widetilde{\mathfrak{g}}$-action.
2. Q acts by $f[-1] \mapsto f[-2]$ and $f[-2] \mapsto 0$. In particular, $Q^2 = 0$, and Q has degree $+1$, which is required by the relations $[Q, Q] = 0$ and $[N, Q] = Q$ in the Lie algebra $\widetilde{g}$.
3. The action of $L_v \in \widetilde{\mathfrak{g}}$ is required to be the action induced by the G-action, i.e., $L_v(f[-1]) = (\mathrm{ad}_v^\vee f)[-1]$ and $L_v(f[-2]) = (\mathrm{ad}_v^\vee f)[-2]$. We let ι_v act on $f[-1]$ by $\iota_v(f[-1]) = \langle v, f\rangle \in \mathbb{R}$, where $\langle\ ,\ \rangle$ is the evaluation pairing. By the Cartan relation $L = [d, \iota_v]$, this determines the ι_v-action on $f[-2]$:

$$\iota_v(f[-2]) = \iota_v d(f[-1]) = (L_v - d\iota_v)(f[-1]) = \mathrm{ad}_v^\vee(f[-2]) - d\langle v, f\rangle = \mathrm{ad}_v^\vee(f[-2])$$

Lemma 6.12. *The Weil algebra W is a model for forms on EG, i.e., W is acyclic, and it has a connection $\theta \in W^1 \otimes \mathfrak{g}$.*

Proof. To prove that W is acyclic, consider the graded derivation $\epsilon\colon W \to W$ of degree -1 determined by $f[-2] \mapsto f[-1]$ and $f[-1] \mapsto 0$. Then $M := [\epsilon, d] = \epsilon d + d\epsilon\colon W \to W$ maps $f[-2]$ to $f[-2]$ and $f[-1]$ to $f[-1]$. Since M is a graded derivation, it follows that for $w \in \Lambda^k(\mathfrak{g}[-1]) \otimes S^l(\mathfrak{g}[-2])$ we have $Mw = (k+l)w$. If W_m denotes the eigenspace of M with eigenvalue $m \in \mathbb{Z}$, the differential d preserves W_m since d commutes with M, and hence

$$H^n(W, d) = \bigoplus_{m\in\mathbb{Z}} H^n(W_m, d)$$

Since $M = \epsilon d + d\epsilon$, the map ϵ can be interpreted as a chain homotopy between M and 0. In particular, M induces the trivial map on cohomology. Since M is just multiplication by m on W_m, this implies that $H^*(W_m) = 0$ except for $m = 0$.

To construct the connection θ, let $\{v_i\}$ be a basis of $\mathfrak{g}$, let $\{f^i\}$ be the dual basis of $\mathfrak{g}^\vee$, and define

$$\theta := \sum_i f^i[-1] \otimes v_i \in \mathfrak{g}[-1] \otimes \mathfrak{g} = W^1 \otimes \mathfrak{g}$$

Then

$$\iota_v\theta = \sum_i \iota_v(f^i[-1]) \otimes v_i = \sum_i \langle v, f^i\rangle v_i = v$$

Moreover, θ is clearly invariant under the G-action on $\mathfrak{g}^\vee \otimes \mathfrak{g}$ (via the canonical isomorphism $\mathfrak{g}^\vee \otimes \mathfrak{g} \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ it corresponds to the identity on $\mathfrak{g}$). □

Our next goal is a description of the super Lie algebra $\widetilde{\mathfrak{g}}$ and its action on $\Omega^\bullet(X)$ for a G-manifold X in terms of supermanifolds.

Proposition 6.13. *The super Lie algebra* $\widetilde{\mathfrak{g}}$ *is the Lie algebra of the super Lie group*

$$\widetilde{G} := \underline{\mathsf{Diff}}(\mathbb{R}^{0|1}) \ltimes \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)$$

Moreover, the action of $\widetilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G})$ *on* $\Omega^\bullet(X) = C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X))$ *described in* **??** *is induced by natural action of* $\widetilde{G}$ *on* $\underline{\mathsf{SM}}(\mathbb{R}^{0|1})$ *described below.*

The super Lie group $\underline{\mathsf{Diff}}(\mathbb{R}^{0|1}) \ltimes \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)$ **and its action on** $\underline{\mathsf{SM}}(\mathbb{R}^{0|1})$**.** Let G be a Lie group and $\mathrm{S}G = \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)$ the generalized supermanifold of maps from $\mathbb{R}^{0|1}$ to G. We recall from **??** that a generalized supermanifold Y is a functor $\mathsf{SM}^{op} \to \mathsf{Set}$. We also recall that for a supermanifold S, the set $Y(S)$ of *S-points of Y* is the image of S under the functor Y. For $Y = \mathrm{S}G$ its set of S-points is given by (see **??**):

$$\mathrm{S}G(S) = \mathsf{SM}(S \times \mathbb{R}^{0|1}, G)$$

To describe the multiplication map $m\colon \mathrm{S}G \times \mathrm{S}G \to \mathrm{S}G$ which is a map of generalized supermanifolds, we need to give a family of maps (between sets)

$$m_S\colon\ \mathrm{S}G(S) \times \mathrm{S}G(S) \to \mathrm{S}G(S)$$

which is natural in S. This is given by

$$(g_1\colon S \times \mathbb{R}^{0|1} \to G, g_2\colon S \times \mathbb{R}^{0|1} \to G) \quad \mapsto \quad S \times \mathbb{R}^{0|1} \xrightarrow{g_1 \times g_2} G \times G \longrightarrow G$$

The monoid $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ acts on $\mathrm{S}G$ from the right. To describe the action map

$$\mu\colon\ \mathrm{S}G \times \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \longrightarrow \mathrm{S}G$$

on S-points, it is useful to think of $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(S) = \mathsf{SM}(S \times \mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ as the set of maps $g\colon S \times \mathbb{R}^{0|1} \to S \times \mathbb{R}^{0|1}$ such that the diagram

$$\begin{array}{ccc} S \times \mathbb{R}^{0|1} & \xrightarrow{g} & S \times \mathbb{R}^{0|1} \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & S & \end{array}$$

is commutative. Then the action map μ is given on S-points by

$$\begin{array}{ccc} \mathrm{S}G(S) \times \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(S) & \xrightarrow{\mu(S)} & \mathrm{S}G(S) \\ (S \times \mathbb{R}^{0|1} \xrightarrow{f} G, S \times \mathbb{R}^{0|1} \xrightarrow{g} S \times \mathbb{R}^{0|1}) & \mapsto & f \circ g \end{array}$$

Explain semi-direct products and the action of $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \ltimes \mathrm{S}G$ on $\mathrm{S}X$.

Lemma 6.14. *The Lie algebra* $\mathfrak{g}$ *of the super Lie group* $\mathrm{S}G$ *is isomorphic to the subalgebra* $\mathfrak{g} \oplus \mathfrak{g}[1] \subset \widetilde{\mathfrak{g}}$.

Proof. We recall that the Lie algebra of the super Lie group $\mathrm{S}G$ is by definition the Lie algebra of left invariant vectorfields on $\mathrm{S}G$, also known as left invariant graded derivations of $C^\infty(\mathrm{S}) = \Omega^\bullet(G)$. Given an element $v \in \mathfrak{g}$, the corresponding elements $L_v, \iota_V \in \mathfrak{g} \oplus \mathfrak{g}[1]$ have an obvious interpretation as graded derivations on $\Omega^\bullet(G)$ ($\iota_v \colon \Omega^\bullet(X) \to \Omega^\bullet(X)$ is the contraction with the left invariant vectorfield v, and L_v is the Lie derivative). It remains to show that these are *left-invariant* vectorfields on $\mathrm{S}G$. Then this construction gives a linear map

$$\mathfrak{g} \oplus \mathfrak{g}[1] \longrightarrow \mathfrak{sg}$$

which is obviously injective and a Lie algebra homomorphism by construction of the super Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}[1] \subset \widetilde{g}$. Since the super dimensions of these two super Lie algebras agree, this map is an isomorphism of super Lie algebras.

We recall that a vectorfield v on G is left-invariant if the following diagram commutes.

$$\begin{array}{ccc} C^\infty(G) & \xrightarrow{m^*} & C^\infty(G) \otimes C^\infty(G) \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle 1\otimes v} \\ C^\infty(G) & \xrightarrow{m^*} & C^\infty(G) \otimes C^\infty(G) \end{array}$$

Now let $v \in \mathfrak{g}$, i.e., v is a left-invariant vector field on G. We need to show that ι_v and L_v are left-invariant vector fields on $\mathrm{S}G$, which means we have to prove commutativity of the corresponding diagram with $C^\infty(G)$ replaced by $C^\infty(\mathrm{S}G) = \Omega^\bullet(X)$, m^* by $(\mathrm{S}m)^*$, and v by L_v resp. ι_v. We note that

$$(\mathrm{S}m)^* \colon C^\infty(\mathrm{S}G) \longrightarrow C^\infty(\mathrm{S}G) \otimes C^\infty(\mathrm{S}G)$$

can be identified with the map

$$m^* \colon \Omega^\bullet(G) \longrightarrow \Omega^\bullet(G) \otimes \Omega^\bullet(G)$$

induced by the multiplication map on differential forms (and identifying $\Omega^\bullet(G \times G)$ with $\Omega^\bullet(G) \otimes \Omega^\bullet(G)$). Hence showing that a vectorfield $W \colon \Omega^\bullet(G) \to \Omega^\bullet(G)$ on $\mathrm{S}G$ is left-invariant amounts to proving

$$(1 \otimes W) m^* \omega = m^*(W\omega) \tag{6.15}$$

for all $\omega \in \Omega^\bullet(G)$. Since the algebra $\Omega^\bullet(G)$ is generated by functions and 1-forms of the form $\omega = df$ for $f \in C^\infty(G)$, it suffices to show that this equality holds for $\omega = f \in \Omega^0(G) = C^\infty(G)$ and $\omega = df \in \Omega^1(G)$.

First we show that the vectorfield $W = \iota_v$ is left-invariant. For $\omega = f$ equation (6.15) holds (both sides are zero since ι_v has degree -1. For $\omega = df$ we calculate

$$\begin{aligned}(1\otimes\iota_v)m^*df &= (1\otimes\iota_v)(d\otimes 1 + 1\otimes d)m^*f = (-d\otimes\iota_v + 1\otimes\iota_v d)m^*f\\ &= (1\otimes\iota_v d)m^*f = (1\otimes v)m^*f = m^*(vf) = m^*(\iota_v df)\end{aligned}$$

The third equality holds since ι_v has degree -1 and hence vanishes on functions. The fourth equality holds since $\iota_v d$ applied to a function h is just vh. The fifth equality expresses the left-invariance of the vectorfield v.

Now we show that the vectorfield $W = L_v$ is left-invariant. For $\omega = f$, the equation (6.15) is just the left-invariance of the vectorfield v. For $\omega = df$ we calculate, using the fact that $L_v = d\iota_v + \iota_v d$ and d commute:

$$\begin{aligned}(1\otimes L_v)m^*df &= (1\otimes L_v)(d\otimes 1 + 1\otimes d)m^*f = (d\otimes 1 + 1\otimes d)(1\otimes L_v)m^*f\\ &= (d\otimes 1 + 1\otimes d)m^*(L_V f) = m^*(dL_v f) = m^*(L_v df)\end{aligned}$$

□

Proof. First we determine the Lie algebra of the super Lie group SG. By definition, this is the vector space $\mathfrak{sg}$ of left invariant vectorfields on SG, also known as left invariant derivations of $C^\infty(\mathrm{S}G) = \Omega^\bullet(G)$. Given an element $v \in \mathfrak{g}$, □

6.4 A geometric interpretation of the Weil algebra

The goal of this section is provide a geometric interpretation of the Weil algebra as the algebra of functions on the supermanifold $\mathcal{A}_G$ of connections on the trivial principal G-bundle over $\mathbb{R}^{0|1}$. Moreover, the super Lie group $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1}) \ltimes \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)$ acts by bundle automorphisms on this bundle and hence the space $\mathcal{A}_G$ of connections. The induced action on $C^\infty(\mathcal{A}_G) \cong W_G$ gives the G^*-algebra structure of the Weil algebra.

We recall that a connection on a principal G-bundle $P \to X$ is a 1-form $\theta \in \Omega^1(P)\otimes\mathfrak{g}$ such that θ is G-invariant (where G acts on $\mathfrak{g}$ via the adjoint action), and $\iota_v\theta = 1\otimes v \in \Omega^0(P)\otimes\mathfrak{g}$. We know what a principal G-bundle over a supermanifold X is (see **??**). In order to generalize the notion of connections to that case, we need to define differential forms on supermanifolds. This can be done in various ways; one is to define 1-forms on a supermanifold X as elements in the module over $C^\infty(X)$ dual to the module $\mathrm{Der}(C^\infty(X))$ of derivations on X. The other one is to define 1-forms in terms of Kähler differentials, and this is the path we want to follow here.

To motivate it, let X be a manifold. Then the differential

$$d\colon C^\infty(X) \longrightarrow \Omega^1(X)$$

is a *derivation* in the sense that the usual Leibniz rule $d(fg) = (df)g + fdg$ holds for $f, g \in C^\infty(X)$ (here we use commutativity of $C^\infty(X)$ to think of $\Omega^1(X)$ as left-module *and* right-module over $C^\infty(X)$ to make sense of the products $(df)g$ and fdg). In fact, d is the *universal* derivation of $C^\infty(X)$ in the sense that *any* derivation

$$D: C^\infty(X) \longrightarrow M$$

factors uniquely through d (the target M is some $C^\infty(X)$-module); the factorizing map sends $fdg \in \Omega^1(X)$ to $fDg \in M$. This algebraic way to think about 1-forms immediately generalizes as follows.

Definition 6.16. Let A be a commutative super algebra and M a graded module over A. A linear map $D: A \to M$ of parity $p(D)$ is a *graded derivation* if the (graded) Leibniz rule

$$D(fg) = (Df)g + (-1)^{p(D)p(f)} fDg$$

holds. There is a universal graded derivation $d: A \to \Omega^1_A$; the elements of Ω^1_A are called *Kähler differentials*. Explicitly, Ω^1_A can be constructed as the A-module generated by elements of the form da for $a \in A$ subject to the following relations:

- $da = 0$ if a is a multiple of the unit $1 \in A$;
- $d(a + b) = da + db$, and
- $d(ab) = (da)b + adb$.

Now these algebraic constructions can be used to define 1-forms and connections on supermanifolds.

Definition 6.17. For a supermanifold X we define $\Omega^1(X)$ to be the Kähler differentials $\Omega^1_{C^\infty(X)}$; this is a graded module over the graded algebra $C^\infty(X)$. Any map $f \in \mathsf{SM}(X, Y)$ induces a commutative diagram

$$\begin{array}{ccc} C^\infty(Y) & \xrightarrow{f^*} & C^\infty(X) \\ \downarrow d & & \downarrow d \\ \Omega^1(Y) & \xrightarrow{f^*} & \Omega^1(X) \end{array}$$

Let G be a (super) Lie group and let $P = X \times G \to X$ be the trivial principal G-bundle over X. A *connection* on P is an element $\theta \in (\Omega^1(X) \otimes \mathfrak{g})^{ev}$.

Definition 6.18. Let $\mathcal{B}$ be the generalized supermanifold of $\mathfrak{g}$-valued 1-forms on $\mathbb{R}^{0|1}$. More precisely, this is a generalized supermanifolds defined by declaring its S-points $\mathcal{B}(S)$ for a supermanifold S to be

$$\begin{array}{rcc} \mathcal{B}(S) := \left(\Omega^1_{S\times\mathbb{R}^{0|1}/S}\otimes\mathfrak{g}\right)^{\mathrm{ev}} & \cong & (C^\infty(S)\otimes\mathfrak{g})^{\mathrm{ev}}\oplus(C^\infty(S)\otimes\mathfrak{g})^{\mathrm{odd}} \\ (\theta_0\eta+\theta_1)d\eta & \leftrightarrow & (\theta_0,\theta_1) \end{array}$$

Both descriptions of the S-points of $\mathcal{B}$ have their advantages: viewing the S-points as pairs (θ_0,θ_1) shows that the generalized supermanifold $\mathcal{B}$ is represented by the supermanifold $\mathfrak{g}\times\Pi\mathfrak{g}$. Looking at S-points as 1-forms makes it obvious that the monoid $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})$ acts on $\mathcal{B}$. On S-points this right action is given by

$$\mathcal{B}(S)\times\mathsf{SM}_S(S\times\mathbb{R}^{0|1},S\times\mathbb{R}^{0|1})\longrightarrow\mathcal{B}(S)\qquad(\theta,d)\mapsto d^*\theta \tag{6.19}$$

More explicitly, the map $d\colon S\times\mathbb{R}^{0|1}\to S\times\mathbb{R}^{0|1}$ which commutes with the projection to S is determined by the elements $d_0\in C^\infty(S)^{\mathrm{ev}}$, $d_1\in C^\infty(S)^{\mathrm{odd}}$ given by

$$d^*\eta=d_0\eta+d_1\in C^\infty(S\times\mathbb{R}^{0|1})^{\mathrm{odd}}\cong(C^\infty(S)\otimes\Lambda[\eta])^{\mathrm{odd}}$$

Given $\theta=(\theta_0\eta+\theta_1)d\eta\in\mathcal{B}(S)$, we have

$$d^*\theta=(\theta_0d^*\eta+\theta_1)dd^*\eta=(\theta_0(d_0\eta+d_1)+\theta_1)d_0d\eta=(\theta_0d_0^2\eta+\theta_0d_1d_0+\theta_1d_0)d\eta$$

So, in explicit terms the action (6.19) is given by

$$(\theta_0,\theta_1),(d_0,d_1)\mapsto(\theta_0d_0^2,\theta_0d_1d_0+\theta_1d_0) \tag{6.20}$$

More explicitly, there is an isom by associating to every compact Lie group G a supermanifold $\mathcal{W}_G$ with a natural action of the super Lie group $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})\ltimes\underline{\mathsf{SM}}(\mathbb{R}^{0|1},G)$ such that the algebra of functions $C^\infty(\mathcal{W}_G)$ as G^*-algebra is isomorphic to the Weil algebra W_G.

Our next goal is to describe explicitly the map induced by the action map (6.19) on functions. To do this we describe functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})$ and $\mathcal{B}$ explicitly in terms of their S-points, recalling that for any generalized supermanifold Y, a function $f\in C^\infty(Y)$ determines a family of maps

$$Y(S)\longrightarrow C^\infty(S)$$

which is natural for $S\in\mathsf{SM}$.

Functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})$ in the S-point formalism. We recall that the generalized supermanifold $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})$ is isomorphic to $\mathbb{R}^{1|1}$ and hence

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})\cong C^\infty(\mathbb{R}^{1|1})\cong C^\infty(\mathbb{R})\otimes\Lambda[\theta]$$

Explicitly,

$$\begin{aligned}\underline{\mathsf{SM}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1})(S) = \mathsf{SM}(S\times\mathbb{R}^{0|1},\mathbb{R}^{0|1}) &\leftrightarrow C^\infty(S)^{ev}\times C^\infty(S)^{odd}\\ d &\leftrightarrow (d_0,d_1),\end{aligned}$$

where the d_i's are determined by $d^*\eta = d_0\eta + d_1$, where

$$d^*\colon C^\infty(\mathbb{R}^{0|1}) = \Lambda[\eta] \longrightarrow C^\infty(S\times\mathbb{R}^{0|1}) = C^\infty(S)[\eta]$$

is the map induced by d. Evaluating the coordinate functions $t\in C^\infty(\mathbb{R})$ and θ on an S-point $(d_0,d_1)\in C^\infty(S)^{ev}\times C^\infty(S)^{odd}$, we have

$$t(d_0,d_1) = d_0 \qquad \theta(d_0,d_1) = d_1$$

Functions on $\mathcal{B}$ in the S-point formalism. We recall that the generalized supermanifold $\mathcal{B}$ is isomorphic to $\mathfrak{g}\times\Pi\mathfrak{g}$ and hence

$$C^\infty(\mathcal{B})\cong C^\infty(\mathfrak{g}\times\Pi\mathfrak{g})\cong C^\infty(\mathfrak{g})\otimes C^\infty(\Pi\mathfrak{g})\supset S(\mathfrak{g}^\vee)\otimes\Lambda(\mathfrak{g}^\vee)$$

Given an element $f\in\mathfrak{g}^\vee$, we denote by $f_0\in S(\mathfrak{g}^\vee)\subset C^\infty(\mathcal{B})$, respectively $f_1\in\Lambda(\mathfrak{g}^\vee)\subset C^\infty(\mathcal{B})$ the corresponding elements. Explicitly, evaluating these elements on an S-point $(\xi_0,\xi_1)\in\mathcal{B}(S)$ we have

$$f_0(\xi_0,\xi_1) = \langle f,\xi_0\rangle\in C^\infty(S)^{ev} \qquad f_1(\xi_0,\xi_1) = \langle f,\xi_1\rangle\in C^\infty(S)^{odd},$$

where $\langle\ ,\ \rangle\colon\mathfrak{g}^\vee\times\mathfrak{g}\to\mathbb{R}$ is the evaluation map.

Lemma 6.21. *$\mu^*f_0 = f_0\otimes t^2$ and $\mu^*f_1 = f_1\otimes t + f_0\otimes t\theta$*

Proof.

$$\begin{aligned}(\mu^*f_0)((\xi_0,\xi_1),(d_0,d_1)) &= f_0(\xi_0 d_0^2,\xi_0 d_0 d_1+\xi_1 d_0) = \langle f,\xi_0\rangle d_0^2\\ &= f_0(\xi_0,\xi_1)\cdot t^2(d_0,d_1) = (f_0\otimes t^2)((\xi_0,\xi_1),(d_0,d_1))\end{aligned}$$

and hence $\mu^*f_0 = f_0\otimes d_0^2$. Moreover,

$$\begin{aligned}(\mu^*f_1)((\xi_0,\xi_1),(d_0,d_1)) &= f_1(\xi_0 d_0^2,\xi_0 d_0 d_1+\xi_1 d_0) = \langle f,\xi_0\rangle d_0 d_1 + \langle f,\xi_1\rangle d_0\\ &= f_0(\xi_0,\xi_1)\cdot t\theta(d_0,d_1) + f_1(\xi_0,\xi_1)\cdot t(d_0,d_1)\\ &= (f_0\otimes t\theta + f_1\otimes t)((\xi_0,\xi_1),(d_0,d_1)),\end{aligned}$$

which implies $\mu^*f_1 = f_1\otimes t + f_0\otimes t\theta$. □

Corollary 6.22. *The generators N, d of the Lie algebra $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ act on $C^\infty(\mathcal{B})$ via*

$$f_0 \xrightarrow{N} 2f_0 \qquad f_1 \xrightarrow{N} f_1 \qquad f_1 \xrightarrow{d} f_2 \qquad f_0 \xrightarrow{d} 0.$$

In particular, with respect to the $\mathbb{N}$-grading determined by the N-action, the elements f_1, f_0 have degree 1 and 2, respectively.

Proof. We recall that $N = \partial_t \in \mathrm{Der}(C^\infty(\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})), \mathbb{R}) = T_e\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ and $d = \partial_\theta \in \mathrm{Der}(C^\infty(\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})), \mathbb{R}) = T_e\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$. Moreover, the induced action of any element $v \in \mathrm{Der}(C^\infty(\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})), \mathbb{R})$ on $C^\infty(\mathcal{B})$ is given by $\mu_*(v) = (1 \otimes v) \circ \mu^*$. Hence

$$\begin{aligned}
\mu_*(N)f_0 &= \mu_*(\partial_t)(f_0) = (1 \otimes \partial_t)(f_0 \otimes t^2) = 2f_0\\
\mu_*(N)f_1 &= \mu_*(\partial_t)(f_1) = (1 \otimes \partial_t)(f_1 \otimes t + f_0 \otimes t\theta) = f_1\\
\mu_*(d)f_0 &= \mu_*(\partial_\theta)(f_0) = (1 \otimes \partial_\theta)(f_0 \otimes t^2) = 0\\
\mu_*(d)f_1 &= \mu_*(\partial_\theta)(f_1) = (1 \otimes \partial_\theta)(f_1 \otimes t + f_0 \otimes t\theta) = f_0
\end{aligned}$$

□

Functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)$ in the S-point formalism. We recall that we have natural bijections

$$\begin{aligned}
\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)(S) = \mathsf{SM}(S \times \mathbb{R}^{0|1}, G) &\leftrightarrow \mathsf{Alg}(C^\infty(G), C^\infty(S) \otimes \Lambda[\eta])\\
g &\leftrightarrow g^*
\end{aligned}$$

Moreover, for $h \in C^\infty(G)$, we can write $g^*(h)$ in the form $g^*(h) = g_0^*(h) + g_1^*(h)\eta$ with $g_0^* \in \mathsf{Alg}(C^\infty(G), C^\infty(S))$ and

$$g_1^* \in \mathrm{Der}(C^\infty_G, C^\infty(S)) \cong \mathrm{Hom}_{C^\infty_G}(\Omega^1_G, C^\infty(S))$$

Here $C^\infty(S)$ is considered as a C^∞_G-module via the algebra homomorphism g_0^*. The isomorphism between derivations $C^\infty_G \to C^\infty(S)$ and C^∞_G-linear maps $\Omega^1_G \to C^\infty(S)$ is a consequence of the universal property of Ω^1_G. In particular, we can identify an S-point $g \in \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)$ with a pair (g_0^*, g_1^*) as above.

We recall that $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)) \cong \Omega^*_G$. Evaluating a function $h \in \Omega^0_G$ or a 1-form $\omega \in \Omega^1_G$ on an S-point (g_0^*, g_1^*) we have

$$h(g_0^*, g_1^*) = g_0^*(h) \qquad \omega(g_0^*, g_1^*) = -g_1^*(\omega)$$

The Maurer-Cartan form. Let $\{v_i\}$ be a basis for the Lie algebra $\mathfrak{g}$, which we identify with the vector space of left invariant vector fields on G. Let $\{\omega^i\}$ be the dual basis of $\mathfrak{g}^\vee$, which can be identified with the space of left-invariant 1-forms on G. The *Maurer-Cartan form* is the 1-form

$$\mathrm{MC} := \sum_i \omega^i \otimes v_i \in \Omega^1_G \otimes \mathfrak{g}$$

Let $\Phi\colon \mathsf{SM}(\mathbb{R}^{0|1}, G) \to \mathcal{B}$ be the map defined via the S-point formalism by

$$\begin{aligned} \Phi_S\colon \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)(S) = \mathsf{SM}(S \times \mathbb{R}^{0|1}, G) \ &\longrightarrow \ \mathcal{B}(S) = \Omega^1_{S\times\mathbb{R}^{0|1}/S} \\ g \ &\mapsto \ g^* \mathrm{MC} \end{aligned}$$

Lemma 6.23. *Let $\omega \in \Omega^1_G$ and and let $\omega_0 \in C^\infty(\mathcal{B})^{ev}$ and $\omega_1 \in C^\infty(\mathcal{B})^{odd}$ be the corresponding even/odd function on $\mathcal{B}$. Then*

$$\Phi^*(\omega_0) = 0 \qquad \text{and} \qquad \Phi^*(\omega_1) = \omega \in \Omega^1_G$$

Proof.

$$\Phi(g_0^*, g_1^*) = g^* \mathrm{MC} = g^*(\sum_i \omega^i \otimes v_i) = \sum_i g_1^*(\omega^i) v_i \in C^\infty(S)^{odd} \otimes \mathfrak{g}$$

The even function ω_0 vanishes on all S-points of $\mathcal{B}$ belonging to $C^\infty(S)^{odd} \otimes \mathfrak{g}$. Hence $(\Phi^*\omega_0)(g_0^*, g_1^*) = \omega_0(\Phi(g_0^*, g_1^*)) = 0$, which implies $\Phi^*\omega_0 = 0$. Pulling back the function ω_1 we obtain

$$(\Phi^*\omega_1)(g_0^*, g_1^*) = \omega_1(\sum_i g_1^*(\omega^i) v_i) = -\sum_i g_1^*(\omega^i)\langle \omega, v_i\rangle = -g_1^*(\omega) = \omega(g_0^*, g_1^*)$$

This implies $\Phi^*\omega_1 = \omega \in \Omega^1_G \subset \Omega^*_G = C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G))$. ☐

Finally, we need to look at the action of $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, G)$ as gauge group on the space $\mathcal{A}$ of connections on the trivial G-bundle over $\mathbb{R}^{0|1}$. The typical way the gauge action is written is as

$$\theta \mapsto g^{-1} dg + \mathrm{Ad}_g(\theta)$$

My interpretation of this is that the first term is $g^* \mathrm{MC}$, the pull-back of the Maurer-Cartan form via g, and the second term is the right action of $\mathrm{S}\, G$ on $\mathrm{S}\,\mathfrak{g} = \mathfrak{g} \times \Pi\mathfrak{g}$ induced by the adjoint action (from the right) of G on its Lie algebra $\mathfrak{g}$. If we write $\mathrm{Ad}\colon \mathfrak{g} \times G \to \mathfrak{g}$ for the action map, then the induced map

$$(\mathrm{S}\,\mathrm{Ad})^*\colon C^\infty(\mathrm{S}\,\mathfrak{g}) = \Omega^*(\mathfrak{g}) \longrightarrow C^\infty(\mathrm{S}\,\mathfrak{g} \times \mathrm{S}\, G) = \Omega^*(\mathfrak{g}) \otimes \Omega^*(G)$$

is just the map induced by Ad on functions, and then Kähler differentials. More directly, we know how the Lie algebra of $\mathrm{S}\, G$ acts on $\Omega^*(X)$ for *any* right G-action on X: for $v \in \mathfrak{g}$, the elements ι_v, L_v in the Lie algebra of $\mathrm{S}\, G$ act on $\Omega^*(X)$ by contraction (resp. Lie derivative) with the vector field $\mu_*(v)$. In the case of a *linear* action, the vector field $\mu_*(v)$ is a linear vector field. In the case of the adjoint action, it is given by

References

[Day] B.J. Day *On closed categories of functors*, Lecture Notes in Mathematics, Vol. 137 (Springer, 1970), 1–38

[DM] P. Deligne, J. Morgan, *Notes on Supersymmetry (following Joseph Bernstein).* Quantum fields and strings: A course for Mathematicians, Volume 1, A.M.S.- I.A.S. 1999, p. 41–98.

[GoSa] J. González and J. de Salas *C^∞-Differentiable Spaces*, Lecture Notes in Mathematics, Vol. 1824, Springer-Verlag, Berlin, 2003. IX+188pp.

[GuSt] V. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Springer-Verlag, Berlin, 1999. xxiv+228 pp.

[LM] H.B. Lawson and M. Michelsohn, *Spin geometry*, Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989. xii+427 pp.

[McL] S. Mac Lane, *Categories for the working mathematician*, Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998. xii+314 pp.

[Mi] Michor, P. ,*Manifolds of smooth maps*, Cahiers Topologie Gom. Diffrentielle 19 (1978), no. 1, 4778.

[Sch] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*, Second edition. Graduate Texts in Mathematics, 3. Springer-Verlag, New York, 1999. xii+346 pp.