Super symmetric field theories and integral modular functions

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Preliminary Version of March 9, 2007

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1 Introduction

The main result of this paper, Theorem thm:main 1, is that the partition function of a 2-dimensional super symmetric quantum field theory is an integral modular function. The bulk of this work consists of incorporating a suitable geometric notion of *super symmetry* into the axiomatic description of quantum field theories going back to Atiyah and Segal. Roughly speaking, we replace the usual functors from Riemannian manifolds to (locally convex) vector spaces by functors from Riemannian super manifolds to super vector spaces and work in families parametrized by complex super manifolds. We end up with an axiomatic/geometric version of what physicists might call a super symmetric quantum field theory with minimal super symmetry.

We point out that these field theories are neither conformal (i.e. the functors depend on the Riemannian metric, not just its conformal class) nor chiral (i.e. the operators associated to the bordisms depend only smoothly, not holomorphically, on the Riemannian structures). As a consequence, it is a priori not clear why the partition function, i.e. the quantum field theory evaluated on 2-dimensional tori, should be $SL_2(\mathbb{Z})$-invariant or holomorphic. This all comes out of the right notion of super symmetry, a fact that seems to be well known in the physics community, and we thank Ed Witten for explaining it to us in the context of the super symmetric σ-model. Our own contribution is a *geometric definition* of super symmetry that implies that a certain square root exists (for an infinitesimal generator $\bar{L}_0$ of the operators associated to cylinders). It is the latter algebraic fact that's usually called super symmetry in the physics literature and it seems well known that it leads to a modular partition function.

In this introduction we will outline our definitions. In addition, we explain the relevance of this result for our program of relating 2-dimensional quantum field theories to the topological modular form spectrum TMF due to Hopkins and Miller.

A d-dimensional quantum field theory in the sense of Atiyah and Segal is a symmetric monoidal functor which associates a locally convex vector space $E(Y)$ to a closed oriented Riemannian manifold Y of dimension $(d-1)$

and a trace class operator $E(\Sigma)\colon E(Y_1) \to E(Y_2)$ to an oriented Riemannian bordism Σ from Y_1 to Y_2. If E is a 2-dimensional quantum field theory, then one obtains a complex valued function Z_E on the upper half plane $\mathfrak{h}$, by defining

$$Z_E(\tau) \stackrel{\text{def}}{=} E(T_\tau),$$

where $T_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is the torus obtained by dividing the complex plane by the lattice $\mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$. This is the *partition function* of the quantum field theory E, compare definition def:partition_function 20.

Atiyah and Segal's only additional requirement for a quantum field theory is that the functor E is *continuous* in the usual sense, as a functor between topological categories. We strengthen this requirement to a notion of *smoothness* in the sense that will be explained in detail in section sec:smooth 3. It means that E can be extended to *smooth families*: for any smooth manifold S there is a functor E_S which associates to a smooth family Y of Riemannian manifolds parametrized by S (i.e. a fiber bundle $Y \to S$ equipped with a Riemannian metric along its fibers) a smooth family $E_S(Y)$ of locally convex vector spaces parametrized by S (i.e. $E_S(Y)$ is a locally trivial bundle of locally convex vector spaces over S). Similarly, E_S associates to a smooth family of Riemannian bordisms parametrized by S a smooth family of trace class operators parametrized by S. The collection of functors E_S is required to depend functorially on S. It turns out to be essential that this definition also takes into account families of *objects* in our categories, not just of morphisms, as is more common in the context of topological categories.

A *super symmetric* quantum field theory is defined completely analogously, only replacing (Riemannian) manifolds in the definition above by super (Riemannian) manifolds. More precisely, a *super symmetric quantum field theory* of dimension $d|q$ gives a symmetric monoidal functor E which associates a locally convex super vector space $E(Y)$ to a super Riemannian manifold of dimension $d-1|q$ and a trace class operator $E(\Sigma)\colon E(Y_1) \to E(Y_2)$ to a super Riemannian bordism Σ of dimension $d|q$ from Y_1 to Y_2. Here a *super Riemannian* manifold is a super manifold equipped with a super Riemannian structure, which we define in section sec:susyQFT 4 for super manifolds of dimension $d|q$ for $d = 1, 2$ and $q = 1$. Our terminology is motivated by the fact that a super Riemannian structure on a super manifold M of dimension $d|1$ induces a Riemannian metric on the reduced manifold M_{red} (an ordinary manifold of dimension d). Our definition is not the obvious generalization of a symmetric 2-tensor to super manifolds but is comes from a physicists point of view, where these structures should be useful to construct classical action functionals for certain super symmetric (classical) field theories (see remarks 54 and 61).

As for quantum field theories we require that the functor E describing a super symmetric quantum field theory is smooth, but it is essential that this smoothness condition is formulated by requiring that E does extend to families parametrized by *complex super manifolds*, see section 4 for a detailed definition. We note that ordinary manifolds give (complex) super manifolds of dimension $d|0$ and that a super symmetric quantum field theory leads to a quantum field theory as explained above.

thm:main **Theorem 1.** *The partition function of a super symmetric quantum field theory of dimension* $2|1$ *is a modular function with integral* q*-expansion. This continues to hold if one only has a super symmetric* flat *quantum field theory, see Remark 2.*

We recall that a modular function is a holomorphic function $f\colon \mathbb{R}^2_+ \to \mathbb{C}$ on the upper half plane, which is meromorphic at ∞ and is invariant under the usual $SL_2(\mathbb{Z})$-action on $\mathbb{R}^2_+$. The invariance implies in particular $f(\tau+1) = f(\tau)$ so that $f(\tau)$ can be expressed in the form

$$f(\tau) = \sum_{i \geq -N} a_i q^i \quad \text{where} \quad q = e^{2\pi i \tau}.$$

This is the *q-expansion* of f; it is *integral* if all coefficients a_i are integers. Our main theorem asserts the $SL_2(\mathbb{Z})$-invariance of Z_E, its holomorphicity, its meromorphicity at ∞, and the integrality of its q-expansion.

We want to point out that none of these four properties of the partition function is obvious; rather all of them are consequences of certain cancellations due to super symmetry. In fact, we believe that none of these statements holds true in general for (non-super symmetric) quantum field theories. Concerning the $SL_2(\mathbb{Z})$-invariance of Z_E, we note that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ the torus $T_{\tau'}$ for $\tau' = \frac{a\tau+b}{c\tau+d}$ is conformally equivalent, but in general not isometric to T_τ. This implies that if E is a *conformal field theory* (for which the operators $E(\Sigma)$ are required to depend only on the conformal structure rather than the Riemannian metric on Σ), then the $SL_2(\mathbb{Z})$-invariance of the partition function is indeed obvious. A similar remark applies to the holomorphicity of the partition function under the assumption that the operators $E(\Sigma)$ depend holomorphically on Σ, i.e. if E is a *chiral* quantum field theory.

Theorem 1 above is an important step in our approach to understand the infinite loop space TMF, proven to exist by Hopkins-Miller and recently constructed by Lurie. This space gives the universal elliptic cohomology theory and is a topological version of modular forms. So by construction, there is a map to the ring MF of integral modular forms:

$$\pi_0 TMF \to \mathrm{MF}$$

which induces a rational isomorphism (and where the kernel and cokernel were completely calculated by Hopkins, Mahowald and Miller). We now believe that instead of studying conformal field theories as in [ST], one should define an infinite loop space $FQFT$ of super symmetric *flat* quantum field theories, together with a continuous map $F : FQFT \to TMF$ which leads to a commutative diagram

$$\begin{array}{ccc} \pi_0 FQFT & \xrightarrow[\cong]{F_*} & \pi_0 TMF \\ & {\scriptstyle Z}\searrow \quad \swarrow & \\ & \mathrm{MF} & \end{array} .$$

This would give a computation of the connected components of the space $FQFT$, otherwise a seemingly impossible task.

rem:flat **Remark 2.** By the Gauss-Bonnet Theorem the only closed orientable 2-manifold with a flat Riemannian structure is the surface of genus one. Moreover, the set of isomorphism classes of such structures agrees with the moduli space of elliptic curves over $\mathbb{C}$ (up to rescaling, i.e. a factor $\mathbb{R}_+$). This is our reason to restrict to flat surface when trying to understand the space TMF.

One aspect that is not addressed in this paper are modular forms of weight $w \neq 0$. There is a notion of quantum field theories of *degree* n, $n \in \mathbb{Z}$, very similar to that in [ST]. We expect that a super symmetric version of these will make Theorem 1 hold true, where the degree and weight are related by the equation $n = 2w$. Moreover, there should be spaces $FQFT_n$ of degree n super symmetric flat quantum field theories that are deloopings of the space $FQFT$, and the above diagram should continue to hold on the level of $\pi_n FQFT \cong \pi_0 FQFT_{-n}$.

A final aspect of quantum field theories that needs to be included in order to define the space $FQFT$ above is the *space-time locality*. In [ST] we gave an approach to this aspect via 2-functors from the 2-category of $0-, 1-$ and

2-dimensional conformal manifolds to a 2-category algebras, bimodules and intertwiners. We believe that this continues to work in the super symmetric Riemannian context.

We also expect that examples of such full fetched (super symmetric, local) quantum field theories are given by the super symmetric σ-models for Riemannian string manifolds as target. In this regard, it is an advantage not to have to check conformal invariance since it only seems to hold for Ricci flat targets. We expect that these σ-models will lead to a continuous map on the infinite loop space that represents string cobordism

$$\Omega^\infty MString \longrightarrow FQFT$$

which is our candidate for the family Witten genus. It should commute with the recently constructed Witten genus of Lurie (with values in TMF) via the map F used in the above diagram.

Acknowledgements. The authors would like to thank the Max-Planck-Institut für Mathematik in Bonn for its hospitality during the academic year 2001/2002 when the project started. The authors also would like to thank Chris Douglas, Dan Freed, Andre Henriques, Henning Hohnhold, Matthias Kreck, Jacob Lurie, Alexander Voronov and Ed Witten for stimulating discussions; particularly Jacob helped us tremendously with various aspects of this project. Both authors are partially supported by the NSF grants, Stephan Stolz by NSF grant DMS-0104077, Peter Teichner by NSF grant DMS-0453957.

2 Field theories a la Atiyah-Segal

As mentioned in the introduction, a quantum field theory as axiomatized by Atiyah-Segal is a functor from a suitable bordism category to a category of locally convex (topological) vector spaces satisfying certain axioms. In this section, we follow Segal's description in [Se2, §4] and our rendition of it in [ST] and define a *d-dimensional quantum field theory* (see Definition 19) as a functor

$$E\colon \mathrm{RB}^d \longrightarrow \mathrm{TV}^\pm$$

from the category of d-dimensional Riemannian spin bordisms to a suitably defined category $\mathrm{TV}^\pm$ of locally convex (topological) vector spaces (more precisely, of pairs of such spaces in duality, hence the notation $\pm$; see subsection 2.1) . This functor is required to be compatible with the symmetric monoidal structure (given by disjoint union in the domain category and tensor product in the range category) and two (anti-) involutions defined on both categories. This definition is preliminary in the sense that we require a QFT to be a *smooth* functor in the sense that will be explained in section 3. The details of these definitions are new and we hope to have captured the essential features required in physics. We were also guided by the desire to prove Theorem 1 in a very general setting.

In the following two subsections we will describe the categories $\mathrm{TV}^{\pm}$ and RB^d. We provide more detail than might be necessary at this point; however, we try to phrase things in such a way that the generalization of these categories to their family versions (discussed in section 3) and their super versions (section 4) is straightforward.

There are many possible variants of the definition of the categories RB^d and $\mathrm{TV}^{\pm}$; e.g., different choices for the geometric structure on the bordisms involved lead to variants of the bordism category RB^d. While the focus of [Se2] and [ST] is on *conformal field theories* (where the bordisms are equipped with conformal structures), in this paper we are interested in *quantum field theories* which corresponds to requiring Riemannian metrics on bordisms. We also require our bordisms to be equipped with a *spin structure* (rather than just an orientation, which is more usual), and we will build our category $\mathrm{TV}^{\pm}$ using $\mathbb{Z}/2$-graded vector spaces. The resulting kind of field theories are more closely related to super symmetric field theories.

2.1 The categories TV and $\mathrm{TV}^{\pm}$

subsec:TV

def:TV **Definition 3. (The category TV)**

objects are $\mathbb{Z}/2$-graded locally convex vector spaces;

morphisms are grading preserving continuous linear maps.

def:projective **Definition 4. (The projective tensor product)** If V, W are locally convex vector spaces, their algebraic tensor product $V \otimes_{\mathrm{alg}} W$ is a vector space which carries again a locally convex topology known as the *projective topology* which is characterized by the property that it is the finest locally convex topology such that the canonical bilinear map

$$V \times W \longrightarrow V \otimes_{\mathrm{alg}} W$$

is continuous (see [Koe, §41.2]). We denote by $V \otimes W$ the *projective tensor product* which is defined to be the completion of $V \otimes_{\text{alg}} W$ w.r.t. the projective topology. A $\mathbb{Z}/2$-grading on V, W induces the usual $\mathbb{Z}/2$-grading on $V \otimes W$. The projective tensor product gives TV the structure of a symmetric monoidal category. As usual for graded vector spaces, the symmetry isomorphism

$$V \otimes W \cong W \otimes V \qquad \text{is given by} \qquad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$$

for homogeneous elements $v \in V$, $w \in W$ of degree $|v|$ and $|w|$.

There is a contravariant endofunctor which sends a locally convex vector space V to its dual V' (consisting of all continuous maps $V \to \mathbb{C}$). This endofunctor is not well-related to the tensor product in the sense that the natural map

$$V' \otimes W' \longrightarrow (V \otimes W)'$$

is usually not an isomorphism (compare [Koe, §45.3(7)]). In the next subsection we will see that the Riemannian bordism category RB^d has an anti-involution $^\vee$ which *is* well-related to the symmetric monoidal structure on RB^d (given by disjoint union). This motivates us to replace the category TV by the following category $\text{TV}^{\pm}$ which like RB^d has an anti-involution $^\vee$, compatible with its symmetric monoidal structure in a strict sense.

def:TV^pm **Definition 5. (The category $\text{TV}^{\pm}$)**

objects are triples $V = (V^+, V^-, \mu_V)$, where $V^{\pm}$ are locally convex vector spaces, and $\mu_V : V^- \otimes V^+ \to \mathbb{C}$ is a continuous linear map.

morphisms from $V = (V^+, V^-, \mu_V)$ to $W = (W^+, W^-, \mu_W)$ are pairs $T = (T^+ : V^+ \to W^+, T^- : W^- \to V^-)$ of continuous linear maps, which are *dual* to each other in the sense that

$$\mu_V(T^- w^- \otimes v^+) = \mu_W(w^- \otimes T^+ v^+) \qquad \forall v^+ \in V^+, w^- \in W^- \tag{6}$$ eq:dual

composition: Let $S = (S^+, S^-): V_1 \to V_2$ and $T = (T^+, T^-): V_2 \to V_3$ be morphisms in TV. Then their composition $T \circ S : V_1 \to V_3$ is given by

$$T \circ S \overset{\text{def}}{=} (T^+ \circ S^+, S^- \circ T^-).$$

symmetric monoidal structure: We define the *tensor product* of two objects $V = (V^+, V^-, \mu_V)$ and $W = (W^+, W^-, \mu_W)$ as

$$V \otimes W \stackrel{\text{def}}{=} (V^+ \otimes W^+, V^- \otimes V^-, \mu_{V \otimes W}),$$

where $V^\pm \otimes W^\pm$ is the projective tensor product, and $\mu_{V\otimes W}$ is given by the composition of the usual (graded) symmetry isomorphism

$$(V^- \otimes W^-) \otimes (V^+ \otimes W^+) \cong V^- \otimes V^+ \otimes W^- \otimes W^+$$

and the linear map

$$V^- \otimes V^+ \otimes W^- \otimes W^+ \xrightarrow{\mu_V \otimes \mu_W} \mathbb{C} \otimes \mathbb{C} = \mathbb{C},$$

On morphisms, we define $(T^+, T^-) \otimes (S^+, S^-) \stackrel{\text{def}}{=} (T^+ \otimes S^+, T^- \otimes S^-)$.

anti-involution $^\vee$: On objects it is given by $(V^+, V^-, \mu_V)^\vee \stackrel{\text{def}}{=} (V^-, V^+, \mu_V^\vee)$, where $\mu_V^\vee$ is the composition

$$V^+ \otimes V^- \xrightarrow{\cong} V^- \otimes V^+ \xrightarrow{\mu_V} \mathbb{C}$$

of the (graded) symmetry isomorphism and μ_V. On morphisms the anti-involution is given by $(T^+, T^-)^\vee \stackrel{\text{def}}{=} (T^-, T^+)$. We note that $V^\vee \otimes W^\vee = (V \otimes W)^\vee$.

involution $^-$: If $V = (V^+, V^-, \mu_V)$ is an object of TV, then $\bar{V}$ is given by complex conjugate vector spaces $\bar{V}^+$, $\bar{V}^-$ and the pairing

$$\bar{V}^- \otimes \bar{V}^+ = \overline{V^- \otimes V^+} \xrightarrow{\bar{\mu}_V} \bar{\mathbb{C}} \cong \mathbb{C}$$

On morphisms, it is given by $\overline{(T^+, T^-)} = (\bar{T}^+, \bar{T}^-)$, where (as for $\bar{\mu}_V$) $\bar{T}^\pm$ is the *same* map as $T^\pm$, but regarded as a complex linear map between the complex conjugate vector spaces.

rem:TVandTVpm **Remark 7.** There is a symmetric monoidal functor $\mathrm{TV}^\pm \to \mathrm{TV}$ given by

$$(V^+, V^-, \mu_V) \mapsto V^+ \qquad \text{and} \qquad (T^+, T^-) \mapsto T^+.$$

This functor is injective on morphism sets (since the non-degeneracy condition for the μ's implies by equation (eq:dual 6) that T^+ determines T^-). However, under this functor the endofunctor ${}^\vee \colon \mathrm{TV}^\pm \to \mathrm{TV}^\pm$ does not correspond to the endofunctor of TV given by $V \mapsto V'$ (V' = continuous dual of V).

There is also a functor $\mathrm{TV} \to \mathrm{TV}^\pm$; on objects, it sends a locally convex vector space V to (V, V', μ), where $\mu \colon V' \otimes V \to \mathbb{C}$ is the natural pairing. On morphisms, it sends a linear map $T \colon V \to W$ to the pair (T, T'), where $T' \colon W' \to V'$ is the continuous linear map dual to T.This is not a *monoidal* functor due to the incompatibility of $\otimes$ and $'$ mentioned earlier. However, we obtain a monoidal functor after restricting to *finite dimensional* vector spaces (with their usual topology). Using this functor, we can interpret finite dimensional vector spaces and linear maps as objects resp. morphisms in $\mathrm{TV}^\pm$. In particular, we just write $\mathbb{C}$ for the object of $\mathrm{TV}^\pm$ corresponding to $\mathbb{C} \in \mathrm{TV}$. These are the monoidal units in our categories.

2.2 The Riemannian bordism category RB^d

subsec:RB

We recall that a *spin structure* on a Riemannian d-manifold M consists of an orientation of M together with a double covering $Spin(M) \to SO(M)$ of the oriented frame bundle

$$SO(M) \overset{\text{def}}{=} \{(x, f) \mid x \in M,\ f \colon \mathbb{R}^d \to T_xM \text{ isometry}\} \xrightarrow{\ p\ } M,$$

such that the restriction to each fiber $p^{-1}(x) \subset SO(M)$ is a non-trivial double covering for $d \geq 2$.

We want to stress that spin structures on a manifold are best viewed as a *groupoid*, whose objects are spin structures as defined above, and whose morphisms are isomorphisms of double coverings. The set of isomorphism classes of spin structures on M is a torsor for $H^0(M; \mathbb{Z}/2) \oplus H^1(M; \mathbb{Z}/2)$ (i.e., this group acts freely and transitively on the set of isomorphism classes; the first summand acts by changing the orientation, the second by tensoring with double coverings pulled back from M). We want to alert the reader that in the following a spin structure is an *object* in this groupoid, not an isomorphism class (the latter usage is quite common in the literature).

Example 8. (Examples of spin structures.) There is a *standard spin structure* on $M = \mathbb{R}^d$ given by the double covering

$$M \times Spin(d) \longrightarrow M \times SO(d) \cong SO(M);$$

here $SO(M)$ is the oriented frame bundle w.r.t. the standard orientation of $M = \mathbb{R}^d$; it is isomorphic to the trivial bundle $M \times SO(d)$ via the standard trivialization of the tangent bundle of $\mathbb{R}^d$.

We note that the translation action is compatible with this trivialization and hence the translation action on $SO(M)$ lifts to a translation action on $M \times Spin(d)$ (which acts trivially on the second summand). In particular if G is a discrete subgroup of $\mathbb{R}^d$, then the spin structure on $\mathbb{R}^d$ induces a spin structure on $\mathbb{R}^d/G$.

We note that up to isomorphism there are *two* spin structures on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ equipped with its standard orientation; the spin structure induced by the standard spin structure on $\mathbb{R}$ is often called the *periodic spin structure* (since sections of the associated spinor bundle can be interpreted as periodic functions on $\mathbb{R}$). The circle equipped with this spin structure represents the non-trivial element in the spin bordism group $\Omega_1^{spin} \cong \mathbb{Z}/2$.

Similarly, up to isomorphism there are four spin structures on the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ equipped with its induced orientation; the spin structure induced from $\mathbb{R}^2$ is often called the periodic-periodic spin structure and equipped with this spin structure the torus represents the non-trivial element in the spin bordism group $\Omega_2^{spin} \cong \mathbb{Z}/2$.

Definition 9. Let M, N be two Riemannian spin manifolds of dimension d. An *isometric spin embedding* from M to N is a pair $(f, \widehat{f_*})$, consisting of an isometric embedding $f\colon M \to N$ and a map $\widehat{f_*}\colon Spin(M) \to Spin(N)$ which covers the $SO(d)$-equivariant map $f_*\colon SO(M) \to SO(N)$ induced by the differential of f.

`x:spin_isometries` **Example 10.** The groups of isometries of $\mathbb{R}^d$ is the semi-direct product $\mathbb{R}^d \rtimes SO(d)$, where $\mathbb{R}^d$ acts by translations and $SO(d)$ by rotations. The group of spin isometries of $\mathbb{R}^d$ (i.e., invertible isometric spin embeddings $\mathbb{R}^d \to \mathbb{R}^d$) is the semi-direct product $\mathbb{R}^d \rtimes Spin(d)$. Here $\mathbb{R}^d$ acts on $Spin(\mathbb{R}^d) = \mathbb{R}^d \times Spin(d)$ by translations on the first factor and trivially on the second, while $Spin(d)$ acts on $\mathbb{R}^d$ by rotations (via the double cover $Spin(d) \to SO(d)$) and on $Spin(d)$ by left-multiplication.

`def:RB` **Definition 11. (The Riemannian spin bordism category RB^d.)** Riemannian spin bordisms of dimension d are *morphisms* in the category RB^d whose precise definition is as follows:

objects are quadruples (U, Y, U^-, U^+), where

> U is a Riemannian spin manifold of dimension d (typically not closed);
>
> Y is a closed codimension 1 smooth submanifold of U;

$U^{\pm}$ are disjoint open subsets of $U \setminus Y$ whose union is $U \setminus Y$. We require that Y is contained in the closure of both, U^+ and U^- (this ensures that U^+ as well as U^- are collars of Y; without this requirement we could have for example $U^+ = U \setminus Y$, $U^- = \emptyset$; or this could be true for one component of U).

We think of Y as the primary datum, and of (U, U^-, U^+) as additional data needed to express the geometry on Y and to obtain a well-defined composition. Consequently, we will often suppress (U, U^-, U^+) in the notation and just write Y instead of (U, Y, U^-, U^+). The Riemannian metric on U induces a Riemannian metric on $Y \subset U$; however, there is more geometric data contained in the pair $Y \subset U$, e.g., the second fundamental form of of $Y \subset U$ (we do not require that the metric on U is a product metric near Y).

morphisms from Y_1 to Y_2 are equivalence classes of Riemannian spin bordisms from Y_1 to Y_2; here a *Riemannian spin bordism* is a triple $(\Sigma, \iota_1, \iota_2)$, where

Σ is a Riemannian spin manifold of dimension d (not necessarily closed), and

$\iota_1 \colon V_1 \hookrightarrow \Sigma$ and $\iota_2 \colon V_2 \hookrightarrow \Sigma$ are isometric spin embeddings, where $V_k \subset U_k$ for $k = 1, 2$ is some open neighborhood of $Y_k \subset U_k$.

We define $V_k^{\pm} \stackrel{\text{def}}{=} U_k^{\pm} \cap V_k$ and require that

- the sets $\iota_1(V_1^+ \cup Y_1)$ and $\iota_2(V_2^- \cup Y_2)$ are disjoint and
- $\Sigma \setminus (\iota_1(V_1^+) \cup \iota_2(V_2^-))$ is compact,

(12) eq:bordism_conditi

In particular, $\Sigma \setminus (\iota_1(V_1^+) \cup \iota_2(V_2^-))$ is a compact manifold with boundary $\iota_1(Y_1) \amalg \iota_2(Y_2)$; i.e., it is a bordism between Y_1 and Y_2 in the usual sense. We note that for the incoming boundary Y_1 the collar V_1^- is inside this bordism, whereas for the outgoing boundary Y_2 the collar V_2^+ is inside. This is our convention for keeping track of the domain and the range of a bordism.

Here is a picture of a Riemannian bordism; we usually draw the domain of the bordism to the right of its range, since we want to read compositions of bordisms, like compositions of maps, from right to left.

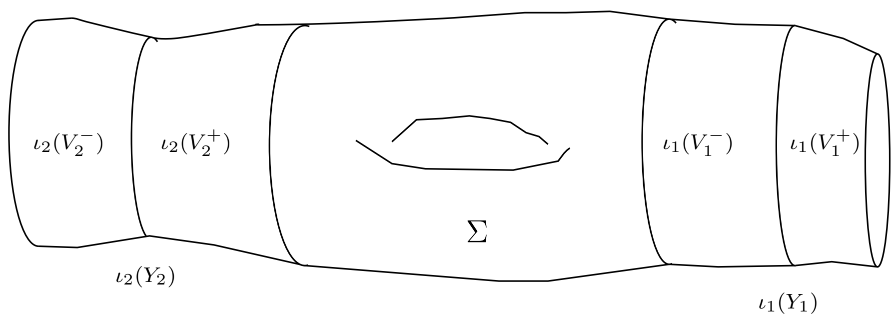


Figure 1: **A Riemannian bordism** fig:bordism

Now suppose that $(\Sigma, \iota_1, \iota_2)$ and $(\Sigma', \iota_1', \iota_2')$ are two Riemannian spin bordisms from Y_1 to Y_2 with $V_1 \subset V_1'$, $V_2 \subset V_2'$ and that there is a spin isometry F that makes the following diagram commutative

$$\begin{array}{ccccc} V_2 & \xhookrightarrow{\iota_2} & \Sigma & \xleftarrow{\iota_1} & V_1 \\ \downarrow & & \downarrow{\scriptstyle F}\,\cong & & \downarrow \\ V_2' & \xhookrightarrow{\iota_2'} & \Sigma' & \xleftarrow{\iota_1'} & V_1' \end{array}$$

Then we declare $(\Sigma, \iota_1, \iota_2)$ as equivalent to $(\Sigma', \iota_1', \iota_2')$. A *morphism* from Y_1 to Y_2 is an equivalence class of Riemannian spin bordisms with respect to the equivalence relation generated by the relation just described.

In particular, if $V_2 \xhookrightarrow{\iota_2} \Sigma \xleftarrow{\iota_1} V_1$ is a Riemannian bordism from Y_1 to Y_2, then replacing V_1 by a smaller bicollar V_1' of $Y_1 \subset U_1$ and Σ by $\Sigma \setminus (\iota_1(V_1^+ \setminus (V_1')^+))$ gives a Riemannian bordism representing the *same* morphism as $(\Sigma, \iota_1, \iota_2)$. Similarly, one can decrease the size of V_2 arbitrarily and hence only germs of these bicollars are relevant for our morphisms.

composition of Riemannian spin bordisms is given by gluing; more precisely, let $(\Sigma', \iota_1', \iota_2')$ be a Riemannian spin bordism from Y_1 to Y_2 and $(\Sigma, \iota_2, \iota_3)$ a Riemannian spin bordism from Y_2 to Y_3. Without loss of generality, we can assume that the domains of the isometries ι_2' and ι_2 agree (both of which are open neighborhoods of $Y_2 \subset U_2$); suppose $V_2 \subset U_2$ is this common domain. Then identifying $\iota_2'(V_2) \subset \Sigma'$ with

$\iota_2(V_2) \subset \Sigma$ via the isometry $\iota_2 \circ (\iota_2')^{-1}$ gives the Riemannian spin manifold

$$\Sigma'' \stackrel{\text{def}}{=} \Sigma \cup_{V_2} \Sigma'.$$

Together with the spin isometries

$$\iota_3'' \colon V_3 \xrightarrow{\iota_3} \Sigma \subset \Sigma'', \qquad \iota_1'' \colon V_1' \xrightarrow{\iota_1'} \Sigma' \subset \Sigma''$$

this is a Riemannian bordism from Y_1 to Y_3.

bjects+morphisms1 **Example 13. (Examples of objects and morphisms of RB^1.)**

the point $\mathrm{pt} \in \mathrm{RB}^1$**.** The quadruple

$$\mathrm{pt} \stackrel{\text{def}}{=} (U, Y, U^-, U^+) = (\mathbb{R}, \{0\}, (-\infty, 0), (0, \infty))$$

is an object of RB^1, where the real line $\mathbb{R}$ is equipped with its standard Riemannian metric and its standard spin structure.

the interval $I^1_\ell \in \mathrm{RB}^1(\mathrm{pt}, \mathrm{pt})$**.** For $\ell \in \mathbb{R}$ consider the pair of spin isometries

$$U = \mathbb{R} \xhookrightarrow{\text{id}} \Sigma = \mathbb{R} \xhookleftarrow{\ell} U = \mathbb{R}\ ,$$

where abusing notation we write $\ell \colon \mathbb{R} \to \mathbb{R}$ is translation by ℓ (i.e., $t \mapsto \ell + t$). For $\ell \in \mathbb{R}_+$ this is a Riemannian spin bordism from pt to itself (for $\ell \leq 0$ conditions eq:bordism_conditions 12 are not satisfied). We will use the notation I^1_ℓ for this *interval of length* ℓ.

the circle $S^1_\ell \in \mathrm{RB}^1(\emptyset, \emptyset)$**.** For $\ell \in \mathbb{R}_+$ let

$$S^1_\ell \stackrel{\text{def}}{=} \mathbb{R}/\ell\mathbb{Z}$$

be the *circle of length* ℓ, equipped with the Riemannian metric (resp. spin structure) induced by the standard Riemannian metric (resp. spin structure) on $\mathbb{R}$. This is a Riemannian bordism from $\emptyset$ to $\emptyset$ and represents the nontrivial element of $\Omega_1^{spin} \cong \mathbb{Z}/2$..

bjects+morphisms2 **Example 14. (Examples of objects and morphisms of RB^2.)** The circle S^1_ℓ does double duty in the bordism categories RB^d as endomorphism of $\emptyset \in \mathrm{RB}^1$ as well as object in RB^2; we will use the same notation in both situations, hoping that the context will make the meaning clear.

the circle $S^1_\ell \in \mathrm{RB}^2$. Consider the quadruple

$$(U, Y, U^+, U^-) = (\mathbb{R}^2, \mathbb{R}, \mathbb{R}^2_+, \mathbb{R}^2_-),$$

where $\mathbb{R}^2$ are equipped with the standard Riemannian metric and standard spin structure, and $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ (resp. $\mathbb{R}^2_- = \{(x, y) \in \mathbb{R}^2 \mid y < 0\}$) is the upper (resp. lower) half plane. This is not an object in RB^2 (since $\mathbb{R}$ is not compact), but we obtain an object by taking the quotient with respect to the action of $\mathbb{Z}\ell \subset \mathbb{R} \subset \mathbb{R}^2$, where $\ell \in \mathbb{R}_+$ and $\mathbb{R}^2$ acts on itself by translations. We note that $\mathbb{R}/\mathbb{Z}\ell$ is a Riemannian circle of length ℓ. This motivates the notation

$$S^1_\ell \stackrel{\text{def}}{=} (\mathbb{R}^2/\mathbb{Z}\ell, \mathbb{R}/\mathbb{Z}\ell, \mathbb{R}^2_+/\mathbb{Z}\ell, \mathbb{R}^2_-/\mathbb{Z}\ell) \in \mathrm{RB}^2\,.$$

the cylinder $C^2_{\ell,\tau} \in \mathrm{RB}^2(S^1_\ell, S^1_\ell)$. For $\ell \in \mathbb{R}_+$ and $\tau \in \mathbb{R}^2_+$ consider the pair of isometric spin embeddings

$$U_2 = \mathbb{R}^2/\mathbb{Z}\ell \xhookrightarrow{\ \mathrm{id}\ } \Sigma = \mathbb{R}^2/\mathbb{Z}\ell \xhookleftarrow{\ \ell\tau\ } U_1 = \mathbb{R}^2/\mathbb{Z}\ell\,,$$

where $\ell\tau \colon \mathbb{R}^2 \to \mathbb{R}^2$ is translation by $\ell\tau \in \mathbb{R}^2$. This is a Riemannian spin bordism from S^1_ℓ to itself (the condition $\tau \in \mathbb{R}^2_+$ guarantees that conditions eq:bordism_conditions ??2 are satisfied). The associated usual bordism $\Sigma \setminus (\iota_2(U_2^-) \cup \iota_1(U_1^+))$ is a cylinder, and so we write $C^2_{\ell,\tau}$ for this endomorphism of S^1_ℓ.

The torus $T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset, \emptyset)$. For $\ell \in \mathbb{R}_+$ and $\tau \in \mathbb{R}^2_+$, consider the closed 2-manifold

$$T_{\ell,\tau} \stackrel{\text{def}}{=} \mathbb{R}^2/\ell(\mathbb{Z}\tau + \mathbb{Z}1).$$

This torus has an Riemannian metric (resp. spin structure) induced by the standard Riemannian metric (resp. spin structure) on $\mathbb{R}^2$, and hence it is a Riemannian spin bordism from $\emptyset$ to $\emptyset$; it represents the non-trivial element of $\Omega_2^{spin} \cong \mathbb{Z}/2$.

Next we want to describe some relations for the morphisms in RB^d described above; these relations (more precisely, their generalization to the super context) will be the geometric core of our proof of the main theorem. One of these relations will involve *self-gluing* of endomorphisms in RB^d, and so this needs to be explained first.

def:self-gluing **Definition 15.** For any object $Y \in \mathrm{RB}^d$, there is a map

$$\widehat{\ }: \mathrm{RB}^d(Y,Y) \longrightarrow \mathrm{RB}^d(Y,Y) \qquad \text{given by} \qquad \left(V \xhookrightarrow{\iota_2} \Sigma \xhookleftarrow{\iota_1} V\right) \mapsto \widehat{\Sigma},$$

where $\widehat{\Sigma}$ is the closed Riemannian manifold given by identifying $\iota_2(V) \subset \Sigma$ with $\iota_1(V) \subset \Sigma$ via the isometry $\iota_2(v) \mapsto \iota_1(v)$ (a look at figure 1 in definition def:RB 11 might be helpful). Here we assume that the open subsets $\iota_1(V)$ and $\iota_2(V)$ are *disjoint*; this can be obtained without loss of generality by replacing V by some smaller bicollar V' of Y. This guarantees that $\widehat{\Sigma}$ is a manifold.

lem:1_relations **Lemma 16.** *For $\ell, \ell' \in \mathbb{R}_+$ the following relations hold in the category RB^1:*

1. $I^1_\ell \circ I^1_{\ell'} = I^1_{\ell+\ell'} \in \mathrm{RB}^1(\mathrm{pt},\mathrm{pt})$;
2. $\widehat{I^1_\ell} = S^1_\ell \in \mathrm{RB}^1(\emptyset,\emptyset)$;

lem:2_relations **Lemma 17.** *For $\ell \in \mathbb{R}_+$, $\tau, \tau' \in \mathbb{R}^2_+$ the following relations hold in the category RB^2:*

1. $C^2_{\ell,\tau} \circ C^2_{\ell,\tau'} = C^2_{\ell,\tau+\tau'} \in \mathrm{RB}^2(S^1_\ell, S^1_\ell)$;
2. $\widehat{C^2_{\ell,\tau}} = T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset,\emptyset)$;
3. $C^2_{\ell,\tau+1} = C^2_{\ell,\tau} \in \mathrm{RB}^2(S^1_\ell, S^1_\ell)$;
4. $T^2_{g(\ell,\tau)} = T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset,\emptyset)$ *for every $g \in SL_2(\mathbb{Z})$;*

Here $g(\ell,\tau) \in \mathbb{R}^2_+$ in part 4 is defined by

$$g(\ell,\tau) \overset{\text{def}}{=} (\ell|c\tau+d|, \frac{a\tau+b}{c\tau+d}) \qquad \text{for} \qquad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right). \tag{18}$$ eq:SL_2(Z)_action

This describes an action of $SL_2(\mathbb{Z})$ on $\mathbb{R}_+ \times \mathbb{R}^2_+$. The space

$$\mathbb{R}_+ \times \mathbb{R}^2_+ \cong GL^+_2(\mathbb{R})/SO(2)$$

is the moduli space of flat Riemannian structures on the torus with a basis of its first homology group via $(\ell,\tau) \mapsto T^2_{\ell,\tau}$. Points in the same orbit lead to isometric tori; the converse holds as well and hence the quotient $SL_2(\mathbb{Z})\backslash(\mathbb{R}_+ \times \mathbb{R}^2_+)$ can be interpreted geometrically as the moduli space of flat Riemannian tori.

The proof of these relations is straightforward. We will prove a family version of these relations (Lemmas 42 and 43) below.

Additional structures on the bordism category RB^d**.** The Riemannian bordism category RB^d has additional structures completely analogous to the additional structures of the category TV discussed in the last subsection.

symmetric monoidal structure: disjoint union gives RB^d the structure of a symmetric monoidal category; the unit object is given by the empty $(d-1)$-manifold.

the anti-involution $^\vee$ On objects the anti-involution $^\vee$ is defined by interchanging U^+ and U^- (which can be thought of as flipping the orientation of the normal bundle to Y in U). If $(\Sigma, \iota_1, \iota_2)$ is a Riemannian bordism from Y_1 to Y_2, then $(\Sigma, \iota_1, \iota_2)^\vee = (\Sigma, \iota_2, \iota_1)$ is a Riemannian bordism from $Y_2^\vee$ to $Y_1^\vee$.

the involution $\bar{\ }$**:** Replacing the spin structure on the bicollars U as well as the bordisms Σ by their opposite defines an involution $\bar{\ }\colon \mathrm{RB}^d \to \mathrm{RB}^d$.

2.3 Quantum field theories, preliminary definition

def:prelim_QFT

Definition 19. (Preliminary.) A *quantum field theory of dimension d* is a symmetric monoidal functor

$$E\colon \mathrm{RB}^d \longrightarrow \mathrm{TV}^\pm$$

which is compatible with the involution $\bar{\ }$ and the anti-involution $^\vee$.

artition_function

Definition 20. Let $E\colon \mathrm{RB}^d \to \mathrm{TV}^\pm$ be a QFT of dimension $d=1$ or $d=2$, and let $E^+\colon \mathrm{RB}^d \to \mathrm{TV}$ be the composition of E with the forgetful functor $\mathrm{TV}^\pm \to \mathrm{TV}$ (see Remark 7). For $d=1$, the *partition function* of E is the function

$$Z_E\colon \mathbb{R}_+ \longrightarrow \mathbb{C} \quad \text{defined by} \quad \ell \mapsto E^+(S^1_\ell),$$

where $S^1_\ell \in \mathrm{RB}^1(\emptyset,\emptyset)$ is the circle of length ℓ (see example 13), and we used that $E^+(\emptyset) = \mathbb{C} \in \mathrm{TV}$ and $\mathrm{TV}(\mathbb{C},\mathbb{C}) = \mathbb{C}$.

For $d=2$, the *extended partition function* of E is the function

$$Z_E\colon \mathbb{R}_+ \times \mathbb{R}^2_+ \longrightarrow \mathbb{C} \quad \text{defined by} \quad (\ell,\tau) \mapsto E^+(T^2_{\ell,\tau}),$$

where $T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset,\emptyset)$ is the torus $\mathbb{R}^2/\ell(\mathbb{Z}\tau + \mathbb{Z}1)$ (see example ex:objects+morphisms2 14). The *partition function* of E is the function $Z_E \colon \mathbb{R}^2_+ \to \mathbb{C}$ obtained by restricting to $\ell = 1$.

The upper half-plane $\mathbb{R}^2_+$ parametrizes all tori up to conformal equivalence (not uniquely); it does not parametrize all tori equipped with Riemannian metrics up to isometry. However, every flat torus is isometric to $T_{\ell,\tau}$ for some $(\ell,\tau) \in \mathbb{R}_+ \times \mathbb{R}^2_+$. Hence, while the upper half plane is the appropriate domain for partition functions for *conformal* field theories, for QFT's it is better to work with the larger domain $\mathbb{R}_+ \times \mathbb{R}^2_+$, since we obtain invariance for the extended partition function defined on $\mathbb{R}_+ \times \mathbb{R}^2_+$ (see Lemma lem:modularity 21).

It is well-known that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ the tori $\mathbb{R}^2/(\mathbb{Z}\tau + \mathbb{Z}1)$ and $\mathbb{R}^2/(\mathbb{Z}g\tau + \mathbb{Z}1)$ are conformally equivalent, where $g\tau = \frac{a\tau+b}{c\tau+d}$. In particular, if E is a *conformal* field theory (i.e., the operators $E(\Sigma)$ depend only on the conformal structure on Σ), then its partition function $Z_E(\tau)$ is invariant under the $SL_2(\mathbb{Z})$-action on the upper half plane $\mathbb{R}^2_+$. However, these tori are not isometric, and hence the partition function of a QFT is usually not invariant.

What is still true is that for $g \in SL_2(\mathbb{Z})$ the torus $T_{g(\ell,\tau)}$ is isometric to $T_{\ell,\tau}$ (see part 4 of Lemma lem:2_relations 17; the $SL_2(\mathbb{Z})$nb-action on $\mathbb{R}_+ \times \mathbb{R}^2_+$ is defined in equation (eq:SL_2(Z)_action 18)). This implies:

lem:modularity **Lemma 21.** *The extended partition function $Z_E \colon \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{C}$ is invariant under the $SL_2(\mathbb{Z})$-action on $\mathbb{R}_+ \times \mathbb{R}^2_+$.*

In section sec:partition_function 5 we will show that for a *super symmetric* QFT E, its extended partition function $Z_E(\ell,\tau)$ is in fact *independent* of ℓ. In particular, the above corollary implies that its partition function $Z_E(\tau) = Z_E(1,\tau)$ is invariant under the $SL_2(\mathbb{Z})$-action.

2.4 Consequences of the axioms

The goal of this subsection is to prove the following results which will be needed in the proof of our main theorem.

prop:nuclear **Proposition 22.** *Let E be a d-dimensional QFT, and let $\Sigma \in \mathrm{RB}^d(Y_1,Y_2)$. Then the associated operator $E(\Sigma)^+ \colon E(Y_1)^+ \to E(Y_2)^+$ is nuclear (see definition def:nuclear 24 below).*

`prop:trace` **Proposition 23.** *Let E be a d-dimensional QFT, let $\Sigma \in \mathrm{RB}^d(Y, Y)$, and let $\widehat{\Sigma}$ be the closed Riemannian spin manifold obtained by gluing the two boundary copies of Σ. If we assume that the locally convex vector space $E(Y)^+$ has the approximation property (see Definition 25), then*

$$E(\widehat{\Sigma})^+ = \operatorname{str} E(\Sigma)^+,$$

where $\operatorname{str} E(\Sigma)^+$ *is the super trace of the operator $E(\Sigma)^+$ (see Definition 25).*

`def:nuclear` **Definition 24. (Nuclear operators).** Let V, W be locally convex vector spaces, and let V' be the dual of V. Then the *finite rank operators* are the continuous linear maps $T: V \to W$ which are in the image of the canonical map

$$\psi: W \otimes_{alg} V' \longrightarrow \mathcal{L}(V, W) \qquad w \otimes f \mapsto (v \mapsto w f(v)).$$

The map ψ is continuous (with the respect to the strong topology on V' and the projective topology on the algebraic tensor product $W \otimes_{alg} V'$). In particular, it extends uniquely to a continuous linear map

$$\bar{\psi}: W \otimes V' \longrightarrow \mathcal{L}(V, W)$$

of the projective tensor product (the completion of $W \otimes_{alg} V'$, see definition 4). The image of $\bar{\psi}$ consists of the *nuclear operators* $\mathcal{N}(V, W) \subset \mathcal{L}(V, W)$ (see [Köe, p. 214]). If W, V are Hilbert spaces, the nuclear operators $\mathcal{N}(V, W)$ are precisely the trace class operators (see [Köe, §42.6]) and the map $\bar{\psi}$ is injective.

`def:trace` **Definition 25. (Approximation property and traces).** In order to define the *trace* of a nuclear operator $T: V \to V$, the locally convex vector space V needs to satisfy a technical condition, namely that the finite rank operators (or equivalently the nuclear operators) are dense in $\mathcal{L}(V, V)$ (with respect to the topology of uniform convergence on all precompact subsets of V), see [Köe, §43.1]. This condition, the *approximation property*, guarantees that the map $\bar{\psi}$ is injective for any locally convex vector space W (see [Köe, §43]). Most locally convex spaces have the approximation property, for example Hilbert spaces and nuclear spaces; in fact, as Koethe mentions on p. 235 of his book, the construction of Banach spaces without the approximation property is quite involved. Grothendieck conjectured the existence of such spaces, but the first examples were constructed only in 1973.

If V is a locally convex vector space that has the approximation property, one can associate to any nuclear operator $T \in \mathcal{N}(V,V)$ a *trace* $\operatorname{tr} T \in \mathbb{C}$ (see [Koe, §42.7]); the map $\operatorname{tr}\colon \mathcal{N}(V,V) \to \mathbb{C}$ is simply the unique map making the following diagram commutative

$$\begin{array}{ccc} V \otimes V' & \xrightarrow{\bar\psi} & \mathcal{N}(V,V) \\ \downarrow \tau & & \downarrow \operatorname{tr} \\ V' \otimes V & \xrightarrow{ev} & \mathbb{C} \end{array}$$

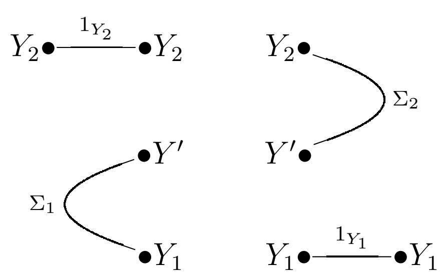
Here τ is the obvious symmetry isomorphism, and $ev\colon V' \otimes V \to \mathbb{C}$, $f \otimes v \mapsto f(v)$ is the evaluation map.

If V is equipped with a $\mathbb{Z}/2$-grading, one defines the *super trace*

$$\operatorname{str}\colon \mathcal{N}(V,V) \longrightarrow \mathbb{C}$$

as the trace above, but using the symmetry isomorphism $\tau\colon V \otimes V' \to V' \otimes V$ appropriate in the graded context, namely $v \otimes f \mapsto (-1)^{|v||f|} f \otimes v$ (where $|v|, |f| \in \{0,1\}$ is the degree of v resp. f). This agrees with the ususal definiton of super trace if V is finite dimensional.

Proof of Proposition 22. We factor the Riemannian spin bordism Σ from Y_1 to Y_2 in the following way



(26) eq:composition

Applying the symmetric monoidal functor $E\colon \mathrm{RB}^d \to \mathrm{TV}^{\pm}$ we obtain an analogous factorization of $E(\Sigma)$ in the form

$$E(Y_1) = \mathbb{C} \otimes E(Y_1) \xrightarrow{T_2 \otimes 1} E(Y_2) \otimes E(Y') \otimes E(Y_1) \xrightarrow{1 \otimes T_1} E(Y_2) \otimes \mathbb{C} = E(Y_2).$$

Then the following algebraic lemma implies the proposition. □

lem:nuclear **Lemma 27.** *Let V, W be objects of $\mathrm{TV}^{\pm}$ and let $T = (T^+, T^-)$ be a morphism from V to W. Assume that T in the symmetric monoidal category $\mathrm{TV}^{\pm}$ can be factored in the form*

$$V = \mathbb{C} \otimes V \xrightarrow{T_2 \otimes 1_V} W \otimes U \otimes V \xrightarrow{1_W \otimes T_1} W \otimes \mathbb{C} = W.$$

Then $T^+ \colon V^+ \to W^+$ is a nuclear map.

For the proof of this, we will need the following direct consequence of the duality relation eq:dual 6.

lem:T^+_formula **Lemma 28.** *If $T = (T^+, T^-)$ is a morphism in $\mathrm{TV}^{\pm}$ from $V = (V^+, V^-, \mu_V)$ to $\mathbb{C} = (\mathbb{C}, \mathbb{C}, \mu_{\mathbb{C}})$, then T^- determines T^+ via the formula*

$$T^+ v^+ = \mu_V(T^-(1) \otimes v^+).$$

Proof of Lemma lem:nuclear *27.* Let $\check{T} \in W^+ \otimes V^-$ be the image of $T_1^-(1) \otimes T_2^+(1)$ under the map

$$U^- \otimes V^- \otimes W^+ \otimes U^+ \xrightarrow{\tau} V^- \otimes U^- \otimes U^+ \otimes W^+ \xrightarrow{1 \otimes \mu_U \otimes 1} W^+ \otimes V^-,$$

where τ is the usual (graded) symmetry isomorphism. We claim that T^+ is image of $\check{T}$ under the map $\bar{\psi}$. To check this, we may assume that $T_1^-(1)$, $T_2^+(1)$ are of the form

$$T_1^-(1) = u^- \otimes v^- \in U^- \otimes V^- \qquad T_2^+(1) = w^+ \otimes u^+ \in W^+ \otimes U^+.$$

Then for $v^+ \in V^+$ we have

$$\begin{aligned} T^+(v^+) &= (1 \otimes T_1^+) \circ (T_2^+ \otimes 1)(v^+) = (1 \otimes T_1^+)(w^+ \otimes u^+ \otimes v^+) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)} w^+ \otimes T_1^+(u^+ \otimes v^+) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)} w^+ \mu_{U \otimes V}(u^- \otimes v^- \otimes u^+ \otimes v^+) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)+|v^-||u^+|} w^+ \mu_U(u^- \otimes u^+) \mu_V(v^- \otimes v^+) \end{aligned}$$

Here we use Lemma lem:T^+_formula 28 to express T_1^+ in terms of $T_1^-(1)$.

Now let us express $\check{T}$ explicitly:

$$\begin{aligned} \check{T} &= (1 \otimes \mu_U \otimes 1) \circ \tau(u^- \otimes v^- \otimes w^+ \otimes u^+) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)+|v^-||u^+|} (1 \otimes \mu_U \otimes 1)(w^+ \otimes u^- \otimes u^+ \otimes v^-) \\ &= (-1)^{|w^+|(|u^-|+|v^-|)+|v^-||u^+|} w^+ \otimes \mu_U(u^- \otimes u^-) v^- \end{aligned}$$

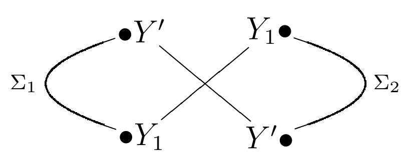
This implies that

$$\bar{\psi}(\check{T})(v^+) = (-1)^{|w^+|(|u^-|+|v^-|)+|v^-||u^+|} w^+ \mu_U(u^- \otimes u^-)\mu_V(v^- \otimes v^+),$$

which proves our claim $T^+ = \bar{\psi}(\check{T})$. In particular T^+ is in the image of $\bar{\psi}$ and hence nuclear. ☐

m:care_with_signs **Remark 29.** In the above proof we are more careful with signs than we need to be at this point; e.g., the map $\mu_U \colon U^- \otimes U^+ \to \mathbb{C}$ is grading preserving, and hence $\mu_U(u^- \otimes u^+) = 0$ unless $|u^-| = |u^+|$, and similarly for $v^\pm$. So in the above proof (as well as in the proof of Lemma lem:trace 30 below) we could make things a little easier by assuming $|u^-| = |u^+|$ and $|v^-| = |v^+|$ without loss of generality. However, we will need more general versions of these results, where the complex vector spaces are replaced by modules over a $\mathbb{Z}/2$-graded algebra Λ, and μ_U, μ_V take values in Λ. Again, μ_U, μ_V are *even*, but this no longer implies $\mu_U(u^- \otimes u^+) = 0$ unless $|u^-| = |u^+|$ (and the analogous statement for $v^\pm$). As long as we avoid the simplifying assumptions $|u^-| = |u^+|$ and $|v^-| = |v^+|$, the proof of these more general results is the same.

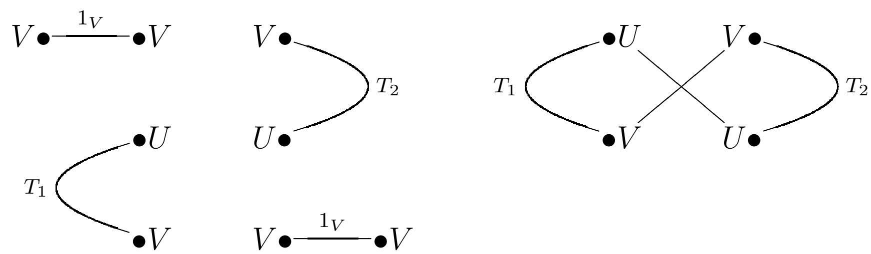
Proof of Proposition prop:trace *23.* Like in the proof of Proposition prop:nuclear 22, we decompose the morphism $\Sigma \in \mathrm{RB}^d(Y_1, Y_2)$ as shown in the figure eq:composition 26. Then the closed manifold $\widehat{\Sigma} \in \mathrm{RB}^d(\emptyset, \emptyset)$ can be written as the following composition in RB^d:



Applying the functor E results in corresponding decompositions of the morphisms $E(\Sigma)$ (resp. $E(\widehat{\Sigma})$) in the category $\mathrm{TV}^\pm$. The following algebraic result then implies the desired statement $E(\widehat{\Sigma})^+ = \operatorname{str} E(\Sigma)^+$, provided $E(Y)^+$ has the approximation property. ☐

lem:trace **Lemma 30.** *Let U, V be objects of* $\mathrm{TV}^\pm$*, let $\tau \colon U \otimes V \cong V \otimes U$ be the symmetry isomorphism, and let $T_1 \colon U \otimes V \to \mathbb{C}$, $T_2 \colon \mathbb{C} \to V \otimes U$ be morphisms in* $\mathrm{TV}^\pm$*. Let $T \colon V \to V$ (resp. $\widehat{T} \colon \mathbb{C} \to \mathbb{C}$) be the following compositions in*

$\mathrm{TV}^{\pm}$ *(these pictures are to be read right to left):*



If V^+ has the approximation property, then

$$\widehat{T}^+ = \operatorname{str}\left(T^+ : V^+ \to V^+\right).$$

Proof. As in the proof of Lemma 27 we can assume

$$T_1^-(1) = u^- \otimes v^- \in U^- \otimes V^- \qquad T_2^+(1) = v^+ \otimes u^+ \in V^+ \otimes U^+;$$

here we write v^+ instead of w^+, since now we have $W = V$. We recall from that proof also that $T = \bar{\psi}(\check{T})$, where

$$\check{T} = (-1)^{|v^+|(|u^-|+|v^-|)+|v^-||u^+|} v^+ \otimes \mu_U(u^- \otimes u^-) v^- \in V^+ \otimes V^-$$

This implies that

$$\tau(\check{T}) = (-1)^{|v^+||u^+|+|v^-||u^+|} \mu_U(u^+ \otimes u^-) v^- \otimes v^+ \in V^- \otimes V^+$$

It follows that

$$\operatorname{str} T = \mu_V(\tau(\check{T})) = (-1)^{|v^+||u^+|+|v^-||u^+|} \mu_U(u^+ \otimes u^-) \mu_V(v^- \otimes v^+).$$

On the other hand, we calculate the composition $\widehat{T}$ as follows:

$$\begin{aligned}
\widehat{T}(1) &= T_1^+ \circ \tau \circ T_2^+(1) = T_1^+ \circ \tau(v^+ \otimes u^+) = (-1)^{|v^+||u^+|} T_1^+(u^+ \otimes v^+) \\
&= (-1)^{|v^+||u^+|} \mu_{U \otimes V}(u^- \otimes v^- \otimes u^+ \otimes v^+) \\
&= (-1)^{|v^+||u^+|+|v^-||u^+|} \mu_U(u^- \otimes u^+) \mu_V(v^- \otimes v^+);
\end{aligned}$$

Here the second to last equality sign is a consequence of Lemma 28 applied to T_1; the last equality is the definition of $\mu_{U \otimes V}$. □

3 QFT's as smooth functors

sec:smooth

The goal of this section is to obtain a better definition of *quantum field theory* (see definition def:smoothQFT 45) by adding another requirement to our preliminary definition def:prelim_QFT 19 of the previous section. We recall that according to that definition a d-dimensional field theory is a symmetric monoidal functor

$$E\colon \mathrm{RB}^d \longrightarrow \mathrm{TV}^{\pm}$$

from the Riemannian spin bordism category to the category of (pairs of) locally convex vector spaces. This functor is required to be compatible with the (anti-) involutions $^-$, $^\vee$ defined in both categories. In this section we add another requirement, namely that the functor E is *smooth* in the sense explained below.

3.1 Smooth categories and functors

To motivate our formal definition of *smooth functor* and *smooth category* below (see Definition def:smooth 33), we want to start with an informal discussion of smoothness of the functor E. Heuristically, smoothness means that the vector space $E(Y)$ associated to a $(d-1)$-manifold as well as the operator $E(\Sigma)\colon E(Y_1) \to E(Y_2)$ associated to a bordism Σ from Y_1 to Y_2 should depend smoothly on Y (resp. Σ); in particular, given a smooth family Y of closed $(d-1)$-manifolds parametrized by a manifold S, we should obtain a smooth family of vector spaces $E_S(Y)$ parametrized by S (by applying E to each member of the family), and similarly for families of bordisms. In other words, if $E\colon \mathrm{RB}^d \longrightarrow \mathrm{TV}^{\pm}$ is a smooth functor, we expect that it induces for every manifold S a functor

$$E_S\colon \mathrm{RB}^d_S \longrightarrow \mathrm{TV}^{\pm}_S,$$

where RB_S (resp. TV_S) is the category whose

objects are *smooth families* of objects of RB_S (resp. $TV^{\pm}_S$) parametrized by S;

morphisms are *smooth families* of morphisms of RB_S (resp. $TV^{\pm}_S$) parametrized by S.

In particular, for $S = \mathrm{pt}$ (the one-point space), the category RB^d_S (resp. $\mathrm{TV}^{\pm}_S$) agrees with the category RB^d (resp. $TV^{\pm}$) defined in the previous section. Definiton def:RB_S 39 (resp. def:TV^pm_S 37) below will provide the precise definition of the family categories RB^d_S (resp. $\mathrm{TV}^{\pm}_S$).

Since the functors E_S are all obtained by applying the same functor E in each fiber, we expect that for any smooth map $f\colon S \to S'$ the diagram of functors

$$\begin{array}{ccc} \mathrm{RB}^d_{S'} & \xrightarrow{E_{S'}} & \mathrm{TV}_{S'} \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{RB}^d_S & \xrightarrow{E_S} & \mathrm{TV}_S \end{array} \tag{31}$$

eq:compatibility

is commutative, where the vertical functors f^* are given by pullback via f.

This can be formalized by saying that we have a commutative diagram

$$\mathrm{MAN}^{op} \underset{\mathrm{TV}}{\overset{\mathrm{RB}^d}{\Downarrow E}} \mathrm{CAT}, \tag{32}$$

eq:natural_transf

where MAN is the category of manifolds, CAT is the category of categories (objects are categories, morphisms are functors), RB^d, TV are the functors given on objects by $S \mapsto \mathrm{RB}^d_S$ (resp. $S \mapsto \mathrm{TV}_S$) and E is the natural transformation between these functors given by $S \mapsto E_S$.

This discussion motivates the following definiton.

def:smooth **Definition 33. (Smooth categories and functors)** A *smooth category* $\mathcal{C}$ is a functor

$$\mathcal{C}\colon\ \mathrm{MAN}^{op} \longrightarrow \mathrm{CAT}$$

from the category of smooth manifolds to the category of categories. If S is a manifold, we will write $\mathcal{C}_S$ for the category $\mathcal{C}(S)$; if $f\colon S' \to S$ is a smooth map, we will write $f^*\colon \mathcal{C}_S \to \mathcal{C}_{S'}$ for the corresponding functor $\mathcal{C}(f)$.

If $\mathcal{C}$ and $\mathcal{D}$ are smooth categories, a *smooth functor* E from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation

$$\mathrm{MAN}^{op} \underset{\mathcal{D}}{\overset{\mathcal{C}}{\Downarrow E}} \mathrm{CAT}, \tag{34}$$

For a manifold S, we will write $E_S\colon \mathcal{C}_S \to \mathcal{D}_S$ for the corresponding functor.

There are obvious variants of the above definition obtained by replacing the category of manifolds by other categories. We define *continuous* (resp. *holomorphic* resp. *super*) categories and functors by substituting for MAN the category of topological spaces (resp. complex analytic manifolds resp. super manifolds).

Remark 35. Super categories and super functors is the appropriate framework for our geometric definition of super symmetric quantum field theories in section sec:susyQFT 4; similarly, holomorphic categories and functors is the right language for holomorphic field theories (also called *chiral*).

subsec:smoothTV

3.2 The smooth category of locally convex vector spaces

As mentioned above, if $V \to S$ is a bundle (smooth, locally trivial), and $f\colon S' \to S$ and $g\colon S'' \to S'$ are maps, then $(fg)^*V$ is not equal to g^*f^*V. To overcome this difficulty, we replace bundle by *quasi bundles* (our ad hoc terminology):

Definition 36. A *quasi bundle* over a smooth manifold S is a pair (h, V), where $h\colon S \to T$ is a smooth map and $V \to T$ is a smooth, locally trivial bundle over T. If (h', V') is another quasi bundle over X, a map from (h, V) to (h', V') is smooth bundle map $F\colon h^*V \to (h')^*V'$.

A smooth map $f\colon S' \to S$ induces a contravariant functor

$$f^*\colon \text{QBUN}(S) \longrightarrow \text{QBUN}(S')$$

from the category of quasi bundles over S to those over S'. It is defined by

$$f^*(h, V) = (f \circ h, V)$$

on objects; on morphisms it is given by the usual pullback via f. In particular, the functor $(fg)^*$ is *equal* to $g^* \circ f^*$.

We note that the category of quasi bundles over X is equivalent to the category of bundles over X; sending (h, V) to h^*V provides the equivalence.

def:TV^pm_S **Definition 37. (The category $\text{TV}^{\pm}_S$.)** Let S be a smooth manifold. We want define the category $\text{TV}^{\pm}_S$ of S-families of (pairs of) locally convex vector spaces in such a way that $\text{TV}^{\pm}_{\text{pt}}$ agrees with the category $\text{TV}^{\pm}$ of definition

def:TV^pm
5. To do this, we can extend that definition word by word, just replacing *locally convex vector spaces* by *quasi bundles of locally convex vector spaces over* S and *continuous linear maps* by *smooth bundle maps*. In particular, objects are triples

$$V = (V^+, V^-, \mu_V),$$

where $V^\pm$ are quasi bundles of locally convex vector spaces over S, and $\mu_V \colon V^- \otimes V^+ \to \mathbb{C}_S$ is a smooth bundle map. Here the *tensor product* of quasi bundles over S is given by

$$(h\colon S \to T, V) \otimes (h'\colon S \to T', V') \overset{\text{def}}{=} (h \times h'\colon S \to T \times T', p_1^* V \otimes p_2^* V'),$$

where $p_1^* V \otimes p_2^* V' \to T \times T'$ is the fiberwise (projective) tensor product, and p_1 (resp. p_2) is the projection onto the first (resp. second) factor. Moreover, $\mathbb{C}_S$ is the quasi bundle $(p\colon S \to \mathrm{pt}, \mathrm{pt} \times \mathbb{C})$; replacing $\mathbb{C}$ by any locally convex vector space V gives a quasi bundle V_S over S. We note that $\mathbb{C}_S$ is the unit for the tensor product of quasi bundles over S.

We observe that a smooth map $f\colon S' \to S$ induces a functor

$$f^*\colon \mathrm{TV}^p\, m_S \longrightarrow \mathrm{TV}^\pm_{S'}$$

via pull-back of quasi bundle; moreover – thanks to using quasi-bundles instead of bundles – if $g\colon S'' \to S'$ is a smooth map, then the functor $(gf)^*$ is *equal* to $g^* f^*$.

The additional structures for the category TV_S (the symmetric monoidal structure, the (anti-) involution and the adjunction transformation) generalize in a straightforward way to the category TV_S. These structures are compatible with the pull-back functor f^*.

def:smoothTV^pm

Definition 38. The smooth category $\mathrm{TV}^\pm$ is the functor

$$\mathrm{MAN}^{op} \longrightarrow \mathrm{CAT}$$

which sends a smooth manifold S to the category $\mathrm{TV}^p\, m_S$ defined below and a smooth map $f\colon S' \to S$ to the pull-back functor $f^*\colon \mathrm{TV}^\pm_S \to \mathrm{TV}^\pm_{S'}$.

3.3 The smooth Riemannian bordism category

def:RB_S

Definition 39. (The category RB^d_S) Let S be a smooth manifold. We want to define the category RB^d_S of S-families of Riemannian bordisms of dimension d in such a way that $\mathrm{RB}^d_{\mathrm{pt}}$ agrees with the category RB^d of definition def:RB

11. We observe that objects and morphisms of the category RB^d were defined in terms of the category Riem^d whose objects are Riemannian spin manifolds of dimension d and morphisms are isometric spin embeddings. Here we need only replace Riem^d be the appropriate category Riem^d_S whose objects (resp. morphisms) are S-families of objects (resp. morphisms) of Riem^d. More precisely,

- an object of Riem^d_S is a smooth quasi bundle $U \to S$ with d-dimensional fibers, equipped with a fiberwise Riemannian metric and spin structure (i.e., a spin structure on the vertical tangent bundle).
- A morphism from U to U' is a smooth quasi bundle map $f\colon U \to U'$ preserving the fiberwise Riemannian metric and spin structure.

def:S-points **Definition 40.** The following examples of objects and morphisms in RB^d_S will be parametrized by smooth maps from S to suitable smooth manifolds M (e.g., $M = \mathbb{R}_+$ or $M = \mathbb{R}^2_+$). It will be convenient to use the notation $M(S)$ for the set of smooth maps from S to M; elements of $M(S)$ are referred to as *S-points of M*. We will make use of this notation in particular later when we use the *functor of points* formalism to describe maps between super manifolds.

objects+morphisms **Example 41. (Examples of objects and morphisms of RB^d_S.)** The following examples of objects and morphisms in the family bordism category RB^d_S will be important to us. They are *family versions* of the examples 13 and 14, parametrized by a smooth manifold S. We shall write these examples in a fairly formal way that will extend to super manifolds without additional work (see example 70). All fiber bundles over S involved here will be topologically trivially; they are either of the form $S \times \mathbb{R}^d \to S$ (with the obvious fiberwise metric and spin structure), or a quotient of this bundle by a discrete subgroup of the group $\mathbb{R}^d(S)$ (of smooth maps from S to $\mathbb{R}^d$, see definition 40). The group $\mathbb{R}^d(S)$ acts by structure preserving bundle automorphisms by associating to $f \in \mathbb{R}^d(S)$ the bundle automorphism

$$S \times \mathbb{R}^d \longrightarrow S \times \mathbb{R}^d \qquad \text{given by} \qquad (s,x) \mapsto (s, f(s)+x).$$

Abusing language, we will write again f for this bundle automorphism.

the point $\mathrm{pt}_S \in \mathrm{RB}^1_S$. The quadruple

$$\mathrm{pt}_S \overset{\text{def}}{=} (U, Y, U^+, U_-) = (S \times \mathbb{R}, S \times \{0\}, S \times (-\infty, 0), S \times (0, \infty))$$

is an object of RB^1_S.

the interval $I^1_\ell \in \mathrm{RB}^1_S(\mathrm{pt}_S, \mathrm{pt}_S)$**.** For $\ell \in \mathbb{R}_+(S)$ the pair of bundle maps

$$U = S \times \mathbb{R} \xhookrightarrow{\mathrm{id}} \Sigma = S \times \mathbb{R} \xhookleftarrow{\ell} U = S \times \mathbb{R}\ ,$$

is a Riemannian spin bordism from pt_S to pt_S. We will use the notation I^1_ℓ for this S-family of intervals whose length is given by the function $\ell\colon S \to \mathbb{R}_+$.

the circle $S^1_\ell \in \mathrm{RB}^1_S(\emptyset, \emptyset)$**.** For $\ell \in \mathbb{R}_+(S)$ the circle bundle

$$S^1_\ell \stackrel{\mathrm{def}}{=} (S \times \mathbb{R})/\mathbb{Z}\ell$$

is a Riemannian spin bordism from $\emptyset$ to $\emptyset$.

the circle $S^1_\ell \in \mathrm{RB}^2_S$**.** For $\ell \in \mathbb{R}_+(S)$ the quadruple

$$S^1_\ell \stackrel{\mathrm{def}}{=} ((S \times \mathbb{R}^2)/\mathbb{Z}\ell, (S \times \mathbb{R})/\mathbb{Z}\ell, (S \times \mathbb{R}^2_+)/\mathbb{Z}\ell, (S \times \mathbb{R}^2_-)/\mathbb{Z}\ell)$$

is an object of RB^2_S.

the cylinder $C^2_{\ell,\tau} \in \mathrm{RB}^2(S^1_\ell, S^1_\ell)$**.** For $\ell \in \mathbb{R}_+(S)$ and $\tau \in \mathbb{R}^2(S)$, consider the following pair of bundle maps preserving the fiberwise metrics and spin structures

$$U_2 = (S \times \mathbb{R}^2)/\mathbb{Z}\ell \xhookrightarrow{\mathrm{id}} \Sigma = (S \times \mathbb{R}^2)/\mathbb{Z}\ell \xhookleftarrow{\ell\cdot\tau} U_1 = (S \times \mathbb{R}^2)/\mathbb{Z}\ell\ .$$

For $\tau \in \mathbb{R}^2_+(S)$ conditions eq:bordism_conditions 12 are satisfied and this is a Riemannian spin bordism from S^1_ℓ to itself.

The torus $T^2_{\ell,\tau} \in \mathrm{RB}^2(\emptyset, \emptyset)$**.** For $t \in \mathbb{R}_+(S)$ and $\tau \in \mathbb{R}^2_+(S)$ the torus bundle

$$T^2_{\ell,\tau} \stackrel{\mathrm{def}}{=} (S \times \mathbb{R}^2)/\ell(\mathbb{Z}\tau + \mathbb{Z}1).$$

is a Riemannian bordism from $\emptyset$ to $\emptyset$.

The following two lemmas are the family versions of the relations among morphisms in RB^d formulated in Lemmas lem:1_relations 16 and lem:2_relations 17.

lem:1_relations_S **Lemma 42.** *Let S be a smooth manifold and $\ell, \ell' \in \mathbb{R}_+(S)$. Then the following relations hold in the category* RB^1_S*:*

1. $I^1_\ell \circ I^1_{\ell'} = I^1_{\ell+\ell'} \in \mathrm{RB}^1_S(\mathrm{pt},\mathrm{pt})$;

2. $\widehat{I^1_\ell} = S^1_\ell \in \mathrm{RB}^1_S(\emptyset,\emptyset)$;

lem:2_relations_S **Lemma 43.** *Let S be a smooth manifold and $\ell \in \mathbb{R}_+(S)$, $\tau,\tau' \in \mathbb{R}^2_+(S)$. Then the following relations hold in the category RB^2_S:*

1. $C^2_{\ell,\tau} \circ C^2_{\ell,\tau'} = C^2_{\ell,\tau+\tau'} \in \mathrm{RB}^2_S(S^1_\ell, S^1_\ell)$;

2. $\widehat{C^2_{\ell,\tau}} = T^2_{\ell,\tau} \in \mathrm{RB}^2_S(\emptyset,\emptyset)$;

3. $C^2_{\ell,\tau+1} = C^2_{\ell,\tau} \in \mathrm{RB}^2_S(S^1_\ell, S^1_\ell)$;

4. $T^2_{g(\ell,\tau)} = T^2_{\ell,\tau} \in \mathrm{RB}^2_S(\emptyset,\emptyset)$ *for every* $g \in SL_2(\mathbb{Z})$;

We will only prove Lemma 43, since the proof two relations of Lemma 42 is completely analogous, but simpler than the proof of the first two relations of Lemma 43.

Proof. To prove the first relation, we use the notation of Definition 11 (where the composition was described) and we write $C^2_{\ell,\tau} = (\Sigma,\iota_2,\iota_1)$ and $C^2_{\ell,\tau'} = (\Sigma',\iota'_2,\iota'_1)$ and arrange this in the diagram

$$\begin{array}{ccccc}
V_3 = (S\times\mathbb{R}^2)/\mathbb{Z}\ell & \xhookrightarrow{\iota_2 = id} & \Sigma = (S\times\mathbb{R}^2)/\mathbb{Z}\ell & \xhookleftarrow{\iota_1 = \ell\tau} & (S\times\mathbb{R}^2)/\mathbb{Z}\ell = V_2 \\
 & & \uparrow{\scriptstyle \iota_1\circ(\iota_2)^{-1} = \ell\tau} & & \\
V_2 = (S\times\mathbb{R}^2)/\mathbb{Z}\ell & \xhookrightarrow{\iota'_2 = id} & \Sigma' = (S\times\mathbb{R}^2)/\mathbb{Z}\ell & \xhookleftarrow{\iota'_1 = \ell\tau'} & (S\times\mathbb{R}^2)/\mathbb{Z}\ell = V_1
\end{array}$$

According to the construction, the composition is given by the bordism $\Sigma'' \stackrel{\mathrm{def}}{=} \Sigma \cup \Sigma'$ (where a point $x' \in \Sigma'$ is identified with its image under $\iota_1 \circ (\iota_2)^{-1}$), together with the embeddings

$$\iota''_3 \colon V_3 \stackrel{\iota_3}{\hookrightarrow} \Sigma \subset \Sigma'' \qquad \text{and} \qquad \iota''_1 \colon V'_1{}^{\iota'_1} into \Sigma' \subset \Sigma''$$

The diagram above shows that we can identify $\Sigma'' \stackrel{\mathrm{def}}{=} \Sigma \cup \Sigma'$ with Σ (by sending $x' \in \Sigma'$ to $\tau(x)$); with this identification, we have $\iota''_3 = id$ and $\iota''_1 = \ell\tau \circ \ell\tau' = \ell\tau + \ell\tau' = \ell(\tau+\tau')$. In other words, the composition $(\Sigma'',\iota''_3,\iota''_1)$ is equal to $C^2_{\ell,\tau+\tau'}$.

The second relation follows from the fact that the projection map

$$(S\times\mathbb{R}^2)/\mathbb{Z}\ell \longrightarrow (S\times\mathbb{R}^2)/\ell(\mathbb{Z}\tau + \mathbb{Z}1) = T^2_{\ell,\tau}$$

induces the desired bundle isomorphism $\widehat{C^2_{\ell,\tau}} \cong T^2_{\ell,\tau}$.

The third relation follows from the fact that the bundle automorphisms

$$\ell\tau, \ell(\tau+1)\colon S \times \mathbb{R}^2 \longrightarrow S \times \mathbb{R}^2$$

induce the *same* bundle automorphism on $(S \times \mathbb{R}^2)/\mathbb{Z}\ell$.

To prove the last relation, let us use the notation

$$\Lambda_{\ell,\tau} \overset{\text{def}}{=} \ell(\mathbb{Z}\tau + \mathbb{Z}1) \subset \mathbb{R}^2_+(S).$$

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ let $R_a\colon \mathbb{R}^2(S) \to \mathbb{R}^2(S) = \mathbb{C}(S)$ be the rotation given by multiplication by $a = \frac{c\tau+d}{|c\tau+d|} \in S^1(S) \subset \mathbb{C}(S)$. We note that

$$\begin{aligned} R_a(\Lambda_{g(\ell,\tau)}) = \frac{c\tau+d}{|c\tau+d|}\Lambda_{g(\ell,\tau)} &= \frac{c\tau+d}{|c\tau+d|}\ell|c\tau+d|\left(\mathbb{Z}\frac{a\tau+b}{c\tau+d} + \mathbb{Z}1\right) \\ &= \ell\left(\mathbb{Z}(a\tau+b) + \mathbb{Z}(c\tau+d)\right) = \Lambda_{\ell,\tau}. \end{aligned}$$

Abusing notation, let us also write R_a for the bundle automorphism

$$R_a\colon S \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad (s,v) \mapsto (s, a(s)v).$$

The calculation above shows that R_a induces a bundle isomorphism

$$T^2_{g(\ell,\tau)} = (S \times \mathbb{R}^2)/\Lambda_{g(\ell,\tau)} \longrightarrow (S \times \mathbb{R}^2)/\Lambda_{g(\ell,\tau)} = T^2_{\ell,\tau}.$$

□

A smooth map $f\colon S' \to S$ induces a pull-back functor $f^*\colon \mathrm{RB}^d_S \to \mathrm{RB}^d_{S'}$ such that $(fg)^* = g^*f^*$. The fiberwise disjoint union gives RB^d_S the structure of a symmetric monoidal category; the involution $^-$, the anti-involution $^\vee$, and the adjunction transformation generalize from RB^d to RB^d_S.

def:smoothRB **Definition 44.** The smooth category RB^d is the functor

$$\mathrm{MAN}^{op} \longrightarrow \mathrm{CAT}$$

which sends a smooth manifold S to the category RB_S and a smooth map $f\colon S' \to S$ to the pull-back functor $f^*\colon \mathrm{RB}^d_S \to \mathrm{RB}^d_{S'}$.

3.4 QFT's of dimension d

def:smoothQFT

Definition 45. A *quantum field theory of dimension* d is a smooth symmetric monoidal functor

$$E\colon\ \mathrm{RB}^d \longrightarrow \mathrm{TV}^{\pm},$$

which is compatible with the involution $\bar{\ }$ and the anti-involution $^\vee$. Here RB^d and $\mathrm{TV}^{\pm}$ are smooth categories (which we don't indicate in our notation).

artition_function

Remark 46. If E is a QFT of dimension 1 we can apply E_S to the universal family of circles $S^1_\ell \in \mathrm{RB}^1_S(\emptyset,\emptyset)$ parametrized by $S = \mathbb{R}_+$, where $\ell \in \mathbb{R}_+(\mathbb{R}_+)$ is the identity function. The result is a smooth function $E(S^1_\ell)\colon \mathbb{R}_+ \to \mathbb{C}$; the compatibility condition (31) (for $S' = \mathrm{pt}$) implies that this is the partition function Z_E of definition 20; in particular Z_E is *smooth.*

Similarly, applying a 2-dimensional QFT E to the universal family of tori $T^2_{\ell,\tau} \in \mathrm{RB}^2_S(\emptyset,\emptyset)$, where $S = \mathbb{R}_+ \times \mathbb{R}^2_+$ ($\mathbb{R}^2_+ \subset \mathbb{R}^2$ is the upper half-plane) and $\ell\colon S \to R_+$, $\tau\colon S \to \mathbb{R}^2_+$ are the projection maps, leads to a smooth function $\mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{C}$ which agrees with the extended partition function Z_E of definition 20. Again, we conclude that the smoothness of the functor E implies the smoothness of Z_E.

4 Super symmetric quantum field theories

sec:susyQFT

As mentioned in the introduction, super symmetric field theories are a variant of the field theories described in the previous section obtained by replacing the Riemannian bordism category RB^d by its super version $\mathrm{RB}^{d|1}$, whose objects are closed super manifolds of dimension $d-1|1$, and whose morphisms are super bordisms of dimension $d|1$ equipped with a super Riemannian metric. The goal of this section is to give a rapid introduction to super manifolds and to define super Riemannian metrics on super manifolds of dimension $d|1$ for $d = 1, 2$. For more details on super manifolds we refer the reader to [Va], [Fr] and [DM].

4.1 Super manifolds

Before giving the sheaf-theoretic definition of super manifolds, we make some motivational remarks. Like schemes, super manifolds are described in terms of their *functions*. In particular, associated to any super manifold M of dimension $n|m$ is a $\mathbb{Z}/2$-graded, graded commutative algebra $C^\infty(M)$, the elements of which we think of as *functions on M*. For example, there is a super manifold denoted $\mathbb{R}^{n|m}$ of dimension $n|m$ with

$$C^\infty(\mathbb{R}^{n|m}) = C^\infty(\mathbb{R}^n) \otimes \Lambda[\theta_1, \dots, \theta_m],$$

where $\Lambda[\theta_1, \dots, \theta_m]$ is the exterior algebra generated by m elements $\theta_1, \dots, \theta_m$ of odd degree. The super manifold $\mathbb{R}^{n|m}$ is the local model for a super manifold of dimension $n|m$ in the same way that $\mathbb{R}^n$ is the local model for a manifold of dimension n or that $\mathrm{spec}(R)$ for a commutative ring R is the local model for a scheme. We note that the algebra $C^\infty(\mathbb{R}^{n|m})$ can be interpreted as the global sections of a sheaf $\mathcal{O}^{n|m}$ of graded commutative algebras over $\mathbb{R}^n$, which on open subsets $U \subset \mathbb{R}^n$ is given by

$$\mathcal{O}^{n|m}(U) \overset{\text{def}}{=} C^\infty(U) \otimes \Lambda[\theta_1, \dots, \theta_m]$$

Definition 47. (Super manifolds.) A super manifold M of dimension $n|m$ is a Hausdorff space M_{red} with countable basis together with a sheaf $\mathcal{O}$ of graded commutative algebras which is locally isomorphic to the sheaf $\mathcal{O}^{n|m}$ over $\mathbb{R}^n$ described above. The sheaf $\mathcal{O}$ is called the *structure sheaf*, its global sections $\mathcal{O}(M_{red})$ is a $\mathbb{Z}/2$-graded algebra denoted $C^\infty(M)$, and the topological space M_{red} is called the *reduced manifold*. To see that M_{red} is in fact a smooth manifold, let $J \subset \mathcal{O}$ be the nilpotent ideal generated by the odd functions in $\mathcal{O}$, and let $\mathcal{O}_{red}$ be the quotient sheaf, which is a sheaf of commutative $\mathbb{R}$-algebras over the topological space M_{red}. We note that any section f of $\mathcal{O}_{red}$ can in fact be interpreted as a *continuous function* on M_{red}, whose value at $x \in M_{red}$ is the unique real number λ such that $f - \lambda$ is not invertible in any neighborhood of x. Since $(M_{red}, \mathcal{O}_{red})$ is locally isomorphic to $(\mathbb{R}^n, \mathcal{O}^{n|m}_{red})$, and $\mathcal{O}^{n|m}_{red}$ is the sheaf of smooth functions on $\mathbb{R}^n$, we see that $\mathcal{O}_{red}$ gives a smooth structure on M_{red}. In particular, if M is a super manifold of dimension $n|0$, then the sheaf $\mathcal{O}$ is equal to $\mathcal{O}_{red}$ and hence a super manifold of dimension $n|0$ can be identified with an ordinary smooth manifold of dimension n.

Many constructions and definitions for manifolds have analogs for super manifolds. To formulate these analogs, we typically try to express the original notion for ordinary manifolds in terms of their smooth functions, and then generalize to super manifolds. For example:

vector field A vector field X on an ordinary manifold N can be viewed as a derivation of the algebra $C^\infty(N)$ of smooth functions on N. A vector field X on a super manifold M is defined as a graded derivation of $C^\infty(M)$, i.e., $X\colon C^\infty(M) \to C^\infty(M)$ is a linear map with the derivation property

$$X(fg) = X(f)g + (-1)^{|X||f|} f X(g) \qquad \text{for} \quad f, g \in C^\infty(M)^\pm.$$

Here $|X|, |f| \in \{0,1\}$ is the degree of X and f, respectively. More precisely, this defines even vector fields ($|X| = 0$) and odd vector fields ($|X| = 1$); a general vector field is the sum of an even and an odd vector field.

vector bundle We recall that the category of smooth vector bundles over an ordinary manifold N is equivalent to the category of sheaves of locally free modules over the sheaf of smooth functions on N (by associating to a vector bundle $E \to N$ its sheaf of smooth sections). We will use this equivalence to identify smooth vector bundles with sheaves of locally free modules. Moreover, if M is a super manifold, we *define* a vector bundle of dimension $p|q$ to be a sheaf E of graded modules over the structure sheaf $\mathcal{O}_M$, which is locally free of rank $p|q$ (i.e., for a sufficiently small open subset U the graded $\mathcal{O}_M(U)$-module $E(U)$ has a basis consisting of p even and q odd elements). In that situation, the quotient $E_{red} \stackrel{\text{def}}{=} E/JE$ is a sheaf of locally free $\mathbb{Z}/2$-graded modules over $\mathcal{O}_{red}$, i.e., $E_{red} = E_{red}^{ev} \oplus E_{red}^{odd}$ is a $\mathbb{Z}/2$-graded vector bundle over the reduced manifold M_{red}.

tangent bundle For an ordinary n-manifold N, the tangent bundle TN interpreted as a sheaf is equal to $\mathrm{Der}(\mathcal{O}_N)$, the sheaf of derivations of the structure sheaf $\mathcal{O}_N$. This is a sheaf of locally free modules of rank n. If M is a super manifold of dimension $n|m$, let $TM \stackrel{\text{def}}{=} \mathrm{Der}(\mathcal{O}_M)$ be the sheaf of vector fields (aka graded derivations) on M; this a vector bundle (aka sheaf of locally free modules over $\mathcal{O}_M$) of rank $p|q$. The associated reduced bundle $TM_{red} = TM_{red}^{ev} \oplus TM_{red}^{odd}$ over M_{red} is a vector bundle of dimension $p|q$; moreover, its even part TM_{red}^{ev} can be identified with the tangent bundle of M_{red}.

ex:vector_fields **Example 48. (Vector fields on $\mathbb{R}^{n|m}$).** Now let us consider vector fields on the super manifold $\mathbb{R}^{n|m}$. We recall that $C^\infty(\mathbb{R}^{n|m}) = C^\infty(\mathbb{R}^n) \otimes \Lambda[\theta_1, \dots, \theta_m]$.

Let $x_1, \dots, x_n \in C^\infty(\mathbb{R}^n)$ be the coordinate functions, and let $\partial_{x_1}, \dots, \partial_{x_n} \in \mathrm{Der}(C^\infty(\mathbb{R}^n))$ be the corresponding partial derivatives. These can be extended to even derivations of $C^\infty(\mathbb{R}^{n|m})$ which commute with the action of $\Lambda[\theta_1, \dots, \theta_m]$. Similarly, there are odd derivations

$$\partial_{\theta_i} \colon C^\infty(\mathbb{R}^{n|m}) \longrightarrow C^\infty(\mathbb{R}^{n|m});$$

these commute with the action of $C^\infty(\mathbb{R}^n)$ and $\partial_{\theta_i}(\theta_i) = 1$, $\partial_{\theta_i}(\theta_j) = 0$ for $i \neq j$; in fact, these properties characterize ∂_{θ_i}. We note that ∂_{x_i} preserves the $\mathbb{Z}/2$-grading of $C^\infty(\mathbb{R}^{n|m})$, while ∂_{θ_i} reverses it. In other words, ∂_{x_i} (resp. ∂_{θ_i}) is an even (resp. odd) element of $\mathrm{Der}(C^\infty(\mathbb{R}^{n|m})$; or, geometrically speaking, ∂_{x_i} is an even vector field on $\mathbb{R}^{n|m}$, while ∂_{θ_i} is an odd vector field.

Now consider $\mathbb{R}^{1|1}$ and let us write $t, \theta \in C^\infty(\mathbb{R}^{1|1})$ for the even (resp. odd) coordinate function. Any function $f \in C^\infty(\mathbb{R}^{1|1}) = C^\infty(\mathbb{R}) \otimes \Lambda[\theta]$ is then of the form

$$f = f_0 + \theta f_1 \qquad \text{with } f_0, f_1 \in C^\infty(\mathbb{R}).$$

Let D be the odd vector field $D = \partial_\theta - \theta\partial_t$. Then

$$Df = (\partial_\theta - \theta\partial_t)(f_0 + \theta f_1) = \partial_\theta \theta f_1 - \theta\partial_t f_0 = f_1 - \theta\partial_t f_0,$$

since the terms $\partial_\theta f_0$ and $\theta\partial_t(\theta f_1) = \theta^2(\partial_t f_1)$ both vanish. In particular,

$$D^2(f_0 + f_1\theta) = -\partial_t f_0 - \partial_t f_1 \theta = -\partial_t(f_0 + f_1\theta).$$

In other words, $D^2 = -\partial_t$.

def:cs-manifolds **Definition 49. (cs-manifolds.)** In the next subsection we will define *super Riemannian structures* on super manifolds of dimension $d|1$ for $d = 1, 2$. More precisely, these structures will be defined for a variant of super manifolds that Deligne and Morgan DM [DM, §4.8] refer to as cs-manifolds which stands for *complex super*. This terminology might be somewhat misleading, since it suggests that the associated reduced manifold is a complex manifold. This is not the case; rather, the adjective *complex* means that we describe the super manifold in terms of its complex valued functions instead of its real valued functions. The precise definition is this: a *cs-manifold* of dimension $n|m$ is a topological space M_{red} together with a sheaf $\mathcal{O}_M$ of graded commutative algebras over the *complex numbers*, which is locally isomorphic to $\mathbb{R}^{n|m}_{cs} \overset{\text{def}}{=} (\mathbb{R}^n, \mathcal{O}^{n|m} \otimes \mathbb{C})$. If $M = (M_{red}, \mathcal{O})$ is a cs-manifold, we denote by $\bar{M} \overset{\text{def}}{=}$

$(M_{red}, \bar{\mathcal{O}}_{red})$ the *complex conjugate* cs-manifold (where $\bar{\mathcal{O}}_{red}$ is the complex conjugate structure sheaf obtained by replacing all the complex vector spaces $\mathcal{O}(U)$ for $U \subset M_{red}$ by the complex conjugate vector spaces; this is compatible with the algebra structure).

We note that a super manifold of dimension $n|m$ leads to a cs-manifold by complexifying its structure sheaf. In fact, we can interpret a super manifold as a cs-manifold M equipped with a *real structure*, i.e., a complex anti-linear involution $\bar{\ }$ on its (complex) structure sheaf $\mathcal{O}_M$. We note that the reduced structure sheaf $\mathcal{O}_{red} = \mathcal{O}/J$ has a *canonical* real structure, since it can be identified with the sheaf of $\mathbb{C}$-valued smooth functions on M_{red} (with respect to the smooth structure on M_{red} determined by this sheaf), which has the complex conjugation involution. In particular, if M is a cs-manifold of dimension $n|0$, then there are no odd elements in the sheaf $\mathcal{O}$ and hence $\mathcal{O} = \mathcal{O}_{red}$ has a canonical real structure. In other words, cs-manifolds of dimension $n|0$ are just ordinary smooth manifolds of dimension n.

def:maps **Definition 50. (Maps between super manifolds)** If M, N are super manifolds, the morphisms from M to N are simply grading preserving algebra homomophisms

$$C^\infty(N) \longrightarrow C^\infty(M).$$

If M, N are cs-manifolds, we require in addition that the reduced map $C^\infty(N_{red};\mathbb{C}) \longrightarrow C^\infty(M_{red};\mathbb{C})$ (obtained by modding out the ideal generated by odd functions) is *real*, i.e., compatible with complex conjugation (this condition guarantees that a map $f\colon M \to N$ between cs-manifolds induces a smooth map $M_{red} \to N_{red}$ between the reduced manifolds).

A convenient way to describe describe maps between cs-manifolds is the *functor of points approach* (see [DM, §2.8, 2.9]). If M, S are cs-manifolds, the S-points of M are the set of morphisms $S \to M$ of cs-manifolds; we will use the notation $M_S \stackrel{\text{def}}{=} \mathrm{SMAN}(S, M)$ for the S-points of M. For example, an S-point of the cs-manifold $\mathbb{R}^{p|q}$ can be identified with a collection $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$ of even (resp. odd) functions on S such that the reductions $(x_k)_{red} \in C^\infty(S_{red};\mathbb{C})$ are *real valued functions*.

4.2 Super Riemannian structures on $1|1$-manifolds

Now we are ready to define super Riemannian structures on cs-manifolds of dimension $d|1$ for $d = 1, 2$. There should be a general notion of super Riemannian structures on general super manifolds (for example along the lines of the paper by John Lott [Lo]), but for the purposes of this paper, the authors prefer the pedestrian approach of first defining this notion for $d = 1$, and then for $d = 2$. We hope that the terminology *super Riemannian structure* won't tempt the reader into thinking that this is some kind of inner product on the tangent bundle of the super manifold at hand. Rather, we think of a Riemannian metric as the structure needed to define an action functional (the usual energy); similarly, the structure on super manifolds of dimension $1|1$ and $2|1$ described below is the structure needed to define analogous action functionals (see Remarks 54 and 61). This motivates us to call this structure a super Riemannian structure. An additional motivation is provided by the fact that a super Riemannian structure on a cs-manifold M of dimension $d|1$ induces a Riemannian metric on the reduced manifold M_{red} (an ordinary manifold of dimension d). In both cases ($d = 1, 2$), we will first give a preliminary definition, provide the standard example of this structure, and motivate this structure by considerations from physics. Then we give a more elaborate definition which allows for more flexibility (which will be needed in the proof of our main result).

Definition 51. Preliminary definition). A *super Riemannian structure* on a cs-manifold M of dimension $1|1$ is an odd vector field D on M such that

(i) the reduction of the even vector field D^2 gives a nowhere vanishing (complex) vector field $(D^2)_{red}$ on M_{red}.

(ii) The complex conjugate of $(D^2)_{red}$ is $-(D^2)_{red}$.

Lemma 52. *A super Riemannian structure on a cs-manifold M of dimension $1|1$ determines a Riemannian metric and a spin structure on M_{red}. Replacing M by the complex conjugate cs-manifold $\bar{M}$ (see definition 49) results in the same metric and the opposite spin structure.*

Proof. Multiplying the complex vector field $(D^2)_{red}$ by $i \in \mathbb{C}$, we obtain the vector field $i(D^2)_{red}$ which is invariant under complex conjugation and can hence be interpreted as a nowhere vanishing *real* vector field on the reduced manifold M_{red}. Since M_{red} is 1-dimensional, $i(D^2)_{red}$ determines an spin structure on M_{red} as well as a Riemannian metric (with respect to which it is a vector field of unit length). Replacing M by $\bar{M}$ results in replacing $i(D^2)_{red}$ by $-i(D^2)_{red}$, which determines the opposite spin structure and the same Riemannian metric on M_{red}. □

dard_structure1|1 **Example 53. (The standard super Riemannian structure on $\mathbb{R}^{1|1}_{cs}$)** Let us write y resp. θ for the even resp. odd coordinate function on $\mathbb{R}^{1|1}_{cs}$ and ∂_y, ∂_θ for the corresponding vector fields. Then the calculation of example 48 shows that the odd vector field

$$D \overset{\text{def}}{=} \partial_\theta - i\theta\partial_y$$

squares to $D^2 = -i\partial_y$. In particular, it satisfies condition (i) and (ii) above. We note that $i(D^2)_{red}$ is the vector field ∂_y on $(\mathbb{R}^{1|1}_{cs})_{red} = \mathbb{R}$; this shows that this super Riemannian structure on $\mathbb{R}^{1|1}_{cs}$ induces the standard Riemannian metric and the standard spin structure on $\mathbb{R}$.

ics_motivation1|1 **Remark 54. (Physics motivation).** From the Lagrangian point of view, a particle moving on geodesics in a Riemannian manifold X can be described by minimizing the energy functional $S(f)$, where $f\colon \mathbb{R} \to X$ describes the *world line* of the particle. From an abstract point of view, the world lines don't need to be parametrized by $\mathbb{R}$; any 1-manifold Σ equipped with a Riemannian metric is sufficient to define the energy of $f\colon \Sigma \to X$.

Similarly, the world line of a super particle moving in X is described by a map $f\colon \Sigma \to X$, where now Σ is a super manifold of dimension $1|1$. If D is an odd vector field such that $(D^2)_{red}$ is nowhere vanishing, there is a well-defined action

$$S(f) \overset{\text{def}}{=} \frac{1}{2}\int_\Sigma \langle D^2 f, Df\rangle \operatorname{vol}_D .$$

Here $D^2 f$ (resp. Df) is the derivative of f in the direction of the vector field D^2 (resp. D); these are sections of the pull-back bundle f^*TX which can paired using the Riemannian metric on X to obtain the function $\langle D^2 f, Df\rangle \in C^\infty(\Sigma)$. We recall that on the super manifold Σ, it is sections of the line bundle $\operatorname{Ber}(T\Sigma)^*$ (the dual of the *Berezinian line bundle* associated to the tangent bundle) which can be integrated over Σ (after an orientation on the reduced manifold Σ_{red} is fixed; see [DM, Prop. 3.10.5]). A short calculation similar to that on p. 663 of [Wi3] shows that there is a canonical isomorphism $\operatorname{Ber}(T\Sigma) \cong \mathcal{D}$, where $\mathcal{D} \overset{\text{def}}{=} \langle D\rangle \subset T\Sigma$ is the odd line bundle spanned by D. In particular, D determines a dual section $\operatorname{vol}_D \in \operatorname{Ber}(T\Sigma)^*$. We note that replacing D by $-D$ doesn't change the action.

If $\Sigma = \mathbb{R}^{1|1}$ and $D = \partial_\theta - \theta\partial_t$, then the action takes the usual form for the action of a super particle moving in a Riemannian manifold X (see [Wi3, Problem FP2] and [Fr, p. 41-43]):

$$S(f) = -\frac{1}{2}\int_{\mathbb{R}^{1|1}} dt d\theta \langle \dot{f}, Df\rangle.$$

The parameter t here should be thought of as *time*; we would like to apply *Wick rotation* and express all quantities involved in terms of $y = it$ (*imaginary time*). In particular, D becomes our standard vector field $\partial_\theta - i\theta\partial_y$ on the cs-manifold $\mathbb{R}^{1|1}_{cs}$ with coordinates (y, θ) (see Example 53). We note that the appearance of i in the formula for D explains our preference for cs-manifolds. Why do we Wick rotate? One reason is that in the quantum theory we want trace class operators e^{-yQ^2} rather than unitary operators e^{itQ^2}.

We note that the action functional above is invariant under any automorphisms of $\mathbb{R}^{1|1}$ that preserve D (up to sign). It is easy to check that the vector field $Q = \partial_\theta + \theta\partial_t$ commutes with D (in the graded sense); in other words, Q is the infinitesimal generator of a group of automorphisms of $\mathbb{R}^{1|1}$ which preserve D. This symmetry of the classical action is a *super* symmetry in the sense that Q is an *odd* vector field. In particular, upon quantization, Q leads to an odd operator acting on the $\mathbb{Z}/2$-graded Hilbert space of the theory.

Definition 55 needs to be modified in two ways:

(a) We want to consider the super Riemannian structure given by an odd vector field D the *same* as the structure given by $-D$.

(b) A super Riemannian structure is only locally given by an odd vector field.

This suggests to define a super Riemannian structure is an *equivalence class* of such D's where we identify D and $-D$. The defect of that definition would be that it is *not local* in the following sense: suppose D_1 and D_2 are sections of the tangent sheaf $TM \to M_{red}$ restricted to open subsets $U_1 \subset M_{red}$ (resp. $U_2 \subset M_{red}$); suppose that the super Riemannian structures determined by D_1 resp. D_2 agree on the intersection $U_1 \cap U_2$ (i.e., $D_1 = \pm D_2$ on $U_1 \cap U_2$). Then there might not be any D on $U_1 \cup U_2$ which agrees with $\pm D_i$ on U_i. In other words, these structure might not fit together to give a structure on $U_1 \cup U_2$ restricting to the given structures on U_1 and U_2. This defect can be fixed by the following more elaborate definition:

per_Riemannian1|1 **Definition 55.** Let M be a cs-manifolds of dimension $1|1$. A *super Riemannian structure* on a M is given by a collection of pairs (U_i, D_i) indexed by some set I, where

- the U_i's are open subsets of M_{red} whose union is all of M_{red};
- the D_i's are sections of the tangent sheaf TM restricted to U_i satisfying the conditions (i) and (ii) of Definition 51;
- the restrictions of D_i and D_j to $U_i \cap U_j$ are equal up to a possible sign.

Two such collection define the *same* structure if their union is again such a structure (this is analogous to saying that two smooth atlases define the same smooth structure if the union of these atlases is again an atlas).

We note that Lemma 52 continues to hold with respect to this more sophisticated notion of *super Riemannian structure*, since while the vector fields D_i might not fit together to give a vector field D on M, the even vector fields D_i^2 do agree on the intersections and hence we have a globally defined even vector field D^2 on M.

We will need some information about the automorphism super group of $\mathbb{R}^{1|1}_{cs}$ equipped with its standard super Riemannian structure. The analogous non-super statement is that the isometry group of $\mathbb{R}$ (equipped with its standard metric) contains $\mathbb{R}$ (acting by translations on itself). To state the analog for $\mathbb{R}^{1|1}_{cs}$, we first define the multiplication

$$\mu\colon \mathbb{R}^{1|1}_{cs} \times \mathbb{R}^{1|1}_{cs} \longrightarrow \mathbb{R}^{1|1}_{cs} \qquad (y_1,\theta_1),(y_2,\theta_2) \mapsto (y_1+y_2+\theta_1\theta_2, \theta_1+\theta_2). \quad (56)$$ eq:1|1_multiplica

This gives $\mathbb{R}^{1|1}_{cs}$ the structure of a *super Lie group*, i.e., a group object in the category of super manifolds.

lem:isometry1 **Lemma 57.** *The left translation action of $\mathbb{R}^{1|1}_{cs}$ on itself preserves the standard super Riemannian structure of Example 53.*

4.3 Super Riemannian structures on $2|1$-manifolds

per_Riemannian2|1 **Definition 58.** Let M be a cs-manifold of dimension $2|1$. A super Riemannian structure on M is given locally by a pair (D, B), consisting of an odd vector field D and an even vector field B on M such that

(i) The vector fields $(D^2)_{red}$, B_{red} on the reduced manifold M_{red} are linearly independent at every point of M_{red}.

(ii) The vector field B_{red} is the complex conjugate of $(D^2)_{red}$.

It should be emphasized that $(D^2)_{red}$ and B_{red} are *complex* vector fields on the reduced manifold M_{red}; i.e., sections of the complexified tangent bundle of M_{red}. As in the case of dimension $1|1$, this needs to be modified by describing an appropriate equivalence relation on such pairs and by making the definition *local* in M_{red}. We call two such pairs (D, B), (D', B') *equivalent* if there is a smooth function $f \colon M \to S^1$ such that $D' = \bar{f} D$ and $B' = f^2 B$.

A *super Riemannian structure* on a cs-manifold M of dimension $2|1$ is given by a collection of triples (U_i, D_i, B_i), $i \in I$, where

- the U_i's are open subsets of M_{red} whose union is all of M_{red};
- the D_i's (resp. B_i's) are sections of the tangent sheaf TM restricted to U_i satisfying conditions (i) and (ii) above;
- the restrictions of (D_i, B_i) and (D_j, B_j) to $U_i \cap U_j$ are equivalent in the sense above.

Two such collection define the *same* structure if their union is again such a structure.

dard_structure2|1 **Example 59. (The standard super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$.)** We define a super Riemannian structure on the cs-manifold $\mathbb{R}^{2|1}_{cs} = (\mathbb{R}^2, \mathcal{O}^{2|1} \otimes \mathbb{C})$ as follows. Let us write $x, y, \theta \in C^\infty(\mathbb{R}^{2|1}_{cs})$ for the coordinate functions, set $z = x + iy \in C^\infty(\mathbb{R}^{2|1}_{cs})$, and consider the following vector fields on $\mathbb{R}^{2|1}_{cs}$:

$$B \overset{\text{def}}{=} \partial_z \overset{\text{def}}{=} \frac{1}{2}(\partial_x - i\partial_y) \qquad \partial_{\bar z} \overset{\text{def}}{=} \frac{1}{2}(\partial_x + i\partial_y) \qquad D \overset{\text{def}}{=} \partial_\theta + \theta\partial_{\bar z}$$

A calculation as in Example ex:vector_fields 48 shows that $D^2 = \partial_{\bar z}$; in particular, $D^2_{red} = \partial_{\bar z}$ is nowhere vanishing on $\mathbb{R}^2$ and $\overline{D^2_{red}} = \overline{\partial_{\bar z}} = \partial_z = B_{red}$ (to keep notation at bay we write ∂_z and $\partial_{\bar z}$ for the vector fields on $\mathbb{R}^{2|1}_{cs}$ as well as for their reductions, which are complex vector fields on $\mathbb{R}^2 = (\mathbb{R}^{2|1}_{cs})_{red}$). This shows that (D, B) is in fact a super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$.

The terminology *super Riemannian structure* is motivated by the following result.

per_Riemannian2|1

Lemma 60. *A super Riemannian structure on a cs-manifold M of dimension $2|1$ induces a Riemannian metric and a spin structure on the reduced manifold M_{red} (an ordinary 2-manifold). Replacing M by the complex conjugate cs-manifold $\bar{M}$ results in the same metric and the opposite spin structure.*

Proof. We first work locally. So let (D, B) as above be a pair of sections of the tangent sheaf $TM \to M_{red}$ over some open subset $U \subset M_{red}$; in particular, the (local) sections D^2_{res}, B_{red} of the complexified tangent bundle $TM_{red} \otimes \mathbb{C}$ are everywhere linearly independent. Hence there there is a unique hermitian metric on the complexified tangent bundle $TM_{red} \otimes \mathbb{C}$ with respect to which the linearly independent sections B_{red} and $\bar{B}_{red} = D^2_{red}$ are perpendicular to each other and both have length $\sqrt{2}$ (a choice of normalization motivated by the standard super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$; see Example ex:standard_structure2|1 59). This restricts to a *real valued* inner product on $TM_{red} \subset TM_{red} \otimes \mathbb{C}$. To see this, let X be a (complex) vector field on M_{red}; it can be written in the form $X = fB_{red} + g\bar{B}_{red}$ where f, g are smooth complex valued functions on M_{red}. Then

$$\bar{X} = \bar{f}\bar{B}_{red} + \bar{g}B_{red},$$

which shows that X is a *real* vector field if and only if $g = \bar{f}$. If $X = fB_{red} + \bar{f}\bar{B}_{red}$ and $Y = hB_{red} + \bar{h}\bar{B}_{red}$ are real vector fields, then

$$\langle X, Y \rangle = \langle fB_{red} + \bar{f}\bar{B}_{red}, hB_{red} + \bar{h}\bar{B}_{red} \rangle = \bar{f}h + f\bar{h}$$

is real-valued.

We note that if (D', B') is another such pair equivalent to (D, B), i.e., $D' = \bar{f}D$ and $B' = f^2B$ for some smooth function $f\colon M_{red} \to S^1$, then $(D')^2_{red} = \bar{f}^2 D^2_{red}$ and $B'_{red} = f^2 B_{red}$ are again perpendicular unit vector fields, and hence the hermitian metrics these pairs determine on $TM_{red} \otimes \mathbb{C}$ agree. Hence our local arguments above are sufficient. □

We remark that the standard super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$ of example ex:standard_structure2|1 59 induces the usual Riemannian metric on $\mathbb{R}^2 = (\mathbb{R}^{2|1}_{cs})_{red}$, since the vector fields $B_{red} = \partial_z$ and $(D^2)_{red} = \partial_{\bar{z}}$ are perpendicular and of constant length $\sqrt{2}$ w.r.t. the usual metric.

ics_motivation2|1

Remark 61. (Physics Motivation.) A string moving in a Riemannian manifold X is described by a map $f\colon \Sigma^2 \to X$ from a Lorentz surface Σ^2 to X (in the simplest case $\Sigma = S^1 \times \mathbb{R}$, where S^1 parametrizes the string and $\mathbb{R}$ parametrizes time); a *super string* moving in X amounts to a map

$f\colon \Sigma^{2|1} \to X$ from a super manifold Σ of dimension $2|1$ to X. Problem FP6 (p. 613) of Witten's homework collection in the IAS proceedings [Wi3] (see also the solution on p. 663) explains that an *action functional* (Lagrangian) for such maps can be defined, provided Σ comes equipped with a pair $(\mathcal{D}, \mathcal{B})$ of subbundles (distributions) $\mathcal{D}, \mathcal{B} \subset T\Sigma$ of dimension $0|1$ (resp. $1|0$). In other words, $\mathcal{D}$ (resp. $\mathcal{B}$) is generated locally by an odd vector field D (resp. and even vector field B); it is required that the vector fields D, D^2 and B span TX. Then the action of a map $f\colon \Sigma \to X$ is defined by

$$S(f) \overset{\text{def}}{=} \int_\Sigma \langle df_{|\mathcal{D}}, df_{|\mathcal{B}} \rangle.$$

Here $df_{|\mathcal{D}}$ (resp. $df_{|\mathcal{B}}$) is the differential of f restricted to $\mathcal{D} \subset T\Sigma$ (resp. $\mathcal{B} \subset T\Sigma$); these are sections of $\mathcal{D}^* \otimes f^*TX$ (resp. $\mathcal{B}^* \otimes f^*TX$) that can be paired using the Riemannian metric on X to obtain $\langle df_{|\mathcal{D}}, df_{|\mathcal{B}} \rangle$, which is a section of $\mathcal{D}^* \otimes \mathcal{B}^*$. The calculation on p. 663 of [Wi3] shows that this line bundle is canonically isomorphic to the Berezinian $\mathrm{Ber}(T\Sigma)^*$, and so its section $\langle df_{|\mathcal{D}}, df_{|\mathcal{B}} \rangle$ can be integrated over Σ.

As in the $1|1$-dimensional case, the automorphisms of $(\Sigma, \mathcal{D}, \mathcal{B})$ give rise to (super) symmetries of the theory. As is well-known, the automorphism group of $(\Sigma, \mathcal{D}, \mathcal{B})$ (super conformal group) is an infinite dimensional super Lie group. We prefer to work with the super Riemannian structure (given by D, B) rather than the *super conformal structure* given by the distributions $\mathcal{D}$, $\mathcal{B}$ they generate, since the conformal invariance of the classical theory might not survive to the quantum theory (conformal anomaly).

A flat model is $\mathbb{R}^{2|1}$ with even coordinates u, v and odd coordinate θ and

$$D = \partial_\theta - \theta\partial_u \qquad B = \partial_v,$$

where, as Witten explains in [Wi2, §2.2], the coordinates u, v are the right- resp. left-moving light cone coordinates on $\mathbb{R}^2$ equipped with the Minkowski metric $ds^2 = du\,dv$. In terms of the more usual coordinates t (time) and x (space), for which the Minkowski metric takes the form $ds^2 = dt^2 - dx^2$, the light cone coordinates are given by

$$u = x + t \qquad v = t - x.$$

For the same reasons as in the $1|1$-dimensional case, we want to do a *Wick rotation* and express all quantities in terms of the *imaginary time* $y = it$. In particular, $u = x + t = x - iy = \bar{z}$ and $v = t - x = -(x + iy) = -z$. In particular, expressed in terms of $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, and ∂_θ we have:

$$D = \partial_\theta - \theta\partial_{\bar{z}} \qquad B = -\partial_z \tag{62}$$

Again the calculation in Example ex:vector_fields 48 shows that $D^2 = -\partial_{\bar{z}}$; in particular, the vector fields D^2 and B reduce to the everywhere linearly independent (complex) vectors fields $-\partial_{\bar{z}}$ (resp. $-\partial_z$) on $(\mathbb{R}^{2|1}_{cs})_{red} = \mathbb{R}^2$ which are complex conjugates of each other (of course with respect to the *new* real structure where the coordinate functions x, y are regarded as real functions). In other words, after a Wick rotation the vectors fields D, B define a super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$.

per_Riemannian2|1 **Definition 63. (Flat super Riemannian structures on 2|1-manifolds).** Let M be a cs-manifold of dimension $2|1$. A super Riemannian structure on M is called *flat* if the pairs of local vector fields (D, B) defining it satisfy the following additional property:

(iii) The graded commutator $[D, B]$ vanishes.

We note that the standard super Riemannian structure (ex:standard_structure2|1 59) is flat and we believe that vice versa, a flat cs-manifold is locally isomorphic to the standard structure.

lem:flat **Lemma 64.** *Let M be a cs-manifold of dimension $2|1$ equipped with a super Riemannian structure which is flat. Then the induced Riemannian metric on M_{red} is flat.*

Proof. Let ∇ be the unique connection on $TM_{red} \otimes \mathbb{C}$ for which the sections D^2_{red} and B_{red} are parallel. We claim that ∇ is the Levi-Civita connection for the induced metric on B_{red} (or more precisely, the connection on the complexified tangent bundle induced by the Levi-Civita connection). It is clear that ∇ is a metric connection since D^2_{red} and B_{red} are both vector fields of constant length. It remains to show that the torsion tensor T vanishes, which is given by

$$T(X, Y) \overset{\text{def}}{=} \nabla_X Y - \nabla_Y X - [X, Y]$$

for complex vector fields X, Y (the expression $\nabla_X Y$ is originally defined for real vector fields X and complex vector fields Y, but we can extend it to complex vector fields X by requiring $\nabla_X Y$ to depend complex linearly on X). Since $T(X,Y)$ is skew-symmetric and complex linear in both variables, to prove $T \equiv 0$, it suffices to show $T(D^2_{red}, B_{red}) = 0$; this is the case since $\nabla_X D^2_{red}$ and $\nabla_X B_{red}$ vanish for every vector field X (this is the condition that D^2_{red} and B_{red} are parallel w.r.t. ∇) and since our flatness-condition $[D,B] = 0$ implies $[D^2, B] = 0$ and hence $[D^2_{red}, B_{red}] = 0$. This shows that ∇ is the (complexified) Levi-Civita connection; in particular, the Levi-Civita connection is flat due to the existence of the parallel sections D^2_{red} and B_{red}. ☐

We can give $\mathbb{R}^{2|1}_{cs}$ the structure of a super Lie group by defining the multiplication

$$\begin{aligned} \mu\colon \mathbb{R}^{2|1}_{cs} \times \mathbb{R}^{2|1}_{cs} &\longrightarrow \mathbb{R}^{2|1}_{cs} \\ (z_1, \bar{z}_1, \theta_1), (z_2, \bar{z}_2, \theta_2) &\mapsto (z_1 + z_2, \bar{z}_1 + \bar{z}_2 + \theta_1\theta_2, \theta_1 + \theta_2) \end{aligned} \tag{65}$$

eq:2|1_multiplica

Here $z = x + iy \in C^\infty(\mathbb{R}^{2|1}_{cs})$ and $\bar{z} = x - iy$, where $x, y \in C^\infty(\mathbb{R}^{2|1}_{cs})$ are the coordinate functions of the cs-manifold $\mathbb{R}^{2|1}_{cs}$; the subscript 1 (resp. 2) indicate the first (resp. second) copy of $\mathbb{R}^{2|1}_{cs}$. We note that we need to work with $\mathbb{R}^{2|1}_{cs}$ in order to make sense of the function z (and hence of the map μ). Moreover, after reduction $\bar{z}$ is the complex conjugate of z, which implies that μ satisfies the reality condition of Definition def:maps 50.

lem:isometry2 **Lemma 66.** *The left translation action of $\mathbb{R}^{2|1}_{cs}$ on itself preserves the standard super Riemannian structure.*

We will need the following result in the proof of our main result.

lem:rotation **Lemma 67.** *Let $a \in \mathbb{C}$ be of unit length. Then the automorphism*

$$R_a\colon \mathbb{R}^{2|1}_{cs} \longrightarrow \mathbb{R}^{2|1}_{cs} \qquad (z, \bar{z}, \theta) \mapsto (a^2 z, \bar{a}^2 \bar{z}, \bar{a}\theta)$$

preserves the standard super Riemannian structure.

Proof. A calculation shows

$$F_*\partial_z = a^2\partial_z \qquad F_*\partial_{\bar{z}} = \bar{a}^2\partial_{\bar{z}} \qquad F_*\partial_\theta = a\partial_\theta \qquad F_*(\partial_\theta - \theta\partial_{\bar{z}}) = \bar{a}(\partial_\theta - \theta\partial_{\bar{z}})$$

This implies that the pairs $(D,B) = (\partial_\theta - \theta\partial_{\bar{z}}, -\partial_z)$ and (F_*D, F_*B) are equivalent in the sense of Definition def:super_Riemannian2|1 58 and hence represent the *same* super Riemannian structure on $\mathbb{R}^{2|1}_{cs}$. ☐

4.4 The super bordism category $\mathrm{RB}^{d|1}$

The goal of this subsection is to define the super category $\mathrm{RB}^{d|1}$. We recall from Definition 33 that this means in particular that for any super manifold S we need to construct a category $\mathrm{RB}_S^{d|1}$.

def:super_RB_S **Definition 68.** The definition of $\mathrm{RB}_S^{d|1}$ is completely analogous to Definition 11 of RB^d (resp. Definition 39 of RB_S^d); we just need to replace Riemannian manifolds of dimension d by cs-manifolds of dimension $d|1$ equipped with super Riemannian structures and allow cs-manifolds as parameter spaces. More precisely, we recall that the objects and morphisms of RB^d (resp. RB_S^d) are defined in terms of the category Riem^d (resp. Riem_S^d) of Riemannian spin d-manifolds (resp. families of such manifolds parametrized by S). We obtain $\mathrm{RB}_S^{d|1}$ by replacing Riem^d by $\mathrm{Riem}_S^{d|1}$, where

- the objects of $\mathrm{Riem}_S^{d|1}$ are smooth quasi bundles of cs-manifolds $U \to S$ with fibers of dimension $d|1$ which are equipped with a fiberwise super Riemannian structure (fiberwise here means that the local vector fields D, B are *vertical*).

- A morphism from U to U' is an embedding $U \hookrightarrow U'$ of cs-manifolds that is a bundle map (i.e., commutes with the projection to S), and preserves the fiberwise super Riemannian structure.

Remark 69. There is a natural *reduction functor*

$$\mathrm{Riem}_S^{d|1} \xrightarrow{red} \mathrm{Riem}_{S_{red}}^d$$

that sends a (quasi) bundle $U \to S$ of super manifolds equipped with a fiberwise super Riemannian structure to the (quasi) bundle $U_{red} \to S_{red}$ equipped with the induced fiberwise Riemannian metric and spin structure (cf. Lemmas 52 and 60). This induces a reduction functor between the corresponding bordism categories

$$\mathrm{RB}_S^{d|1} \xrightarrow{red} \mathrm{RB}_{S_{red}}^d .$$

objects+morphisms **Example 70. (Examples of objects and morphisms in $\mathrm{RB}_S^{d|1}$).** The following examples extend the examples discussed in (41) to the super setting in the sense that their images under the reduction functor yields the examples discussed there. All fiber bundles over the cs-manifold S described below are either of the form $S \times \mathbb{R}^{d|1}_{cs} \to S$ (with the obvious fiberwise super Riemannian structure), or a quotient of this bundle by a discrete subgroup of $\mathbb{R}^{d|1}_{cs}(S) = \mathrm{SMAN}(S, \mathbb{R}^{d|1}_{cs})$ (the group of smooth maps from S to $\mathbb{R}^{d|1}_{cs}$ with eq:1|1_multiplication group structure induced by the multiplication map μ of equation (56) resp. eq:2|1_multiplication (65)); this group acts by structure preserving bundle automorphisms by associating to $f\colon S \to \mathbb{R}^{d|1}_{cs}$ the bundle automorphism given by the composition

$$S \times \mathbb{R}^{d|1}_{cs} \xrightarrow{\Delta\times 1} S \times S \times \mathbb{R}^{d|1}_{cs} \xrightarrow{1\times f\times 1} S \times S \times \mathbb{R}^{d|1}_{cs} \xrightarrow{1\times\mu} S \times \mathbb{R}^{d|1}_{cs}.$$

Abusing language we again write f for this bundle automorphism. Lemma lem:isometry1 lem:isometry2 57 resp. 66 imply that f preserves the fiberwise super Riemannian structure.

the super point $\mathrm{spt}_S \in \mathrm{RB}^{1|1}_S$**.** The quadruple

$$\mathrm{spt} \stackrel{\text{def}}{=} (U, Y, U^+, U_-) = (S \times \mathbb{R}^{1|1}_{cs}, S \times \mathbb{R}^{0|1}_{cs}, S \times \mathbb{R}^{1|1}_{cs,-}, S \times \mathbb{R}^{1|1}_{cs,+})$$

is an object of $\mathrm{RB}^{1|1}_S$; here $\mathbb{R}^{1|1}_{cs,\pm} \subset \mathbb{R}^{1|1}_{cs}$ is the super submanifold whose reduced manifold is $\mathbb{R}^1_\pm \subset \mathbb{R}^1$.

the super interval $I^{1|1}_\ell \in \mathrm{RB}^{1|1}_S(\mathrm{spt}_S, \mathrm{spt}_S)$**.** For $\ell \in \mathbb{R}^{1|1}_{cs,+}(S)$ the pair of bundle maps

$$U = S \times \mathbb{R}^{1|1}_{cs} \xhookrightarrow{\mathrm{id}} \Sigma = S \times \mathbb{R}^{1|1}_{cs} \xhookleftarrow{\ell} U = S \times \mathbb{R}^{1|1}_{cs},$$

is a super Riemannian bordism from spt_S to spt_S. We will use the notation $I^{1|1}_\ell$ for this morphism.

the super circle $S^{1|1}_\ell \in \mathrm{RB}^{1|1}_S(\emptyset, \emptyset)$**.** For $\ell \in \mathbb{R}^{1|1}_{cs,+}(S)$ the bundle

$$S^{1|1}_\ell \stackrel{\text{def}}{=} (S \times \mathbb{R}^{1|1}_{cs})/\mathbb{Z}\ell$$

is a Riemannian bordism from $\emptyset$ to $\emptyset$.

the super circle $S^{1|1}_\ell \in \mathrm{RB}^{2|1}$**.** For $\ell \in \mathbb{R}^1_{cs,+}(S)$ the quadruple

$$S^{1|1}_\ell \stackrel{\text{def}}{=} ((S \times \mathbb{R}^{2|1}_{cs})/\mathbb{Z}\ell, (S \times \mathbb{R}^{1|1})/\mathbb{Z}\ell, (S \times \mathbb{R}^{2|1}_{cs,+})/\mathbb{Z}\ell, (S \times \mathbb{R}^{2|1}_{cs,-})/\mathbb{Z}\ell)$$

is an object of $\mathrm{RB}^{2|1}_S$. We remark that we need to restrict ℓ to be an element of $\mathbb{R}^1_{cs,+}(S) \subset \mathbb{R}^{2|1}_{cs,+}(S)$ (rather than $\mathbb{R}^{1|1}_{cs,+}(S)$), since otherwise the translation ℓ doesn't preserve the subspace $S \times \mathbb{R}^{1|1}_{cs} \subset S \times \mathbb{R}^{2|1}_{cs}$.

the super cylinder $C^{2|1}_{\ell,f} \in \mathrm{RB}^{2|1}_S(S^{1|1}_\ell, S^{1|1}_\ell)$. For $\ell \in \mathbb{R}^1_{cs,+}(S)$, $f \in \mathbb{R}^{2|1}_{cs,+}(S)$ the bundle automorphism f commutes with ℓ and hence we have the following pair of morphisms in $\mathrm{Riem}^{d|1}_S$

$$V_2 = (S \times \mathbb{R}^{2|1}_{cs})/\mathbb{Z}\ell \xrightarrow{\mathrm{id}} \Sigma = (S \times \mathbb{R}^{2|1}_{cs})/\mathbb{Z}\ell \xleftarrow{\ell f} V_1 = (S \times \mathbb{R}^{2|1}_{cs})/\mathbb{Z}\ell \cdot$$

This is a super Riemannian bordism from $S^{1|1}_\ell$ to itself; we use the notion $C^{2|1}_{\ell,f}$ for this endomorphism of $S^{1|1}_\ell \in \mathrm{RB}^{2|1}_S$.

The super torus $T^{2|1}_{\ell,f} \in \mathrm{RB}^2_S(\emptyset, \emptyset)$. For $\ell \in \mathbb{R}^1_{cs,+}(S)$ and $f \in \mathbb{R}^{2|1}_{cs,+}(S)$ the bundle

$$T^{2|1}_{\ell,f} \overset{\mathrm{def}}{=} (S \times \mathbb{R}^{2|1}_{cs})/\ell(\mathbb{Z}f + \mathbb{Z}1).$$

is a super Riemannian bordism from $\emptyset \in \mathrm{RB}^{2|1}_S$ to itself.

All the relations of Lemmas lem:1_relations 42 and lem:2_relations_S 43 generalize as follows.

lem:1|1_relations **Lemma 71.** *Let S be a cs-manifold and $\ell, \ell' \in \mathbb{R}^{1|1}_{cs,+}(S)$. Then the following relations hold in the category $\mathrm{RB}^{1|1}_S$:*

1. $I^{1|1}_\ell \circ I^{1|1}_{\ell'} = I^{1|1}_{\ell+\ell'} \in \mathrm{RB}^{1|1}_S(\mathrm{spt}, \mathrm{spt})$*;*
2. $\widehat{I^{1|1}_\ell} = S^{1|1}_\ell \in \mathrm{RB}^{1|1}_S(\emptyset, \emptyset)$*;*

lem:2|1_relations **Lemma 72.** *Let S be a cs-manifold and $\ell \in \mathbb{R}_+(S)$, $f, f' \in \mathbb{R}^{2|1}_{cs,+}(S)$. Then the following relations hold in the category $\mathrm{RB}^{2|1}_S$:*

1. $C^{2|1}_{\ell,f} \circ C^{2|1}_{\ell,f'} = C^{2|1}_{\ell,\mu(f,f')} \in \mathrm{RB}^{2|1}_S(S^{1|1}_\ell, S^{1|1}_\ell)$*;*
2. $\widehat{C^{2|1}_{\ell,f}} = T^{2|1}_{\ell,f} \in \mathrm{RB}^{2|1}_S(\emptyset, \emptyset)$*;*
3. $C^{2|1}_{\ell,f+1} = C^{2|1}_{\ell,f} \in \mathrm{RB}^{2|1}_S(S^{1|1}_\ell, S^{1|1}_\ell)$*;*
4. $T^{2|1}_{g(\ell,f)} = T^{2|1}_{\ell,f} \in \mathrm{RB}^{2|1}_S(\emptyset, \emptyset)$ *for every $g \in SL_2(\mathbb{Z})$;*

The proof of these results is completely analogous to the proof of Lemma lem:2_relations_S 43. We only want to mention that the rotation $R_a \colon \mathbb{R}^2 \to \mathbb{R}^2$ now needs to be replaced by the structure preserving automorphism $R_a \colon \mathbb{R}^{2|1}_{cs} \to \mathbb{R}^{2|1}_{cs}$ of Lemma lem;rotation 67 (more precisely, a has to be replaced by its square root).

4.5 Extending TV to a super category

Definition 73. (The category $TV_S^\pm$ for cs-manifolds S.) def:TV^pm_S In Definition 37 we described the category $\mathrm{TV}_S^\pm$ if S is an ordinary manifold in terms of bundles of locally convex vector spaces. To extend that definition to the case that S is a cs-manifold, we can use the same definition, provided we explain how to define such bundles over cs-manifolds. This can be done by describing vector bundles in terms of their associated sheaf of sections; that definition extends immediately to super manifolds. We recall that if V is a locally convex vector space, the space of smooth V-valued functions on a manifold S is the projective tensor product $C^\infty(S) \otimes V$. This motives the following definition. A *vector bundle* over a cs-manifold S is a sheaf of modules over the structure sheaf $\mathcal{O}_S$ which is locally isomorphic to the (projective, graded) tensor product $\mathcal{O}_S \otimes V$, where V is a $\mathbb{Z}/2$-graded locally convex vector space. These modules are equipped with a locally convex topology and the local isomorphism is bi-continuous. Here $C^\infty(S)$ comes with its usual Frechet topology. A *vector bundle map* is a continuous map between these sheaves.

def:superTV^pm **Definition 74.** The super category $\mathrm{TV}^\pm$ is the functor

$$\mathrm{SMAN}^{op} \longrightarrow \mathrm{CAT}$$

which sends a super manifold S to $\mathrm{TV}_S^\pm$ and a smooth map $f\colon S' \to S$ to the pull-back functor $f^*\colon\ \mathrm{TV}_S^\pm \to \mathrm{TV}_{S'}^\pm$.

4.6 Super symmetric quantum field theories

def:superQFT **Definition 75.** A *super symmetric quantum field theory of dimension* $d|1$ for $d = 1, 2$ is a super functor

$$E\colon\ \mathrm{RB}^{d|1} \longrightarrow \mathrm{TV}^\pm,$$

compatible with the symmetric monoidal structure, the involution $\bar{\ }$ and the anti-involution $^\vee$. When speaking of a QFT of dimension $d|1$, we mean a super symmetric theory and we usually drop the adjective super symmetric.

ex:QFT1|1 **Example 76. (Example of a QFT of dimension $1|1$.)** Let M be a compact spin manifold, let V the $\mathbb{Z}/2$-graded Hilbert space of L^2-spinors on M, $\mu\colon V \otimes \bar{V} \to \mathbb{C}$ in inner product and $D\colon V \to V$ be the Dirac operator on M (an unbounded self-adjoint operator). Then M determines a QFT E_M

of dimension $1|1$ with the following properties: for any cs-manifold S and $t \in C^\infty(S)^{ev}$, $\theta \in C^\infty(S)^{odd}$ the functor

$$(E_M)_S \colon \mathrm{RB}^{1|1}_S \longrightarrow \mathrm{TV}^{\pm}_S$$

maps

$$\mathrm{spt}_S \mapsto (S \times V, S \times \bar{V}, \mu) \qquad \text{and} \qquad I_{t,\theta} \mapsto e^{-tD^2+\theta D},$$

where the operator valued function $e^{-tD^2+\theta D}$ is defined using spectral calculus.

:partition_QFTd|1 **Definition 77. (Partition function of a QFT of dimension $d|1$.)** Let E be a QFT of dimension $d|1$ and $E^+ \colon \mathrm{RB}^{2|1} \to \mathrm{TV}$ the composition of E with the forgetful functor $\mathrm{TV}^{\pm} \to \mathrm{TV}$ (see Remark rem:TVandTVpm 7). Then its partition function is obtained by applying E to a suitable S-family Σ of closed Riemannian super manifolds of dimension $d|1$; in other words, Σ is an endomorphism of the object $\emptyset \in \mathrm{RB}^{2|1}_S$. Then $E_S(\Sigma)$ is an even endomorphism of the trivial bundle $S \times \mathbb{C}$, which may be regarded as an element $E_S(\Sigma) \in C^\infty(S)^{ev}$.

For $d = 1$ we define

$$Z_E \overset{\text{def}}{=} E_S(S^{1|1}_\ell) \in C^\infty(S),$$

where $S = \mathbb{R}_+$ and $\ell \colon S \to \mathbb{R}_+$ is the identity.

For $d = 2$ we define

$$Z_E \overset{\text{def}}{=} E^+_S(T^{2|1}_{\ell,\tau}) \in C^\infty(S),$$

where $S = \mathbb{R}_+ \times \mathbb{R}^2_+$ and $\ell \colon S \to \mathbb{R}_+$, $\tau \colon S \to \mathbb{R}^2_+ \hookrightarrow \mathbb{R}^{2|1}_{cs,+}$ are the obvious maps.

Remark 78. It might seem that there is a better way to define the partition function of a super symmetric QFT of dimension $2|1$ by including odd parameters by setting

$$Z_E \overset{\text{def}}{=} E^+_S(T^2_{\ell,\tau}) \in C^\infty(S)^{ev},$$

where $S = \mathbb{R}_+ \times \mathbb{R}^{2|1}_+$ and $\ell \colon S \to \mathbb{R}_+$, $\tau \colon S \to \mathbb{R}^{2|1}_{cs,+}$ are the projection maps. However, $C^\infty(\mathbb{R}_+ \times \mathbb{R}^{2|1}_{cs,+}) = C^\infty(\mathbb{R}_+ \times \mathbb{R}^2_+; \mathbb{C}) \otimes \Lambda[\theta]$, where θ is odd, and hence

$$C^\infty(\mathbb{R}_+ \times \mathbb{R}^{2|1}_{cs,+})^{ev} = C^\infty(\mathbb{R}_+ \times \mathbb{R}^2_+),$$

which shows that the function associated to this more general family of super tori contains the same information as the partition function described above. An analogous remark applies to QFT's of dimension $1|1$.

5 Partition functions of susy QFT's

artition_function

The goal of this section is the proof of our main theorem thm:main 1 concerning the partition functions of super symmetric QFT's of dimension $2|1$; this is done in subsection subsec:partition_function2|1 5.2. As a warm-up we prove in the first subsection an analogous result for super symmetric QFT's of dimension $1|1$.

5.1 Partition functions of QFT's of dimension $1|1$

thm:main1

Theorem 79. *Let E be a super symmetric quantum field theory of dimension $1|1$. Then its partition function $Z_E\colon \mathbb{R}_+ \to \mathbb{C}$ is an integer valued constant function.*

The proof of this result will be based on using the following algebraic data obtained by applying E or rather the composite functor

$$E^+\colon\ \mathrm{RB}^{1|1} \xrightarrow{E} \mathrm{TV}^{\pm} \longrightarrow \mathrm{TV}$$

(see Remark rem:TVandTVpm 7) to certain objects resp. morphisms of the Riemannian bordism category $\mathrm{RB}^{1|1}$:

- the locally convex vector space $H \overset{\text{def}}{=} E^+_{\mathrm{pt}}(\mathrm{spt})$ associated to the *super point* spt (an object of the bordism category $\mathrm{RB}^{1|1}_{\mathrm{pt}}$);

- The function $E^+_S(S^{1|1}_f) \in \mathbb{R}^{1|1}_+(S) = \mathrm{SMAN}(S, \mathbb{R}^{1|1}_+)$ associated to the family of *super circles* $S^{1|1}_f \in \mathrm{RB}^{1|1}_S(\emptyset,\emptyset)$ determined by $f \in \mathbb{R}^{1|1}_+(S)$ (see Example ex:super_objects+morphisms 70).

- The function $E^+_S(S^{1|1}_f) \in \mathcal{N}(H,H)(S) = \mathrm{SMAN}(S, \mathcal{N}(H,H))$ associated to the family of *super intervals* $S^{1|1}_f \in \mathrm{RB}^{1|1}_S(\emptyset,\emptyset)$ determined by $f \in \mathbb{R}^{1|1}_+(S)$ (see Example ex:super_objects+morphisms 70). We recall that $S^{1|1}_f$ is an endomorphism of $p_S^*\,\mathrm{spt} \in RB^{1|1}_S$; hence $E_S(S^{1|1}_f)$ is an endomorphism of the trivial bundle $E_S(p_S^*\,\mathrm{spt}) = p_S^*(H) = S \times H$. This in turn can be reinterpreted as a smooth map $S \to \mathcal{N}(H,H)$ to the nuclear endomorphisms of H, i.e., $E^+_S(S^{1|1}_f) \in \mathcal{N}(H,H)(S) = \mathrm{SMAN}(S, \mathcal{N}(H,H))$.

Geometric relations then imply algebraic relations for the associated algebraic data. In particular, gluing the incoming with the outgoing super point in the family of super intervals $I_f^{1|1}$ results in the family of super circles $S_f^{1|1}$, and hence by proposition 23 (or rather its generalization to $\mathrm{RB}_S^{d|1}$) we have

$$E_S^+(S_f^{1|1}) = \operatorname{str} E_S^+(I_f^{1|1}). \tag{80}$$

Similarly, the geometric relation $I_f^{1|1} \circ I_{f'}^{1|1} = I_{\mu(f,f')}^{1|1}$ (see part 1 of Lemma 71) implies the algebraic relation

$$E_S^+(I_f^{1|1}) \circ E_S^+(I_g^{1|1}) = E_S^+(I_{\mu(f,g)}^{1|1}). \tag{81}$$

We note that the map

$$\mathbb{R}_{cs,+}^{1|1}(S) \longrightarrow \mathcal{N}(H,H)(S) \qquad \text{given by} \qquad f \mapsto E_S^+(S_f^{1|1})$$

depends functorially on S by commutativity of Diagram 31. In other words, the above describes a map of (generalized) super manifolds $\mathbb{R}_{cs,+}^{1|1} \to \mathcal{N}(H,H)$ in the S-point formalism. Identifying elements $f \in \mathbb{R}_{cs,+}^{1|1}$ with pairs (y,θ) of functions $\theta \in C^\infty(S)^{odd}$, $y \in C^\infty(S)^{ev}$ with $y_{red}(s) \in \mathbb{R}_+$ for all $s \in S_{red}$, we can write $E_S^+(S_{y,\theta}^{1|1})$ in the form

$$E_S^+(S_{y,\theta}^{1|1}) = A(y) + \theta B(y).$$

Here $A, B\colon \mathbb{R}_+ \to \mathcal{N}(H,H)$ are smooth maps, described via the S-point formalism by

$$\mathbb{R}_+(S) = C^\infty(S)^{ev} \longrightarrow \mathcal{N}(H,H)(S) \qquad y \mapsto A(y) \quad (\text{resp. } B(y)).$$

Lemma 82. *The relation (81) implies the following relations for the operators $A(y)$, $B(y)$:*

$$\begin{aligned} A(y_1)A(y_2) &= A(y_1+y_2)\\ A(y_1)B(y_2) &= B(y_1)A(y_2) = B(y_1+y_2)\\ B(y_1)B(y_2) &= -A'(y_1+y_2) \end{aligned} \tag{83}$$

Proof. Writing out the left hand side of equation 26 for $f = (y_1,\theta_1)$ and $g = (y_2,\theta_2)$, we obtain

$$\begin{aligned} &E_S^+(I_{y_1,\theta_1}^{1|1}) \circ E_S^+(I_{y_1,\theta_1}^{1|1})\\ =&(A(y_1)+\theta_1 B(y_1))(A(y_2)+\theta_2 B(y_2))\\ =&A(y_1)A(y_2) + \theta_1 B(y_1)A(y_2) + \theta_2 A(y_1)B(y_2) - \theta_1\theta_2 B(y_1)B(y_2). \end{aligned}$$

Here the minus sign is a consequence of permuting the odd elements θ_1 and $B(y_1)$. In order to expand the right hand side, we recall from equation (eq:1|1_multiplication 56) that

$$\mu((y_1,\theta_1),(y_2,\theta_2)) = (y_1 + y_2 + \theta_1\theta_2, \theta_1 + \theta_2).$$

It follows that the right hand side of equation eq:1|1_composition 81 is equal to

$$\begin{aligned}
&E_S^+(I^{1|1}_{\mu((y_1,\theta_1),(y_2,\theta_2))})\\
=&A(y_1 + y_2 + \theta_1\theta_2) + (\theta_1 + \theta_2)B(y_1 + y_2 + \theta_1\theta_2)\\
=&A(y_1 + y_2) + A'(y_1 + y_2)\theta_1\theta_2 + (\theta_1 + \theta_2)(B(y_1 + y_2) + B'(y_1 + y_2)\theta_1\theta_2)\\
=&A(y_1 + y_2) + \theta_1 B(y_1 + y_2) + \theta_2 B(y_1 + y_2) + \theta_1\theta_2 A'(y_1 + y_2).
\end{aligned}$$

Comparing coefficients then yields the desired relations. □

We note that

$$Z_E(y) = E^+_{\mathbb{R}_+}(S^{1|1}_{y,0}) = \operatorname{str} E^+_{\mathbb{R}_+}(I^{1|1}_{y,0}) = \operatorname{str} A(y),$$

where the first equality is the definition of the partition function (see Definiton def:partition_QFTd|1 77), and the second equality is equation eq:1|1_trace 80. Hence Theorem thm:main1 79 is a consequence of the following algebraic result.

Proposition 84. *Let $A(y)$, $B(y)$ be smooth families of nuclear operators parametrized by $y \in \mathbb{R}_+$ satisfying relations (eq:1|1_relations 83). Let H_1 be the eigenspace of $A(1)$ with eigenvalue $+1$, and let $\operatorname{sdim} H_1$ be its super dimension. Then*

$$\operatorname{str} A(y) = \operatorname{sdim} H_1.$$

We recall that a nuclear operator is compact; in particular, any (generalized) eigenspace of a nuclear operator corresponding to a non-zero eigenvalue is finite dimensional.

Proof. The third of the relations eq:1|1_relations 83 implies that

$$A'(y) = -B(\frac{y}{2})B(\frac{y}{2}) = -\frac{1}{2}[B(\frac{y}{2}), B(\frac{y}{2})],$$

where $[B(\frac{y}{2}), B(\frac{y}{2})]$ is the graded commutator of the odd operator $B(\frac{y}{2})$ with itself. Taking the super trace, we obtain

$$\frac{d}{dy}\operatorname{str} A(y) = -\frac{1}{2}\operatorname{str}[B(\frac{y}{2}), B(\frac{y}{2})] = 0,$$

since the super trace vanishes on graded commutators. This shows that $\operatorname{str} A(y)$ is independent of $y \in \mathbb{R}_+$.

To relate $\operatorname{str} A(1)$ to the super dimension of the eigenspace H_1, we apply the spectral calculus developed by Wrobel [Wr, Thm. 2.3] to the compact operator $A = A(1)$. The spectrum of any compact operator is a countable bounded subset $\sigma \subset \mathbb{C}$ whose only possible accumulation point is $0 \in \mathbb{C}$ (cf. [Ed, Thm. 9.10.2]). For $\lambda \in \sigma \setminus \{0\}$ the corresponding (generalized) eigenspace H_λ is finite dimensional. Since the eigenspace with eigenvalue 0 doesn't contribute to the super trace of A we have

$$\operatorname{str} A = \sum_{\lambda \in \sigma \setminus \{0\}} \operatorname{str}(A_{|H_\lambda})$$

To calculate $\operatorname{str}(A_{|H_\lambda})$, we choose a basis of H_λ such that the matrix corresponding to A is an upper triangular matrix with diagonal entries λ. Then $\operatorname{str}(A_{|H_\lambda}) = \lambda \operatorname{sdim} H_\lambda$ and similarly $\operatorname{str}(A^2_{|H_\lambda}) = \lambda^2 \operatorname{sdim} H_\lambda$.

Using Wrobel's spectral calculus (cf. [Wr, Thm. 2.3]), projection operators onto the generalized eigenspaces H_λ for $\lambda \neq 0$ can be constructed by functional calculus out of the operator A. Since $A = A(1)$ commutes with the operators $A(y)$, $B(y)$ for all $y \in \mathbb{R}_+$ (by the relations 83), also the projection operator onto H_λ commutes with $A(y)$ and $B(y)$. In particular, the operators $A(y)$, $B(y)$ map the subspace H_λ to itself and we can apply the argument above to the subspace H_λ to conclude that the super trace of $A(y)$ restricted to H_λ is independent of y.

Now let us calculate $\operatorname{str} A^2_{|H_\lambda}$ in two different ways. On one hand, $A^2 = A(1)A(1) = A(2)$ (by the first of the relations 83, and hence

$$\operatorname{str} A^2_{|H_\lambda} = \operatorname{str} A(2)_{|H_\lambda} = \operatorname{str} A_{|H_\lambda} = \lambda \operatorname{sdim} H_\lambda.$$

On the hand, we calculated above $\operatorname{str}(A^2_{|H_\lambda}) = \lambda^2 \operatorname{sdim} H_\lambda$. This implies $\operatorname{sdim} H_\lambda = 0$ for $\lambda \neq 1$ and hence $\operatorname{str} A = \operatorname{str} A_{|H_1} = \operatorname{sdim} H_1$. □

5.2 Partition functions of QFT's of dimension 2|1

In this section we will prove our main result theorem 1 concerning the partition function of a super symmetric quantum field theory of dimension 2|1. The proof is entirely analogous to the proof of the corresponding result 79 for field theories of dimension 1|1; it is based by analyzing the algebraic data obtained by applying the QFT to the certain objects and morphisms of the

Riemannian bordism category $\mathrm{RB}^{2|1}$. More precisely, if $E\colon \mathrm{RB}^{2|1} \to \mathrm{TV}^{\pm}$ is the QFT at hand, we evaluate the functor

$$E^+_S\colon \mathrm{RB}^{2|1}_S \xrightarrow{E_S} \mathrm{TV}^{\pm} \longrightarrow \mathrm{TV}$$

on certain objects and morphisms of $\mathrm{RB}^{2|1}_S$ described in example ex:super_objects+morphisms 70.

- The locally convex vector space $H \overset{\text{def}}{=} E^+_{\mathrm{pt}}(S^{1|1}_1)$ associated to the *super circle* of length 1 and the vector bundle over $\mathbb{R}_+$ given by $E^+_{\mathrm{pt}}(S^{1|1}_\ell)$, where $\ell\colon \mathbb{R}_+ \to \mathbb{R}_+$ is the identity. The fiber of this bundle at $1 \in \mathbb{R}_+$ is H, and we fix a trivialization $E^+_{\mathrm{pt}}(S^{1|1}_\ell) \cong S \times H$ which is the identity at $1 \in \mathbb{R}_+$.

- The function $E^+_S(T^{2|1}_{\ell,f}) \in C^\infty(S)^{ev}$ associated to the family of *super tori* $T^{2|1}_{\ell,f} \in \mathrm{RB}^{2|1}_S(\emptyset,\emptyset)$ determined by $f \in \mathbb{R}^{2|1}_+(S)$.

- The function $E^+_S(C^{2|1}_{\ell,f}) \in \mathrm{SMAN}(S, \mathcal{N}(H,H))$ associated to the family of *super cylinders* $C^{2|1}_{\ell,f} \in \mathrm{RB}^{2|1}_S(S^{1|1}_\ell, S^{1|1}_\ell)$ determined by $\ell \in \mathbb{R}_+(S)$, $f \in \mathbb{R}^{2|1}_{cs,+}(S)$.

We recall from part 2 of Lemma lem:2|1_relations 72 that $T^{2|1}_{\ell,f} = \widehat{C^{2|1}_{\ell,f}}$ (i.e., the family of super tori $T^{2|1}_{\ell,f}$ is obtained by gluing the domain and range of the family of super cylinders $C^{2|1}_{\ell,f}$). Hence Proposition prop:trace 23 (or rather its generalization to $\mathrm{RB}^{2|1}_S$) implies

$$E^+_S(T^{2|1}_{\ell,f}) = \operatorname{str} E^+_S(C^{2|1}_{\ell,f}). \tag{85}$$

eq:2|1_trace

The relation $C^{2|1}_{\ell,f} \circ C^{2|1}_{\ell,f'} = C^{2|1}_{\ell,\mu(f,f')}$ (see Lemma lem:2|1_relations 72 implies

$$E^+_S(C^{2|1}_{\ell,f}) \circ E^+_S(C^{2|1}_{\ell,g}) = E^+_S(C^{2|1}_{\ell,\mu(f,g)}) \tag{86}$$

eq:2|1_composition

Identifying $f \in \mathbb{R}^{2|1}_+(S)$ with triples (x,y,θ) of functions $x,y \in C^\infty(S)^{ev}$, $\theta \in C^\infty(S)^{odd}$ with $y_{red}(s) > 0$ for all $s \in S_{red}$, we write $E^+_S(C_{\ell,x,y,\theta})$ in the form

$$E^+_S(C_{\ell,x,y,\theta}) = A(\ell,x,y) + \theta B(\ell,x,y), \tag{87}$$

eq:AandB

where $A, B\colon \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathcal{N}(H,H)$ are smooth maps. Fixing ℓ for now, we will write $A(x,y)$, $B(x,y)$ instead of $A(\ell,x,y)$, $B(\ell,x,y)$.

Lemma 88. *Relation* eq:2|1_composition *86 implies the following relations for the functions A, B:*

$$\begin{aligned}
&A(x_1,y_1)A(x_2,y_2) = A(x_1+x_2,y_1+y_2)\\
&A(x_1,y_1)B(x_2,y_2) = B(x_1,y_1)A(x_2,y_2) = B(x_1+x_2,y_1+y_2)\\
&B(x_1,y_1)B(x_2,y_2) = -\frac{\partial A}{\partial \bar z}(x_1+x_2,y_1+y_2)
\end{aligned} \tag{89}$$

eq:2|1_relations

Proof. Writing out the left hand sice of equation (eq:2|1_composition 86) for $f=(x_1,y_1,\theta_1)$ and $g=(x_2,y_2,\theta_2)$ we obtain

$$\begin{aligned}
&E_S^+(C^{1|1}_{\ell,f})\circ E_S^+(C^{1|1}_{\ell,g})\\
=&(A(\tau_1)+\theta_1 B(\tau_1))(A(\tau_2)+\theta_2 B(\tau_2))\\
=&A(\tau_1)A(\tau_2)+\theta_1 B(\tau_1)A(\tau_2)+\theta_2 A(\tau_1)B(\tau_2)-\theta_1\theta_2 B(\tau_1)B(\tau_2),
\end{aligned} \tag{90}$$

eq:2|1_lhs

where we write τ_i instead of $(x_i,y_i)\in\mathbb{R}^2_+$.

In order to expand the right hand side, we need to write the multiplication map $\mu\colon \mathbb{R}^{2|1}_{cs}\times\mathbb{R}^{2|1}_{cs}\to\mathbb{R}^{2|1}_{cs}$ explicitly in terms of the coordinate functions x,y,θ; rewriting equation (eq:2|1_multiplication 65) (which is written in terms of $z=x+iy$, $\bar z = x-iy$ and θ) we obtain:

$$\mu((x_1,y_1,\theta_1),(x_2,y_2,\theta_2)) = (x_1+x_2+\frac{1}{2}\theta_1\theta_2, y_1+y_2+\frac{i}{2}\theta_1\theta_2,\theta_1+\theta_2).$$

It follows that the right hand side of equation eq:2|1_composition 86 is equal to

$$\begin{aligned}
&E_S^+(C^{2|1}_{\ell,\mu(f,g)})\\
=&A(x+\frac{1}{2}\theta_1\theta_2,y+\frac{i}{2}\theta_1\theta_2)+(\theta_1+\theta_2)B(x+\frac{1}{2}\theta_1\theta_2,y+\frac{i}{2}\theta_1\theta_2),
\end{aligned}$$

where we abbreviate x_1+x_2 by x and y_1+y_2 by y. Now we use Taylor expansion around the point (x,y) to rewrite the first term as follows:

$$\begin{aligned}
&A(x+\frac{1}{2}\theta_1\theta_2,y+\frac{i}{2}\theta_1\theta_2)\\
=&A(x,y)+\frac{\partial A}{\partial x}(x,y)\frac{1}{2}\theta_1\theta_2+\frac{\partial A}{\partial y}(x,y)\frac{i}{2}\theta_1\theta_2\\
=&A(x,y)+\frac{\partial A}{\partial \bar z}(x,y)\theta_1\theta_2,
\end{aligned}$$

where as usual $\frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. We note that all higher order terms in the Taylor expansion vanish since $\theta_i^2 = 0$. Similarly, we obtain a Taylor expansion for $B(x+\frac{1}{2}\theta_1\theta_2, y+\frac{i}{2}\theta_1\theta_2)$. Putting the terms together, we obtain:

$$E_S^+(C^{2|1}_{\ell,\mu(f,g)}) = A(x,y) + \frac{\partial A}{\partial \bar z}(x,y)\theta_1\theta_2 + B(x,y)\theta_1 + B(x,y)\theta_2 \qquad (91)$$

eq:2|1_rhs

Comparing coefficients in equations (90) and (91) then yields the desired relations. □

The following algebraic result is the key step in the proof of our main theorem 1.

prop:key

Proposition 92. *Let $A(\tau)$, $B(\tau)$ be smooth families of nuclear operators parametrized by $\tau \in \mathbb{R}^2_+ \subset \mathbb{C}$ satisfying relations (89) and $A(\tau+1) = A(\tau)$. Then the function* $\operatorname{str} A(\tau)$ *is a holomorphic function with an expansion of the form*

$$\operatorname{str} A(\tau) = \sum_{k\in\mathbb{Z}} a_k q^k,$$

where $q = e^{2\pi i\tau}$, the coefficients a_k are integers and $a_k = 0$ for sufficiently negative k. More generally, if $A(\ell,\tau)$, $B(\ell,\tau)$ is a family of such operators depending smoothly on some parameter $\ell \in \mathbb{R}_+$, then $\operatorname{str} A(\ell,\tau)$ *is in fact independent of ℓ.*

Assuming this statement for now, we next prove our main theorem.

Proof of Theorem 1. We note that

$$Z_E(\ell,\tau) = E_S^+(T^{2|1}_{\ell,\tau,0}) = \operatorname{str} E_S^+(C^{2|1}_{\ell,\tau,0}) = \operatorname{str} A(\ell,\tau),$$

where the first equality is the definition of the partition function (see Definiton 77), and the second equality is equation 85.

If $A, B\colon \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathcal{N}(H,H)$ are the smooth families of nuclear operators defined by equation (87), these satisfy the relations (89) by Lemma 72. Moreover, the equality $C^{2|1}_{\ell,x+1,y,\theta} = C^{2|1}_{\ell,x,y,\theta} \in \mathrm{RB}^{2|1}_S(S^{1|1}_\ell, S^{1|1}_\ell)$ (see part 3 of Lemma 72) implies in particular $A(\ell,x+1,y,\theta) = A(\ell,x,y,\theta)$. Then the above proposition implies the holomorphicity of its partition function, the integrality of its q-expansion, and the fact that the coefficients a_k vanish for

$k << 0$. To see that $Z_E(1,\tau)$ is invariant under the $SL_2(\mathbb{Z})$-action, we note that for $g \in SL_2(\mathbb{Z})$ we have

$$Z_E(\ell, g\tau) = Z_E(\ell|c\tau + d|, g\tau) = Z_E(g(\ell,\tau)) = Z_E(\ell,\tau).$$

Here the first equality comes from independence of $\operatorname{str} A(\ell,\tau)$ of ℓ, the second is by definition of the $SL_2(\mathbb{Z})$-action on $\mathbb{R}_+ \times \mathbb{R}^2_+$, and the third is Lemma lem:modularity 21 (or rather its generalization to QFT's of dimension $2|1$ which is straightforward). □

Proof of Proposition prop:key *92.* We first prove holomorphicity of $\operatorname{str} A(\tau)$. The third of the relations (eq:2|1_relations 89) implies that

$$\frac{\partial A}{\partial \bar z}(\tau) = -B^2(\frac{\tau}{2}) = -\frac{1}{2}[B(\frac{\tau}{2}), B(\frac{\tau}{2})]$$

(where [,] is the *graded* commutator) is again a trace class operator and hence we can calculate:

$$\frac{\partial}{\partial \bar z}\operatorname{str} A(\tau) = -\operatorname{str}(B^2(\frac{\tau}{2})) = -\frac{1}{2}\operatorname{str}[B(\frac{\tau}{2}), B(\frac{\tau}{2})] = 0$$

This shows that $\operatorname{str} A(\tau)$ is a holomorphic function on the upper half plane $\mathbb{R}^2_+$.

We observe that the relations (eq:2|1_relations 89) imply that the compact operators $A(\tau)$ for various $\tau \in \mathbb{R}^2_+$ all commute with each other. In particular, we can consider simultaneous generalized eigenspaces H_λ for the family of operators $A(\tau)$, $\tau \in \mathbb{R}^2_+$, where $\lambda\colon \mathbb{R}^2_+ \to \mathbb{C}$ is the corresponding eigenvalue. We note that the first of the relations (eq:2|1_relations 89) imply that λ is an exponential map, i.e.,

$$\lambda(\tau_1 + \tau_2) = \lambda(\tau_1)\cdot\lambda(\tau_2).$$

It follows that λ is either *identically* equal to zero, or its image is contained in $\mathbb{C}^\times$ (the non-zero complex numbers); in the latter case, λ can be written in the form $\lambda(\tau) = e^{\widetilde\lambda(\tau)}$, where $\widetilde\lambda\colon \mathbb{R}^2_+ \to \mathbb{C}$ is a homomorphism (of additive semigroups). The continuity of $\widetilde\lambda$ (which follows from the fact that $A\colon \mathbb{R}^2_+ \to \mathcal{N}(H,H)$ is smooth) implies that $\widetilde\lambda$ is the restriction of an $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$. It will be convenient to write $\widetilde\lambda$ in the form $\widetilde\lambda(\tau) = 2\pi i(a\tau - b\bar\tau)$ for $a, b \in \mathbb{C}$; in other words,

$$\lambda(\tau) = e^{2\pi i(a\tau - b\bar\tau)} = q^a \bar q^b \qquad (93)$$ eq:eigenvalue

We note that the condition $A(\tau+1) = A(\tau)$ implies $a - b \in \mathbb{Z}$. Let us denote by $H_{a,b} \subset H$ the generalized eigenspace corresponding to the eigenvalue function $\lambda(\tau)$ given by equation (93). We note that the spaces $H_{a,b}$ are *finite dimensional*, since the operators $A(\tau)$ are trace class and hence compact; in particular, any generalized eigenspace with non-zero eigenvalue is finite dimensional.

Since only the non-zero eigenspaces contribute to the super trace of $A(\tau)$, we have

$$\operatorname{str} A(\tau) = \sum_{a,b} \operatorname{str}(A(\tau)_{|H_{a,b}}).$$

It is straightforward to calculate the super trace of $A(\tau)$ restricted to $H_{a,b}$; $A(\tau)$ is an even operator and hence it maps the even (resp. odd) part of H to itself, and we can calculate the trace of $A(\tau)$ acting $H^{\pm}_{a,b}$ separately. There is a basis of $H^{\pm}_{a,b}$ such that the matrix corresponding to $A(\tau)$ is upper triangular with diagonal entries $\lambda_{a,b}(\tau)$. It follows that

$$\operatorname{str}\left(A(\tau)_{|H_{a,b}}\right) = \lambda_{a,b}(\tau) \operatorname{sdim} H_{a,b}.$$

We note that the argument proving the holomorphicity of $\operatorname{str} A(\tau)$ continues to hold if we restrict $A(\tau)$ to the subspace $H_{a,b}$ (the projection map onto $H_{a,b}$ is built by *functional calculus* from the operators $A(\tau)$; hence any operator that commutes with all $A(\tau)$'s – like $B(\tau/2)$ – will also commute with the projection operator and hence preserve the subspace $H_{a,b}$). We note that the function $\lambda_{a,b}(\tau)$ is holomorphic if and only if $b = 0$. It follows:

$$\operatorname{sdim} H_{a,b} = 0 \qquad \text{for} \quad b \neq 0.$$

In particular, the only contribution to the super trace of $A(\tau)$ comes from the space $H_{a,0}$, which forces a to be an integer. We conclude

$$\operatorname{str} A(\tau) = \sum_{k\in\mathbb{Z}} \operatorname{str}\left(A(\tau)_{|H_{k,0}}\right) = \sum_{k\in\mathbb{Z}} \lambda_{k,0}(\tau) \operatorname{sdim} H_{k,0} = \sum_{k\in\mathbb{Z}} q^k \operatorname{sdim} H_{k,0}.$$

We note that the eigenspaces $H_{k,0}$ must be trivial for sufficiently negative k (otherwise the corresponding eigenvalues q^k are arbitrarily large), and hence $a_k = \operatorname{sdim} H_{k,0}$ is zero. □

References

DM [DM] P. Deligne and J. Morgan, *Classical fields and supersymmetry* Quantum fields and strings: a course for mathematicians, Vol. 1 (Princeton, NJ, 1996/1997), AMS (1999) 41 – 98.

Ed [Ed] R. E. Edwards, *Functional analysis. Theory and applications*, Holt, Rinehart and Winston, New York-Toronto-London 1965

Fr [Fr] D. Freed, *Five lectures on supersymmetry.* AMS 1999.

HKST [HKST] H. Hohnhold, M. Kreck, S. Stolz and P. Teichner *DeRham cohomology via super symmetric field theories* in preparation

HST [HST] H. Hohnhold, S. Stolz and P. Teichner *The K-theory spectrum: from minimal geodesics to super symmetric field theories* Preprint, available at http://www.nd.edu/ stolz/preprint.html

Ho [Ho] M. Hopkins, *Algebraic Topology and Modular Forms.* Plenary Lecture, ICM Beijing 2002.

Koe [Koe] Gottfried Köthe, *Topological Vector Spaces II*, Grundlehren der mathematischen Wissenschaften Vol. 237, Springer Verlag (1979)

Lo [Lo] John Lott, *The Geometry of Supergravity Torsion Constraints* preprint arXiv:math.DG/0108125

Se1 [Se1] G. Segal, *Elliptic Cohomology.* Séminaire Bourbaki 695 (1988) 187–201.

Se2 [Se2] G. Segal, *The definition of conformal field theory.* Topology, geometry and quantum field theory, 423 – 577, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.

St [St] S. Stolz, *A conjecture concerning positive Ricci curvature and the Witten genus.* Math. Ann. 304 (1996), no. 4, 785–800

ST [ST] S. Stolz and P. Teichner, *What is an elliptic object?* Topology, geometry and quantum field theory, 247–343, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.

Va [Va] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction* Volume 11 of Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 2004.

Wi1 [Wi1] E. Witten, *The index of the Dirac operator on loop space.* Elliptic curves and modular forms in alg. top., Princeton Proc. 1986, LNM 1326, Springer, 161–181.

Wi2 [Wi2] E. Witten, *Index of Dirac operators.* Quantum fields and strings: a course for mathematicians, Vol. 1 (Princeton, NJ, 1996/1997), AMS (1999) 475–511.

Wi3 [Wi3] E. Witten, *Homework.* Quantum fields and strings: a course for mathematicians, Vol. 1 (Princeton, NJ, 1996/1997), AMS (1999) 609 – 717.

Wr [Wr] Volker Wrobel, *Spektraltheorie stetiger Endomorphismen eines lokalkonvexen Raumes*, Math. Ann. 234 (1978), no. 3, 193–208.