Math 80440 (Topics in Topology), Spring 2011

Tuesday/Thursday 3:30 - 4:45 pm in 215 DeBartolo

The $\widehat{A}$-genus and Witten genus in topology, geometry and physics

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The $\widehat{A}$-genus $\widehat{A}(M) \in \mathbb{Q}$ is a classical invariant associated to a closed oriented manifold M, defined by Hirzebruch in the 50's. In the 80's the physicst Witten defined the genus $W(M) \in \mathbb{Q}[[q]]$, the *Witten genus*. This is a refinement of the $\widehat{A}$-genus in the sense that $\widehat{A}(M)$ is the constant term of the power series $W(M)$. Both invariants are *genera*, that is, these invariants agree if two manifolds are bordant, and the invariant of the Cartesian product of two manifolds is the product of the invariant of the factors.

While $\widehat{A}(M)$ and $W(M)$ are defined topologically, their relevance is revealed by index theory: If M is a spin manifold, then $\widehat{A}(M)$ is the index of the *Dirac operator* on M (the index of an operator D is $\dim\ker D - \dim\operatorname{coker} D$). Conjecturally, the Witten genus can be interpreted as the S^1-equivariant index of the Dirac operator on the free loop space LM (the space of smooth maps $S^1 \to M$); unfortunately, the Dirac operator on LM has so far not been constructed rigorously. This interpretation of $\widehat{A}(M)$ leads to consequences in geometry: if M is a spin manifold with positive scalar curvature, then $\widehat{A}(M)$ vanishes. Similarly, it has been conjectured that $W(M)$ vanishes if the Ricci curvature of M is positive.

From a physics point of view, the invariants $\widehat{A}(M)$ (resp. $W(M)$) can be interpreted as the *partition function* of a supersymmetric quantum field theory associated to M called the non-linear σ-model of M of dimension 1 (resp. 2). Again, this is only a conjecture in the latter case, since the 2-dimensional non-linear σ-model has not been constructed yet mathematically rigorous.

Topics to be covered:

- Chern classes and Pontryagin classes, genera, $\widehat{A}(M)$, $W(M)$;
- Index theory: construction of (twisted) Dirac operators, the index theorem for twisted Dirac operators, S^1-equivariant index theorem, $W(M)$ as S^1-equivariant index of Dirac operator on LM, Weitzenböck formula
- quantum field theories a la Segal and their partition functions;

The pace of the class and the precise choice of topics to be covered will be determined by the background and the interests of the participants.