Math 80440 (Topics in Topology), Spring 2011
Tuesday/Thursday 3:30 - 4:45 pm in 215 DeBartolo

The $\hat{A}$-genus and Witten genus in topology, geometry and physics

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The $\hat{A}$-genus $\hat{A}(M) \in \mathbb{Q}$ is a classical invariant associated to a closed oriented manifold $M$, defined by Hirzebruch in the 50’s. In the 80’s the physicist Witten defined the genus $W(M) \in \mathbb{Q}[[q]]$, the Witten genus. This is a refinement of the $\hat{A}$-genus in the sense that $\hat{A}(M)$ is the constant term of the power series $W(M)$. Both invariants are genera, that is, these invariants agree if two manifolds are bordant, and the invariant of the Cartesian product of two manifolds is the product of the invariant of the factors.

While $\hat{A}(M)$ and $W(M)$ are defined topologically, their relevance is revealed by index theory: If $M$ is a spin manifold, then $\hat{A}(M)$ is the index of the Dirac operator on $M$ (the index of an operator $D$ is $\dim \ker D - \dim \coker D$). Conjecturally, the Witten genus can be interpreted as the $S^1$-equivariant index of the Dirac operator on the free loop space $LM$ (the space of smooth maps $S^1 \to M$); unfortunately, the Dirac operator on $LM$ has so far not been constructed rigorously. This interpretation of $\hat{A}(M)$ leads to consequences in geometry: if $M$ is a spin manifold with positive scalar curvature, then $\hat{A}(M)$ vanishes. Similarly, it has been conjectured that $W(M)$ vanishes if the Ricci curvature of $M$ is positive.

From a physics point of view, the invariants $\hat{A}(M)$ (resp. $W(M)$) can be interpreted as the partition function of a supersymmetric quantum field theory associated to $M$ called the non-linear $\sigma$-model of $M$ of dimension 1 (resp. 2). Again, this is only a conjecture in the latter case, since the 2-dimensional non-linear $\sigma$-model has not been constructed yet mathematically rigorous.

Topics to be covered:

• Chern classes and Pontryagin classes, genera, $\hat{A}(M)$, $W(M)$;

• Index theory: construction of (twisted) Dirac operators, the index theorem for twisted Dirac operators, $S^1$-equivariant index theorem, $W(M)$ as $S^1$-equivariant index of Dirac operator on $LM$, Weitzenböck formula

• quantum field theories a la Segal and their partition functions;

The pace of the class and the precise choice of topics to be covered will be determined by the background and the interests of the participants.