Multiple Choice

1.(5 pts.) Find the solution to the differential equation with initial value:

$$x^2y' + xy = 3x^3 + 1 \qquad y(1) = 0$$

(a) $y = e^x(x^2 - 3x + 2)$ (b) $y = x^2 + \dfrac{\ln x}{x} - \dfrac{1}{x}$ (c) $y = \ln(x + e^x - 1) - 1.$

(d) $y = \dfrac{3x^4}{4} + x - \dfrac{7}{4}$ (e) $y = e^x - e$

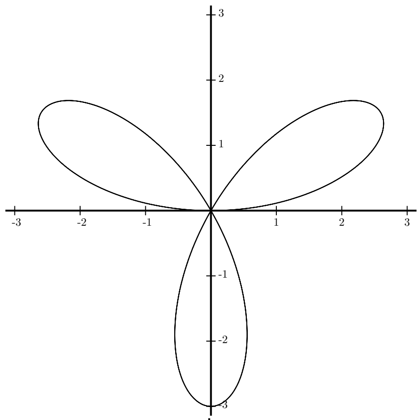
2.(5 pts.) Which integral below computes the area above the x–axis and below the parameterized curve $x(t) = t - \sin(t) + e^t$ and $y(t) = \sin(t)$ for $0 \le t \le \pi$.

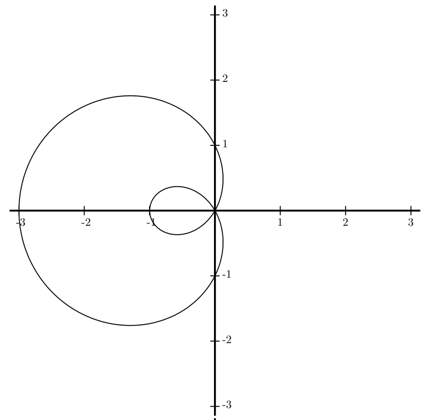
(a) $\displaystyle\int_0^\pi \cos(t)(1 - \cos(t) + e^t)\ dt$ (b) $\displaystyle\int_0^\pi \cos(t)(1 - \sin(t) + e^t)\ dt$

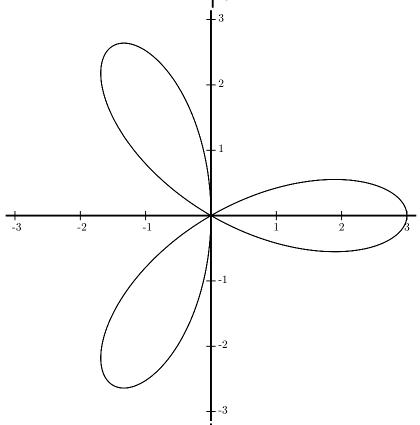
(c) $\displaystyle\int_0^\pi \sin(t)(1 + 2\cos(t) + e^t)\ dt$ (d) $\displaystyle\int_0^\pi \sin(t)(1 - \sin(t) + e^t)\ dt$

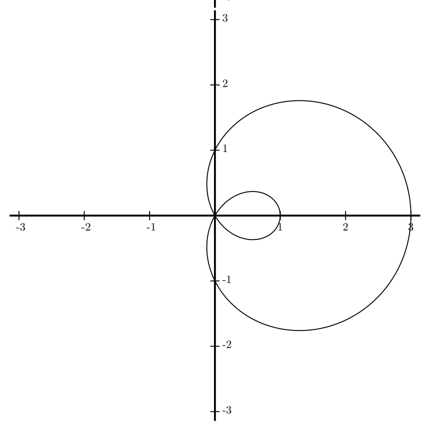
(e) $\displaystyle\int_0^\pi \sin(t)(1 - \cos(t) + e^t)\ dt$

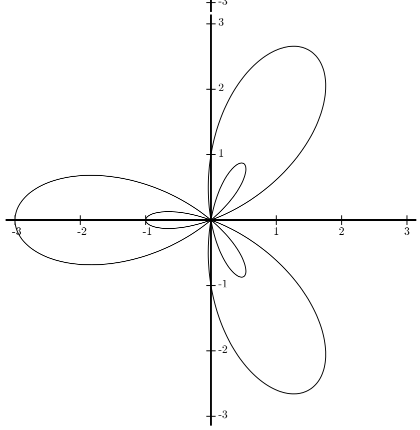
3.(5 pts.) Which graph below is the graph of the polar equation $r = 3\sin(3\theta)$.

(a) 

(b) 

(c) 

(d) 

(e) 

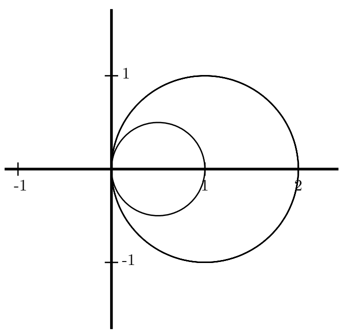
4.(5 pts.) Which integral below computes the length of the parameterized curve $x(t) = t + e^t$ and $y(t) = t^2 + e^t$ for $0 \leq t \leq \pi$.

(a) $\int_0^{\pi} \sqrt{t^2 + 2te^t + 2e^{2t} + t^4 + 2t^2e^t}\ dt$ (b) $\int_0^{\pi} \sqrt{t^2 + 6te^t + 2e^{2t} + 4t^2}\ dt$

(c) $\int_0^{\pi} \sqrt{1 + 2e^t + 2e^{2t} + t^4 + 2t^2e^t}\ dt$ (d) $\int_0^{\pi} \sqrt{1 + 2e^t + 2e^{2t} + 4t^2 + 4te^t}\ dt$

(e) $\int_0^{\pi} \sqrt{1 + 2e^t - 4t^2 - 4te^t}\ dt$

5.(5 pts.) Which integral below is the area of the region inside the circle $r = 2\cos\theta$ and outside the circle $r = \cos\theta$.



(a) $\frac{1}{2}\int_0^{\pi} 5\cos^2\theta\ d\theta$ (b) $\frac{1}{2}\int_0^{\pi} 3\cos^2\theta\ d\theta$ (c) $\frac{1}{2}\int_0^{\pi} 4\cos^2\theta\ d\theta$

(d) $\frac{1}{2}\int_0^{\pi} 3\sin^2\theta\ d\theta$ (e) $\frac{1}{2}\int_0^{\pi} 4\sin^2\theta\ d\theta$

6.(5 pts.) Find the length of the polar spiral $r = e^{\theta}$, $0 \le \theta \le 3$.

(a) $\sqrt{2}e^3 - 1$ (b) $e^3 - 1$ (c) $e^3 + 1$

(d) $\sqrt{2}\,(e^3 + 1)$ (e) $\sqrt{2}\,(e^3 - 1)$

7.(5 pts.) Which statement below is true about the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2 + e^n}$

(a) $\lim_{n\to\infty} \frac{e^n}{n^2+e^n}$ does not exist so the series converges.

(b) $\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 0$ so the series diverges.

(c) $\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 0$ so the series converges.

(d) $\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 1$ so the series diverges.

(e) $\lim_{n\to\infty} \frac{e^n}{n^2+e^n} = 1$ so the series converges.

8.(5 pts.) Sum the series $\sum_{n=2}^{\infty} \frac{2^n}{5^{2n}}$.

(a) $\frac{4}{23 \cdot 25}$ (b) $\frac{92}{25}$ (c) $\frac{5}{3}$ (d) $\frac{25}{23}$ (e) $\frac{100}{23}$

9.(5 pts.) Which of the statements below is true about the three series

I) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ II) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ III) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

(a) I) diverges, II) conditionally converges and III) absolutely converges.

(b) They all conditionally converge.

(c) I) absolutely converges, II) conditionally converges and III) diverges.

(d) I) conditionally converges; II) and III) absolutely converge

(e) They all absolutely converge.

10.(5 pts.) Consider the following two series:

$$\text{I) } \sum_{n=2}^{\infty} \frac{1}{n^3-1} \quad \text{and} \quad \text{II) } \sum_{n=2}^{\infty} \frac{1}{n^3}.$$

Which statement below is true?

(a) Series II) converges but series I) diverges.

(b) None of the other statements are true.

(c) Both series converge.

(d) Both series diverge.

(e) Series I) converges but series II) diverges.

11.(5 pts.) Which series below conditionally converges?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^2}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^n}{\sqrt[5]{n^2}}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[5]{n^4}}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n^4}}$

Partial Credit

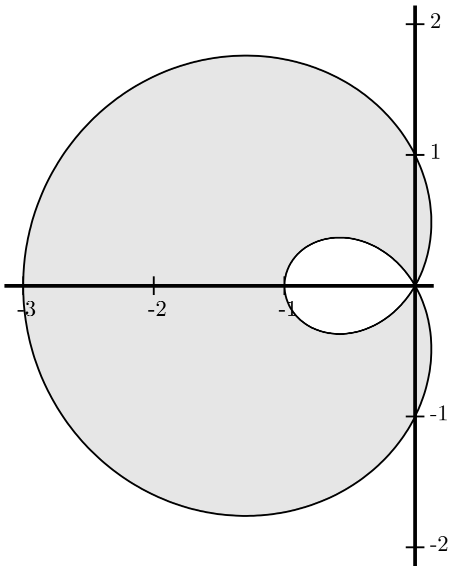
You must show your work on the partial credit problems to receive credit!

12.(15 pts.) Use the Integral Test to show that the series $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges.

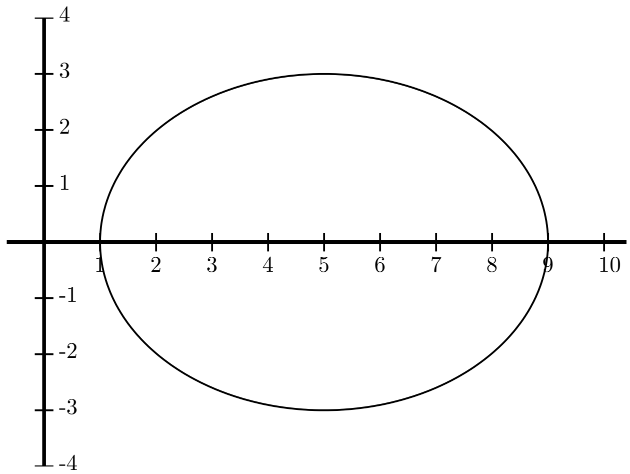
Remark: Be sure to check that the Integral Test can be applied.

13.(15 pts.) Set up an integral which computes the shaded area. The polar equation of the region is $r = 1 - 2\cos\theta$.

Remarks: $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. The answer is the difference of two areas.



14.(15 pts.) Set up an integral which computes the surface area of the surface obtained by rotating the parameterized ellipse $x(t) = 5 + 4\cos(t),\ y(t) = 3\sin(t),\ 0 \le t \le 2\pi$ about the $\underline{y\text{–axis}}$.



Name: ______________________________

Instructor: ANSWERS ______________

Math 126, Exam III
April 20, 2004

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for one hour.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 7 pages of the test.

Good Luck!

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(•)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(•)
3.	(•)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(•)	(e)
5.	(a)	(•)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(•)
7.	(a)	(b)	(c)	(•)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(•)	(e)
10.	(a)	(b)	(•)	(d)	(e)
11.	(a)	(b)	(•)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
12.	________
13.	________
14.	________
Total	________

12. Seven points were allocated for the correct verification of the hypotheses in the statement of the Integral Test; eight points were given for successfully applying the theorem.

The appropriate function to use in the Integral Test is clearly $f(x) = \ln(x)/x$ since $f(n) = a_n$ for integer n. The hypotheses for using this test are:

(1) (1 point) **Continuity**: $\ln(x)$ and $1/x$ are well–known to be continuous for all $x > 0$. Clearly their product is continuous on $[3, \infty)$.
(2) (1 point) **Positivity**: $\ln(x)$ is positive for $x > 1$, while $1/x$ is positive for $x > 0$. Clearly their product is positive for $[3, \infty)$.
(3) (5 points) **Decreasing**: The derivative of f is

$$f'(x) = \frac{(1/x)x - \ln(x)1}{x^2} = \frac{1 - \ln(x)}{x^2}.$$

Since the denominator is always positive for $x \neq 0$, f' is negative where $1-\ln(x) < 0$, i.e. for $x > e$. Since $e < 3$, f' is negative on $[3, \infty)$ and f is decreasing for this interval.

The Integral Test states that if $\int_3^\infty f(x)\ dx$ diverges then so does $\sum_{n=3}^\infty f(n)$, which is $\sum_{n=3}^\infty a_n$. If we let $u = \ln(x)$ then $du = (1/x)dx$ and

$$\int_3^\infty \frac{\ln(x)}{x}\ dx = \int u\ du = \frac{1}{2}(\ln(x))^2 \Big|_{x=3}^{x=\infty}$$

which diverges since $\ln(x) \to \infty$ as $x \to \infty$.

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13. This was a complicated problem to grade because there are so many correct answers. Many people computed the shaded area above the x–axis and doubled it, which got full marks if done correctly. Others computed the area directly. The crux of the problem is to work out intervals for θ which sweep out the various pieces.

From the first remark, $\theta = \frac{\pi}{3}$ is one of the angles where the curve goes through the pole (the origin). As θ increases, it gets to $\frac{\pi}{2}$ which turns out to be the point $(0, 1)$ in Cartesian coordinates. When $\theta = \pi$ we are at the point $(-3, 0)$. By the time $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$, we have traced the outside curve once and hence the integral $\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos\theta)^2\ d\theta$ computes the total area *inside* the outer curve.

As θ continues to increase past $\frac{5\pi}{3}$, we begin to trace the inside loop. At $\theta = 2\pi$ we are at $(-1, 0)$ and by $\theta = 2\pi + \frac{\pi}{3} = \frac{7\pi}{3}$ we have traced out the entire inner loop so one answer is

$$\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos\theta)^2\, d\theta - \frac{1}{2}\int_{\frac{5\pi}{3}}^{\frac{7\pi}{3}} (1-2\cos\theta)^2\, d\theta$$

Because the curve is periodic with period 2π the limits cab be changed: a popular choice was to write the area of the inner loop as $\frac{1}{2}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} (1-2\cos\theta)^2\, d\theta$.

Another approach is to start with $\frac{1}{2}\int_0^{2\pi} (1-2\cos\theta)^2\, d\theta$. This is the area inside the outer curve plus the area inside the inner curve so

$$\frac{1}{2}\int_0^{2\pi} (1-2\cos\theta)^2\, d\theta - 2\cdot\frac{1}{2}\int_{\frac{5\pi}{3}}^{\frac{7\pi}{3}} (1-2\cos\theta)^2\, d\theta$$

also computes the requested area.

Computing the area above the x–axis works out as follows. From $\frac{\pi}{3}$ to π the top half of the outer curve is traced: from 0 to $\frac{\pi}{3}$ the bottom loop of the inner curve is traced. Hence another answer is

$$2\left(\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (1-2\cos\theta)^2\, d\theta - \frac{1}{2}\int_0^{\frac{\pi}{3}} (1-2\cos\theta)^2\, d\theta\right)$$

One can also do

$$2\left(\frac{1}{2}\int_0^{\pi} (1-2\cos\theta)^2\, d\theta - 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} (1-2\cos\theta)^2\, d\theta\right)$$

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14. For revolution about the y-axis, surface area is

$$A = \int 2\pi x ds = \int_0^{2\pi} 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\ .$$

$\frac{dx}{dt} = \frac{d}{dt}(5+4\cos t) = -4\sin t$ and $\frac{dy}{dt} = \frac{d}{dt}(3\sin t) = 3\cos t$. Thus

$$\begin{aligned} A &= \int_0^{2\pi} 2\pi(5+4\cos t)\sqrt{(-4\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_0^{2\pi} 2\pi(5+4\cos t)\sqrt{16\sin^2 t + 9\cos^2 t}\, dt. \end{aligned}$$