Goals for today

Space Curves

Vector valued functions

Parameter: another name for independent variable.

Often t.

Vector-valued Function:

- 2D:
  - $\vec{f}(t) = \langle x(t), y(t) \rangle$
  - Studied in second semester calculus as parametrized curves.
- 3D:
  - $\vec{f}(t) = \langle x(t), y(t), z(t) \rangle$

Domain of $\vec{f}$ is intersection of the domains of the component functions.

Here is an example.

$$\vec{f}(t) = \left\langle \sqrt{4-t^2}\cos(\pi t), \sqrt{4-t^2}\sin(\pi t), t \right\rangle$$

Domain is $t \in [-2, 2]$.

Lives on the sphere $x^2 + y^2 + z^2 = 2^2$ and spirals around twice.

Usually the curve is difficult to describe other than by the formulas.

The reading for today gave you an example of a curve lying on a cylinder and another lying on a cone, and we just saw one lying on a sphere.

Here is an example where I have no particularly good way to describe it other than the formulas which we understand pretty well.

$$\vec{f}(t) = \left\langle \cos(\pi t), e^{2t}, \sin(\pi t)e^{-t^2} \right\rangle$$

Generically, two solids in 3D intersect in a curve. We have seen how two planes intersect in a line and we even know how to parametrize that line.

After that, things get more exciting. Later we will see how to get much information that we want without having a parametrization. But for now ...

Sphere $x^2 + y^2 + z^2 = 13$ and $z = 2$.

Then $x^2 + y^2 + 4 = 13$ so the intersection curve is a circle of radius 3 lying in the $z = 2$ plane and centered at $(0, 0, 2)$.

$x^2 + y^2 + z^2 = 9$ and $x^2 + (y-2)^2 + z^2 = 0$.

With luck, the intersection lies in a plane parallel to the xz-plane with y half way between the two centers, so $y = 1$. As above we can compute the intersection of this plane with the first sphere and get $\left\langle \sqrt{8}\cos(t), 1, \sqrt{8}\sin(t) \right\rangle$. Check that this curve lies on both surfaces.

The check proves the curve lies in the intersection. It seems likely that it is the complete intersection but we have not shown this conclusively. A similar comment applies to the example worked in the video. Time permitting we will return to this point later.

Limits and continuity

- Sum Rule
- Product Rule
- Quotient Rule
- Composition Rule: $g(t)$ a real-valued function, $\vec{f}$ a vector-valued function: $(f \vec{\circ} g)(t) = \vec{f}\big(g(t)\big)$ is composition.
- Dot Product Rule
- Cross Product Rule

Continuity: Given a point a in the domain of $\vec{f}$, exactly one of three things can happen: the domain of $\vec{f}$

(1) contains points on both sides of a;

(2) contains points only for $t \geqslant a$;

(3) contains points only for $t \leqslant a$.

Then $\vec{f}$ is continuous at a point $t = a$ if

(1) $\lim\limits_{t \to a} \vec{f}(t) = \vec{f}(a)$

(2) $\lim\limits_{t \to a^+} \vec{f}(t) = \vec{f}(a)$

(3) $\lim\limits_{t \to a^-} \vec{f}(t) = \vec{f}(a)$

Parametrize the intersection of a cylinder and a surface.

Cylinder: $h(x, y) = 0$ (does not depend on z). Parametrize h: $\vec{h}(t) = \langle x(t), y(t)\rangle$, t in some interval I. Graphs, circles, ellipses, $\ldots$.

Explicit Surface: $z = f(x, y)$.

$$\vec{r}(t) = \Big\langle x(t), y(t), f\big(x(t), y(t)\big)\Big\rangle \text{ for } t \in I$$

Implicit Surface: $F(x, y, z) = 0$. Solve for z.

Norman Window

$\vec{r}(t)$, $t \in [0, 9]$.

$[0, 2]$ $\vec{r}(t) = \langle t, 0 \rangle$

$[2, 5]$ $\vec{r}(t) = \langle 2, t - 2 \rangle$

$[5, 6]$ $\vec{r}(t) = \Big\langle \cos\big(\pi(t-5)\big) + 1, \sin\big(\pi(t-5)\big) + 3 \Big\rangle$

$[6, 9]$ $\vec{r}(t) = \langle 0, 9 - t \rangle$

The problem I was having in class was that I was trying to make the interval for each piece have length 2. Life is easier if you use the natural intervals for each piece: 2 for the bottom horizontal line, 3 for the two vertical lines and 1 for the arc at the top.