Goals for today

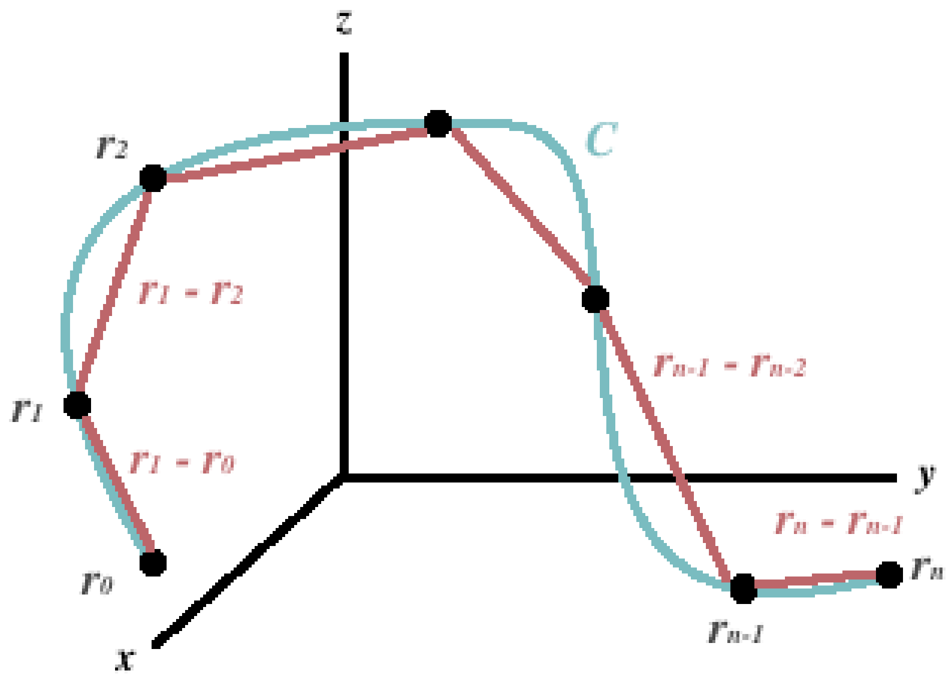
Arc length

Let $\vec{r}(t)$ be a curve which is differentiable and whose derivative is continuous on some interval $[a, b]$. (It suffices for $\vec{r}(t)$ to be continuous and $\vec{r}\,'(t)$ to be piecewise continuously differentiable on $[a, b]$.)

Then

$$\int_a^b |\vec{r}\,'(t)|\, dt$$

exists. But what is it measuring?



The sum of the lengths of the vectors r_i - t_{i-1} approximate the length of C.

There is a value of t_i in the i^{th} interval so that $|\vec{r}\,'(t_i)|$ is the absolute value of the slope so $|\vec{r}\,'(t_i)|\Delta_i$ is the length of the chord and approximates the length of the curve.

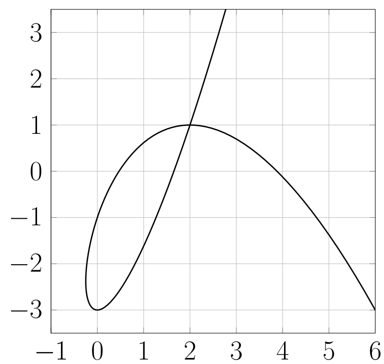
Hence

$$\int_a^b |\vec{r}\,'(t)|\, dt$$

is the distance traveled by a particle moving along the curve with parametrization $\vec{r}(t)$.

This will be the length of the curve as long as the particle does not go over a part of the curve which has positive length more than once.

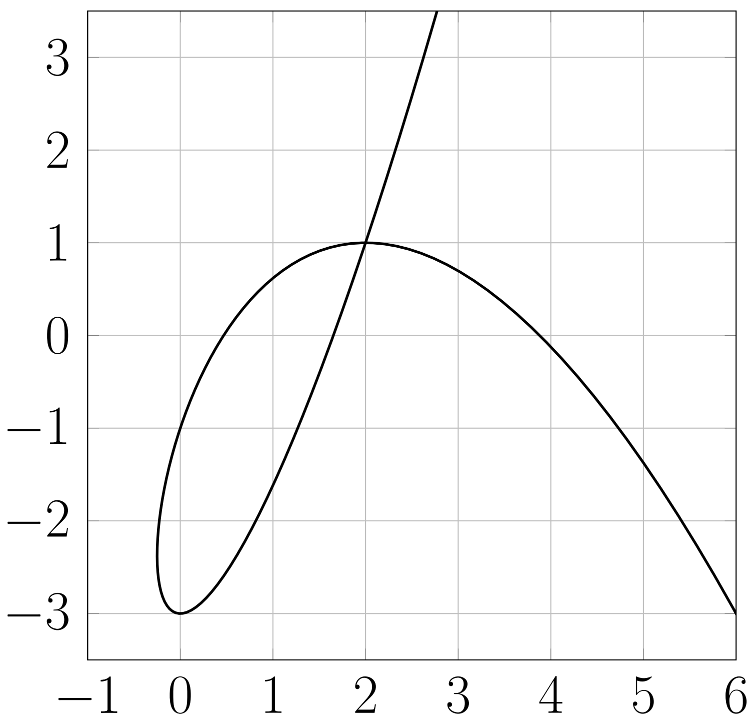
Going around a circle twice BAD,



OK.

The curve on the last slide is parametrized by

$$\vec{r}(t) = \left\langle t^2 - t, t^3 - 3t - 1 \right\rangle.$$



Intersection point is $(2, 1)$ which occurs when $t = -1$ and $t = 2$. The point on the curve at the lower right occurs at $t = -2$ and the local min occurs when $t = 1$.

$$\vec{r}\,'(t) = \left\langle 2t - 1, 3t^2 - 3 \right\rangle$$

The length of the curve from $t = -2$ to $t = 3$ is given by

$$\int_{-2}^{3} \sqrt{(2t-1)^2 + (3t^2-3)^2}\, dt = \int_{-2}^{3} \sqrt{9t^4 - 14t^2 - 4t + 10}\, dt$$

which we would need to evaluate numerically.

Just as in 1st year calculus when you used the Fundamental theorem to define the log function, we can define the *arc length function* for any parametrization which yields length by

$$s(t) = \int_a^t |\vec{r}\,'(u)| du$$

An important formula is

$$\frac{ds(t)}{dt} = |\vec{r}\,'(t)| \; .$$

As long as $|\vec{r}\,'(t)|$ is piecewise positive, $s(t)$ has an inverse function $t(s)$ and

$$\vec{R}(s) = \vec{r}\big(t(s)\big)$$

is an *arc length parametrization* of the curve.

Arc length can rarely be computed in practice but one can always try.

$$\vec{r}(t) = \left\langle t\cos(t), t\sin(t), \frac{1}{3}(2t)^{3/2} \right\rangle$$

$$\vec{r}\,'(t) = \left\langle \big(-t\sin(t) + \cos(t)\big), \big(t\cos(t) + \sin(t)\big), (2t)^{1/2} \right\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\big(-t\sin(t) + \cos(t)\big)^2 + \big(t\cos(t) + \sin(t)\big)^2 + \big((2t)^{1/2}\big)^2}$$

$$|\vec{r}\,'(t)| = \sqrt{t^2 + 2t + 1} = t + 1$$

$$s(t) = \int_0^t (u+1)du = (u^2/2 + u)\Big|_0^t = t^2/2 + t$$

The inverse function in this case is found as follows.

$s = t^2/2 + t$, $2s = t^2 + 2t$, $2s + 1 = t^2 + 2t + 1 = (t+1)^2$,

$t + 1 = \pm\sqrt{2s+1}$

$$t = \sqrt{2s+1} - 1$$

since t should be positive.

$$\vec{r}(s) = \big\langle (\sqrt{2s+1} - 1)\cos(\sqrt{2s+1} - 1),$$

$$(\sqrt{2s+1} - 1)\sin(\sqrt{2s+1} - 1),$$

$$\frac{1}{3}\big(2(\sqrt{2s+1} - 1)\big)^{3/2} \Big\rangle$$

The first is $\vec{\mathbf{T}} = \dfrac{d\vec{r}(s)}{ds} = \dfrac{\frac{d\vec{r}(t)}{dt}}{\frac{ds}{dt}}$.

and by one of earlier formulas

$\vec{\mathbf{T}} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

Hence $\vec{\mathbf{T}}$ is the unit tangent vector to the curve at the point and we can compute it.

Since $\vec{\mathbf{T}} \bullet \vec{\mathbf{T}} = 1$, $\vec{\mathbf{T}} \bullet \frac{d\vec{\mathbf{T}}}{ds} = 0$.

Our second vector in the Frenet-Serret frame is the *normal vector*, the unit vector $\vec{\mathbf{N}}$ so that $\frac{d\vec{\mathbf{T}}}{ds} = \kappa(s)\vec{\mathbf{N}}$ for $\kappa(s) \geqslant 0$.

The function $\kappa(s)$ is called the *curvature*.

The third vector is the binormal, defined by $\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$.

The three vectors are mutually orthogonal, have unit length, and $\vec{\mathbf{T}}$ - $\vec{\mathbf{N}}$ - $\vec{\mathbf{B}}$ gives the right hand rule.

There are more elaborate formula to do calculations but we will not pursue them here.

$$\vec{r}(t) =$$

$$\langle \sin(t) - 2\cos(t) + 1, -2\sin(t) - 2\cos(t) + 2, 2\sin(t) - \cos(t) - 5 \rangle$$

$$\vec{r}\,'(t) = \langle \cos(t) + 2\sin(t), -2\cos(t) + 2\sin(t), 2\cos(t) + \sin(t) \rangle$$

$$|\vec{r}\,'(t)| =$$

$$\sqrt{\left(\cos(t) + 2\sin(t)\right)^2 + \left(-2\cos(t) + 2\sin(t)\right)^2 + \left(2\cos(t) + \sin(t)\right)^2}$$

$$|\vec{r}\,'(t)| = 3$$

Hence

$\vec{\mathbf{T}(t)} = \dfrac{1}{3} \langle \cos(t) + 2\sin(t), -2\cos(t) + 2\sin(t), 2\cos(t) + \sin(t) \rangle.$

$$\frac{d\,\vec{\mathbf{T}(t)}}{ds} = \frac{\dfrac{d\,\vec{\mathbf{T}(t)}}{dt}}{\dfrac{ds}{dt}} =$$

$$\frac{1/3\,\langle -\sin(t) + 2\cos(t), 2\sin(t) + 2\cos(t), -2\sin(t) + \cos(t) \rangle}{\dfrac{1}{3}} =$$

$$\langle -\sin(t) + 2\cos(t), 2\sin(t) + 2\cos(t), -2\sin(t) + \cos(t) \rangle$$

$$\vec{r}(t) =$$

$$\langle \sin(t) - 2\cos(t) + 1, -2\sin(t) - 2\cos(t) + 2, 2\sin(t) - \cos(t) - 5 \rangle$$

$$\vec{\mathbf{T}}(t) = \frac{1}{3} \langle \cos(t) + 2\sin(t), -2\cos(t) + 2\sin(t), 2\cos(t) + \sin(t) \rangle$$

$$\frac{d\,\vec{\mathbf{T}}(t)}{ds} = \langle -\sin(t) + 2\cos(t), 2\sin(t) + 2\cos(t), -2\sin(t) + \cos(t) \rangle$$

$$\left| \frac{d\,\vec{\mathbf{T}}(t)}{ds} \right| = 3$$

so the curvature is constant and is 3.

$$\vec{\mathbf{N}}(t) = \frac{1}{3} \langle -\sin(t) + 2\cos(t), 2\sin(t) + 2\cos(t), -2\sin(t) + \cos(t) \rangle$$

$$\langle \cos(t) + 2\sin(t), -2\cos(t) + 2\sin(t), 2\cos(t) + \sin(t)\rangle \times$$

$$\langle -\sin(t) + 2\cos(t), 2\sin(t) + 2\cos(t), -2\sin(t) + \cos(t)\rangle =$$

$$\langle A, B, C\rangle$$

$$A = \begin{vmatrix} -2\cos(t) + 2\sin(t) & 2\cos(t) + \sin(t) \\ 2\sin(t) + 2\cos(t) & -2\sin(t) + \cos(t) \end{vmatrix} = -6$$

$$B = -\begin{vmatrix} \cos(t) + 2\sin(t) & 2\cos(t) + \sin(t) \\ -\sin(t) + 2\cos(t) & -2\sin(t) + \cos(t) \end{vmatrix} = 3$$

$$C = \begin{vmatrix} \cos(t) + 2\sin(t) & -2\cos(t) + 2\sin(t) \\ -\sin(t) + 2\cos(t) & 2\sin(t) + 2\cos(t) \end{vmatrix} = 6$$

Hence

$$\vec{\mathbf{B}}(t) = \frac{1}{9}\langle -6, 3, 6\rangle = \frac{1}{3}\langle -2, 1, 2\rangle$$

Normal Plane: This plane is perpendicular to $\vec{r}$. It is determined by $\vec{\mathbf{N}}$ and $\vec{\mathbf{B}}$: $\vec{\mathbf{T}}$ is a normal vector for the plane.

Osculating Plane: This plane best captures the motion of the curve. It is determined by $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$: $\vec{\mathbf{B}}$ is a normal vector for the plane.

Rectifying Plane: This plane determined by $\vec{\mathbf{T}}$ and $\vec{\mathbf{B}}$: $\vec{\mathbf{N}}$ is a normal vector for the plane. We won't bother with this one.

The binormal is constant if and only if the curve lies in the Osculating Plane.

Hence the curve

$$\vec{r}(t) =$$

$$\langle \sin(t) - 2\cos(t) + 1, -2\sin(t) - 2\cos(t) + 2, 2\sin(t) - \cos(t) - 5 \rangle$$

is planar, that is, it lies in the plane

$$\langle -2, 1, 2 \rangle \bullet \langle x, y, x \rangle = \langle -2, 1, 2 \rangle \bullet \langle -1, 0, -6 \rangle = -10$$

Arc length formula

$$s(t) = \int_a^t \sqrt{|\vec{r}\,'(u)|}du \ .$$

Derivative of arc length

$$\frac{ds(t)}{dt} = |\vec{r}\,'(t)| \ .$$

Unit tangent vector

$$\vec{\mathbf{T}} = \frac{d\vec{r}(s)}{ds} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \ .$$

Unit normal vector and curvature.

$$\frac{d\vec{\mathbf{T}}}{ds} = \kappa(s)\vec{\mathbf{N}} \ .$$

$$\frac{d\vec{\mathbf{T}}}{ds} = \frac{\frac{d\vec{\mathbf{T}}}{dt}}{|\vec{r}\,'(t)|} \ .$$

Unit binormal vector

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$$

Normal plane at t_0: normal vector $\vec{\mathbf{T}}(t_0)$, point $\vec{r}(t_0)$.

Osculating plane at t_0: normal vector $\vec{\mathbf{B}}(t_0)$, point $\vec{r}(t_0)$.