Goals for today

$\vec{\mathbf{T}}$-$\vec{\mathbf{N}}$-$\vec{\mathbf{B}}$

Velocity and acceleration

Arc length formula

$$s(t) = \int_a^t \sqrt{|\vec{r}'(u)|}du \ .$$

Derivative of arc length

$$\frac{ds(t)}{dt} = |\vec{r}'(t)| \ .$$

Unit tangent vector

$$\vec{\mathbf{T}} = \frac{d\vec{r}(s)}{ds} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \ .$$

Unit normal vector and curvature.

$$\frac{d\vec{\mathbf{T}}}{ds} = \kappa(s)\vec{\mathbf{N}} \ .$$

$$\frac{d\vec{\mathbf{T}}}{ds} = \frac{\frac{d\vec{\mathbf{T}}}{dt}}{|\vec{r}'(t)|} \ .$$

Unit binormal vector

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$$

Normal plane at t_0: normal vector $\vec{\mathbf{T}}(t_0)$, point $\vec{r}(t_0)$.

Osculating plane at t_0: normal vector $\vec{\mathbf{B}}(t_0)$, point $\vec{r}(t_0)$.

Over the weekend we posted a handout in Canvas/Files giving an easy way to compute $\vec{\mathbf{T}}$-$\vec{\mathbf{N}}$-$\vec{\mathbf{B}}$. Today we are going to reprise this in terms of velocity and acceleration.

One way to interpret a parametrized curve $\vec{r}(t)$ is that it describes a particle moving along a curve which is at the point $\vec{r}(t)$ at time t.

In this case $\vec{r}'(t)$ is the *velocity vector*, $\vec{\mathbf{v}}(t)$, at time t. The *speed* is $|\vec{r}'(t)|$. Since we are no longer on a line, we can no longer use the sign of the derivative to determine the direction of motion so we split the idea of velocity into a direction, $\vec{\mathbf{T}}(t)$, and a magnitude, speed $v(t) = |\vec{r}'(t)|$.
Likewise, acceleration becomes a vector $\vec{\mathbf{a}}(t) = \vec{r}''(t)$.

Since $\vec{\mathbf{v}}$ and $\vec{r}'$ are the same thing,

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{v}}(t)}{|\vec{\mathbf{v}}(t)|} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Also

$$\vec{r}'(t) \times \vec{r}''(t) = \vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)$$

and in the handout we saw that $\vec{\mathbf{B}}(t)$ points in the same direction as $\vec{r}'(t) \times \vec{r}''(t) = \vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)$. This gives an easy way to compute $\vec{\mathbf{B}}(t)$ and then we can use $\vec{\mathbf{N}}(t) = \vec{\mathbf{B}}(t) \times \vec{\mathbf{T}}(t)$.

Using the $\vec{\mathbf{v}}(t)$-$\vec{\mathbf{a}}(t)$ interpretation, we can use physics terminology to describe the relations we derived in the handout. Any vector $\vec{x}$ based at $\vec{r}(t_0)$ can be written uniquely as

$$\vec{x} = x_T \vec{\mathbf{T}}(t_0) + x_N \vec{\mathbf{N}}(t_0) + x_B \vec{\mathbf{B}}(t_0)$$

for scalars x_T, x_N, x_N.

Apply this remark to $\vec{\mathbf{v}}(t)$:

$$\vec{\mathbf{v}}(t) = v(t)\vec{\mathbf{T}}(t)$$

with 0 as the coefficient of $\vec{\mathbf{N}}$ and of $\vec{\mathbf{B}}$.

Then $\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + v(t)\vec{\mathbf{T}}'(t)$ and

$$\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + v(t)^2\kappa(t)\vec{\mathbf{N}}(t)$$

so a_T is the derivative of speed, which is what we often mean by acceleration, and $a_N = v(t)^2\kappa(t)$.

Given a formula for speed, a_T is just its derivative. A second way to compute it is

$$a_T(t) = \frac{\vec{\mathbf{a}}(t) \bullet \vec{\mathbf{v}}(t)}{v(t)}$$

so there is no need to differentiate anything once you work out $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$.

The normal component a_N is conceptually more difficult to compute We could try taking the dot product with $\vec{\mathbf{N}}$ but this is the most difficult of the $\vec{\mathbf{T}}$-$\vec{\mathbf{N}}$-$\vec{\mathbf{B}}$ to compute. But taking the cross product with $\vec{\mathbf{v}}$ yields

$$a_N(t) = \frac{|\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)|}{|\vec{\mathbf{v}}(t)|}$$

While it is always true that $\vec{\mathbf{v}}$ and $\vec{\mathbf{T}}$ point in the same direction, this is rarely true of $\vec{\mathbf{a}}$ and $\vec{\mathbf{N}}$. However, $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$ span the same plane as $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ and so $\vec{\mathbf{v}} \times \vec{\mathbf{a}}$ points in the same direction as $\vec{\mathbf{B}}$.