Goals for today

Real valued functions of several variables.

- Understanding shapes in 3D
- Limits
- Continuity

Understanding shapes in 3D

Graphs: $z = f(x, y)$.

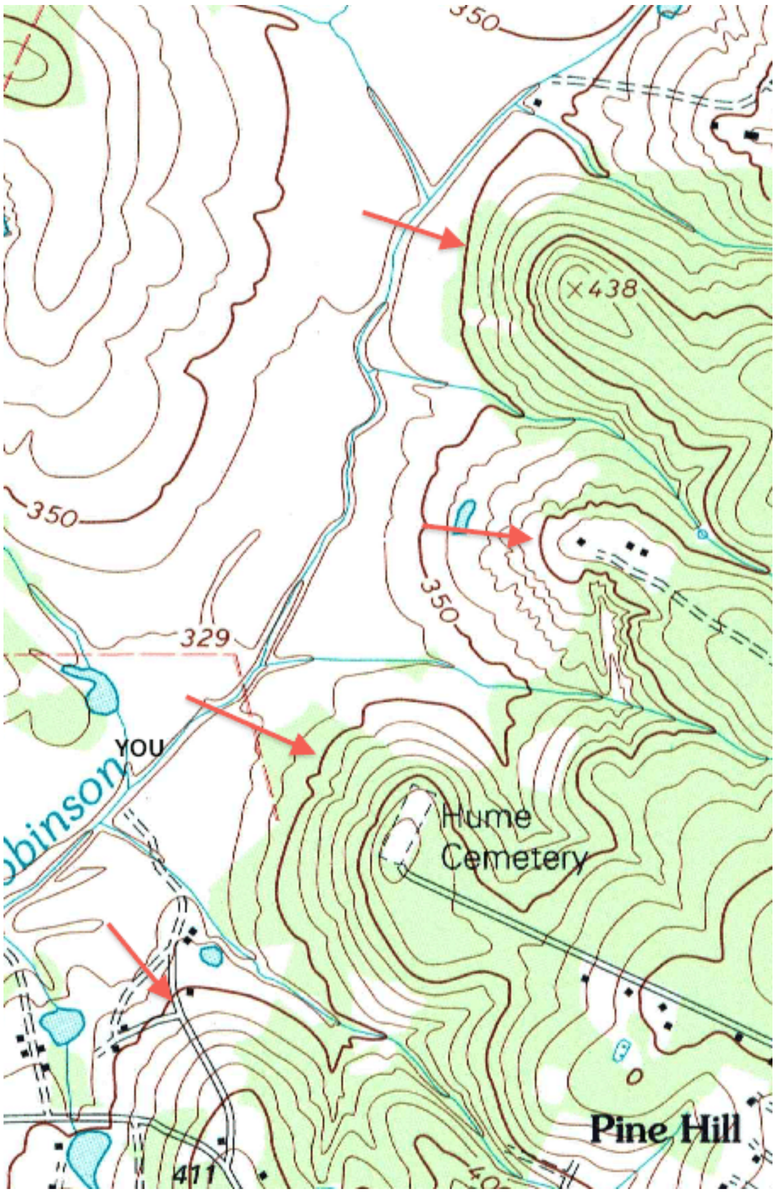
Surfaces: $F(x, y, z) = 0$.

- Graphs are surfaces.
- To see a surface as a graph, solve $F(x, y, z) = 0$ for z.

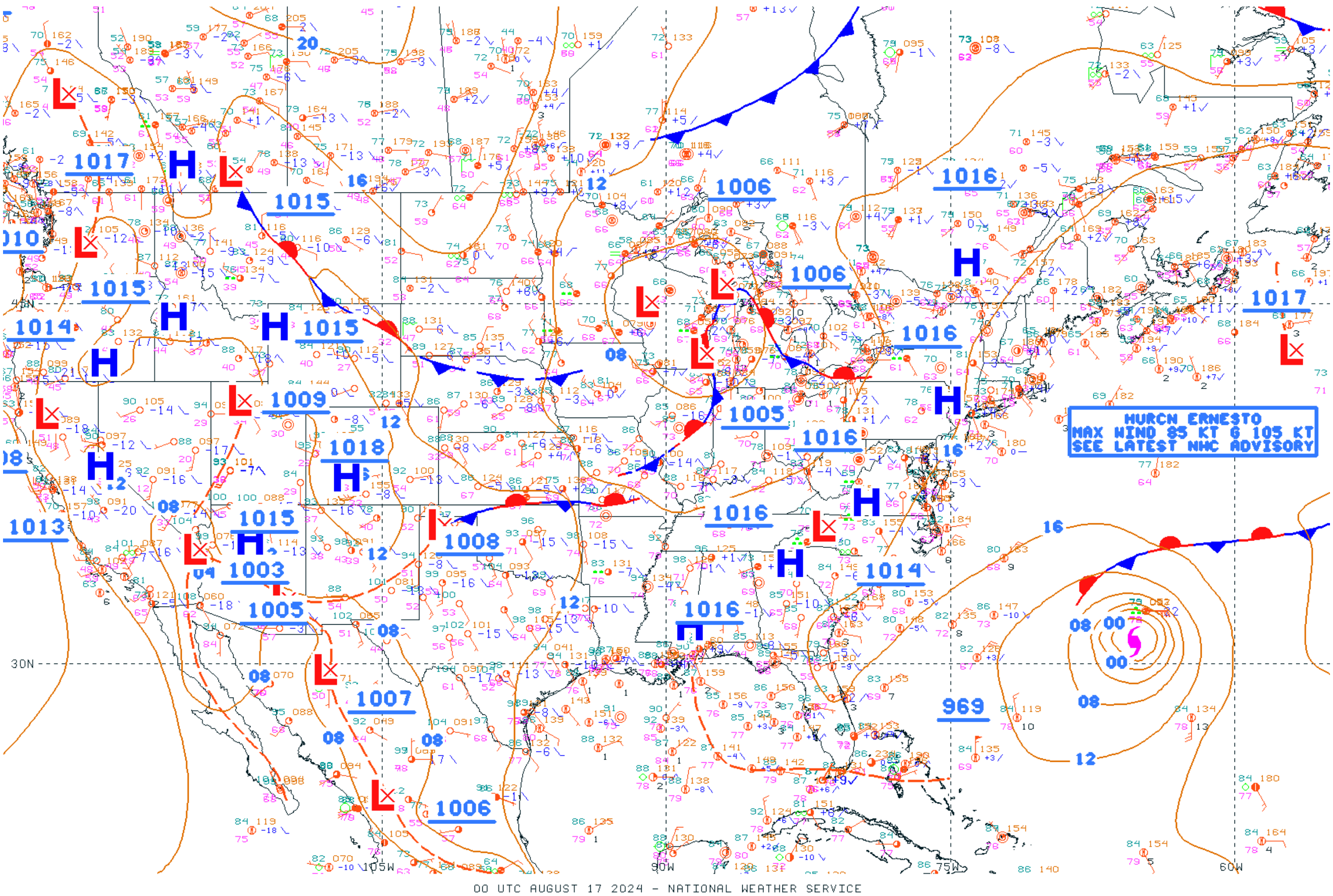
By analogy, in the plane $y = f(x)$ is a graph (or explicit function), $F(x, y) = 0$ is a curve (or implicit function). Every graph is a curve, but not every curve is a graph.

We can try to draw pictures of $y = f(x, y)$ or $F(x, y, z) = 0$ but most of us aren't very good at this so we adopt a different approach, Level Curves.

Here is one example of level curves that you may be familiar with.



Here is another.



Level curves in general are a collection of curves in the plane

Given a surface $F(x, y, z) = 0$, choose some values of z and for each of these values, c, draw the curve $F(x, y, c) = 0$.

Draw some level curves for $x^2 + 4y^2 + z = 0$.

$z = -1$: $x^2 + 4y^2 = 1$ which is an ellipse.

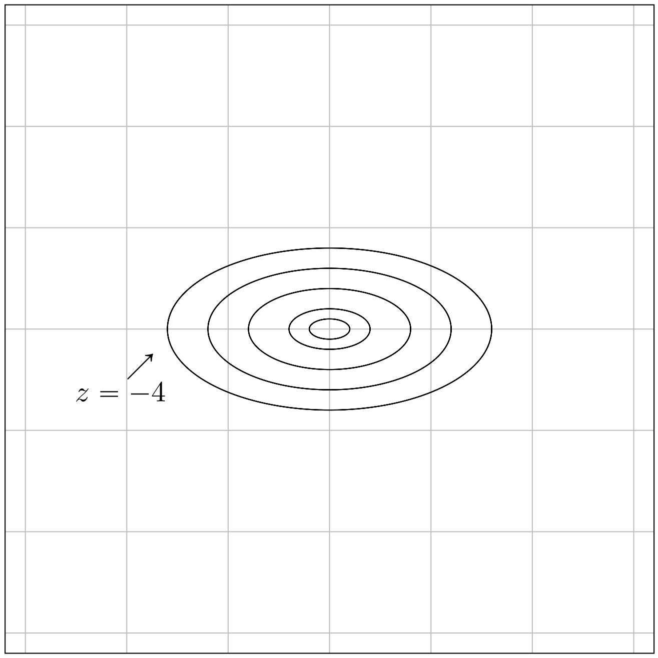
$z = 0$: $x^2 + 4y^2 = 0$ which is the origin.

$z > 0$: $x^2 + 4y^2 < 0$ which is empty.

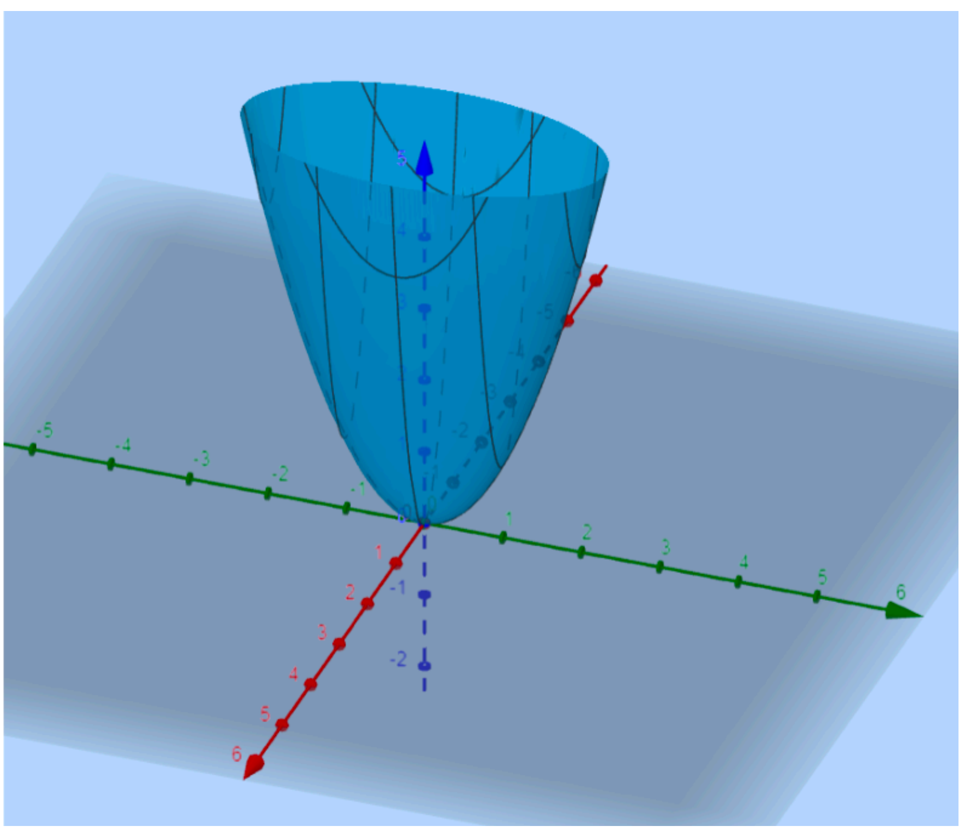
$z = -4$: $x^2 + 4y^2 = 4$ which is an ellipse; $\dfrac{x^2}{2^2} + \dfrac{y^2}{1^2} = 1$.

As z gets smaller, the ellipses are all centered at the origin with major axis along the x-axis and minor axis along the y-axis and keep getting bigger.

$$x^2 + 4y^2 + z = 0$$



$$x^2 + 4y^2 - z = 0$$



Limits

$$\lim_{(x,y)\to(x_0,y_0)} F(x,y) = L$$

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} F(x,y,z) = L$$

$$\vdots$$

Decide how close you want the outputs of the function F to be to L, say $\epsilon > 0$. Then you must find a number δ such that for all (x, y) in the domain of F and such that $0 < d\big((x, y), (x_0, y_0)\big) < \delta$, $|F(x, y) - L| < \epsilon$.

Technical remark. For all $\delta > 0$, the domain of F must contain some points (x, y) with $0 < d\big((x, y), (x_0, y_0)\big) < \delta$.

Theorems are very important because the definition is very hard to use.

- Sum Theorem
- Product Theorem
- Quotient Theorem
- Composition Theorem (to be stated carefully in a bit)
- Limit of constant functions
- Limit of coordinate functions (x, y, $\ldots$)

There is a Squeeze Theorem but inequalities in several variables are difficult to verify.

There is no version of l'Hospital's Rule.

Theorem: Let $\vec{r}(t)$ be a continuous curve with $\vec{r}(t_0) = (x_0, y_0)$, or $\vec{r}(t_0) = (x_0, y_0, z_0)$ etc. and such that $\vec{r}(t) \neq \vec{r}(t_0)$ for $t \neq t_0$.

IF $\lim\limits_{(x,y)\to(x_0,y_0)} F(x, y) = L$, then $\lim\limits_{t\to t_0} F\big(\vec{r}(t)\big) = L$.

Consider

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2 + (x-y)^2}$$

$$IF \quad \lim_{(x,y)\to(x_0,y_0)} F(x,y) = L, \text{ then } \lim_{t\to t_0} F\big(\vec{r}(t)\big) = L.$$

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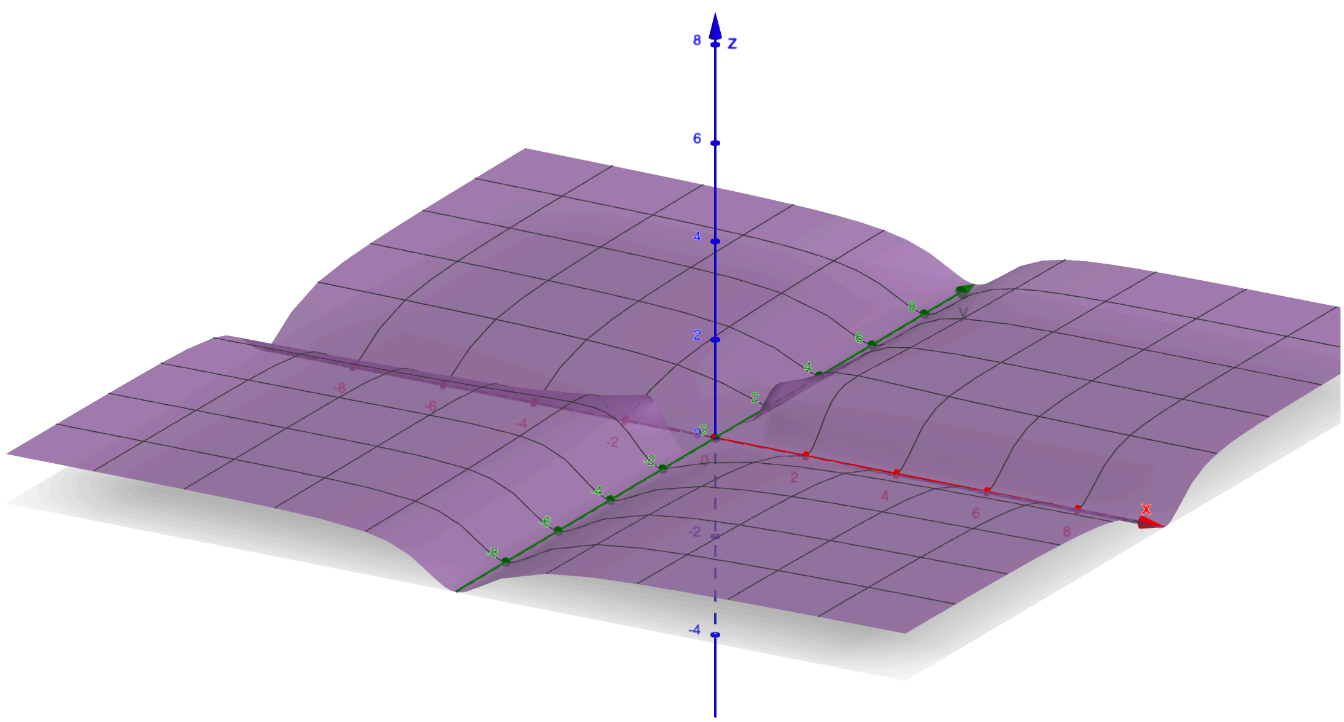
$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2 + (x-y)^2}$$



https://www.geogebra.org/3d?lang=en

Continuity

A function $F(x, y)$, $F(x, y, z)$, ... is *continuous* at a point (x_0, y_0) provided

- $\lim_{(x,y)\to(x_0,y_0)} F(x, y) = L$ for some number L.
- F is defined at (x_0, y_0).
- $F(x_0, y_0) = L$.

Theorems are very important because computing the required limits can be hard.

- Constant functions are continuous
- Coordinate functions are continuous
- Sum Theorem
- Product Theorem
- Quotient Theorem
- Composition Theorem

The Composition Theorem for both limits and continuity has the following form. First, compositions are more complicated than in one-variable. EG

$$F\big(x(t,u,v), y(t,u,v)\big) \text{ or } F\big(x(s,t), y(s,t), z(s,t)\big) \text{ etc.}$$

Suppose F is a function of n variable, $x_1, x_2, \ldots, x_n$. Suppose each x_i is a function of m variables, $t_1, \ldots, t_m$. Then the composition is a function of the t_i,

$$H(t_1, \cdots, t_m) = F\big(x_1(t_1, \cdots, t_m), \cdots, x_n(t_1, \cdots, t_m)\big)$$

Next suppose $\lim_{(t_1,\cdots,t_m)\to(a_1,\cdots,a_m)} x_i(t_1, \cdots, t_m) = L_i$ for $i = 1, \ldots n$.

Finally assume F is continuous at $(L_1, \cdots, L_n)$.

THEN

$$\lim_{(t_1,\cdots,t_m)\to(a_1,\cdots,a_m)} H(t_1, \cdots, t_m) = F(L_1, \cdots, L_n)$$