Goals for today

Partial Derivatives

Chain Rule

Let $f(x_1, \cdots, x_n)$ be a function and $(a_1, \cdots, a_n)$. Pick one of the variables, x_i. Define the partial derivative of f with respect to x_i at $(a_1, \cdots, a_n)$ by

$$f_{x_i}(a_1, \cdots, a_i, a_{i+1}, \cdots, a_n) =$$
$$\lim_{x_i \to a_i} \frac{f(a_1, \cdots, x_i, a_{i+1}, \cdots, a_n) - f(a_1, \cdots, a_i, a_{i+1}, \cdots, a_n)}{x_i - a_i}$$

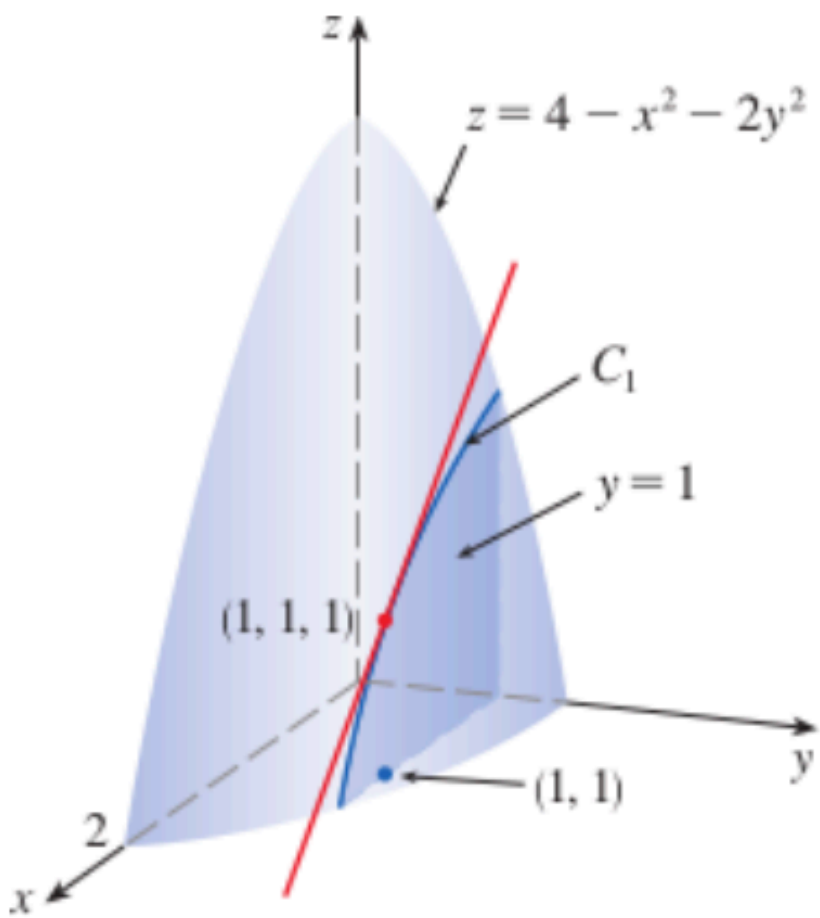
Other notation is

$$f_{x_i}(a_1, \cdots, a_n) = \left.\frac{\partial f}{\partial x_i}\right|_{x_i = a_i} = D_i f(a_1, \cdots, a_n)$$

Remark: Unlike one-variable calculus, just because the partial derivatives exist the function may not be differentiable. We will not define differentiable here because it's a bit technical but it does occur as an hypothesis in many results going forward. The definition is not useful to check if something is differentiable but there is a theorem:

A function is differentiable if all of its partial derivatives exist and are continuous.

Given $z = f(x, y)$, if you hold one of the variables, say y fixed, you get a parametrized curve $\vec{r}(x) = \langle x, y, f(x, y)\rangle$ and $\left\langle 1, 0, \dfrac{\partial f}{\partial x}(x_0, y_0, f(x_0, y_0))\right\rangle$ is its tangent vector at the point on the curve $\vec{r}(x_0)$.



$\vec{r}(t) = (t, 1, 2 - t^2)$ with $\vec{r}(1) = (1, 1, 1)$.

Chain Rule

Suppose F is a function of n variable, $x_1, x_2, \ldots, x_n$. Suppose each x_i is a function of m variables, $t_1, \ldots, t_m$. Then the composition is a function of the t_i,

$$H(t_1, \cdots, t_m) = F\big(x_1(t_1, \cdots, t_m), \cdots, x_n(t_1, \cdots, t_m)\big)$$

If you change t_i keeping the other t_j fixed, all the x_k can change. The change due to the change in x_ℓ is $\dfrac{\partial F}{\partial x_\ell}\dfrac{\partial x_\ell}{\partial t_i}$, and the total change is found by summing up all the changes for each x_ℓ.

Here is one way to organize the work.

Let $\vec{\nabla} F(x_1, \cdots, x_n) = \left\langle \frac{\partial F}{\partial x_1}(x_1, \cdots, x_n), \cdots, \frac{\partial F}{\partial x_n}(x_1, \cdots, x_n) \right\rangle$.

Let $\vec{\mathbf{x}}(t_1, \cdots, t_m) = \langle x_1(t_1, \cdots, t_m), \cdots, x_n(t_1, \cdots, t_m)\rangle$

and define

$$\frac{\partial\, \vec{\mathbf{x}}}{\partial\, t_i}(t_1, \cdots, t_m) = \left\langle \frac{\partial\, x_1}{\partial\, t_i}(t_1, \cdots, t_m), \cdots, \frac{\partial\, x_n}{\partial\, t_i}(t_1, \cdots, t_m) \right\rangle$$

$$\frac{\partial\, H}{\partial\, t_i}(t_1, \cdots, t_m) =$$

$$\vec{\nabla} F(x_1, \cdots, x_n)\Big|_{(x_1, \cdots, x_n) = (x_1(t_1, \cdots, t_m), \cdots, x_n(t_1, \cdots, t_m))} \bullet \frac{\partial\, \vec{\mathbf{x}}}{\partial\, t_i}(t_1, \cdots, t_m)$$

If you hold all of the t_ℓ fixed except for t_i, $\vec{r}(t_i) = \vec{\mathbf{x}}(t_1, \cdots, t_m)$ is a curve and $\frac{\partial\, \vec{\mathbf{x}}}{\partial\, t_i}(t_1, \cdots, t_m)$ is its tangent vector.

It looks worse than it is.

Step 1. Compute $\vec{\nabla} F(x_1, \cdots, x_n)$, which will be an expression in the x_k.

Step 2. Replace each x_k in this formula by $x_k(t_1, \cdots, t_m)$ so you have a function of the t_ℓ's.

Calc I Chain Rule: differentiate the outside function and plug in the inside function.

Step 3. Compute $\dfrac{\partial \vec{\mathbf{x}}}{\partial t_i}(t_1, \cdots, t_m)$ which is already a function of the t_ℓ's.

Calc I Chain Rule: differentiate the inside function.

Step 4: Take the dot product.

Calc I Chain Rule: Multiply the two functions.

Here F is the "outside function", $\vec{\nabla} F$ is its "derivative"; $\vec{\mathbf{x}}$ is the "inside function" and $\dfrac{\partial \vec{\mathbf{x}}}{\partial t_i}$ is its "derivative".

So the general form of the Chain Rule reads: differentiate the outside function; plug in the inside function; differentiate the inside function and multiply the two answers.