Goals for today

Implicit differentiation

Directional Derivatives

Chain Rule Review

Suppose F is a function of n variable, $x_1, x_2, \ldots, x_n$. Suppose each x_i is a function of m variables, $t_1, \ldots, t_m$. Then the composition is a function of the t_i,

$$H(t_1, \cdots, t_m) = F(x_1(t_1, \cdots, t_m), \cdots, x_n(t_1, \cdots, t_m))$$

Let $\vec{\mathbf{x}} = \langle x_1, \cdots, x_n \rangle$ and $\vec{\mathbf{t}} = \langle t_1, \cdots, t_m \rangle$ and write

$F(\vec{\mathbf{x}}) = F(x_1, \cdots, x_n)$ and

$\vec{\mathbf{x}}(\vec{\mathbf{t}}) = \langle x_1(t_1, \cdots, t_m), \cdots, x_n(t_1, \cdots, t_m) \rangle$.

Write

$$H(\vec{\mathbf{t}}) = F\big(\vec{\mathbf{x}}(\vec{\mathbf{t}})\big)$$

Let $\nabla F(\vec{\mathbf{x}}) = \left\langle \frac{\partial F}{\partial x_1}(x_1, \cdots, x_n), \cdots, \frac{\partial F}{\partial x_n}(x_1, \cdots, x_n) \right\rangle$.

Define

$$\frac{\partial \vec{\mathbf{x}}}{\partial t_i}(\vec{\mathbf{t}}) = \left\langle \frac{\partial x_1}{\partial t_i}(t_1, \cdots, t_m), \cdots, \frac{\partial x_n}{\partial t_i}(t_1, \cdots, t_m) \right\rangle$$

If F is *differentiable* then

$$\frac{\partial H}{\partial t_i}(\vec{\mathbf{t}}) = \nabla F(\vec{\mathbf{x}})\Big|_{\vec{\mathbf{x}}=\vec{\mathbf{x}}(\vec{\mathbf{t}})} \cdot \frac{\partial \vec{\mathbf{x}}(\vec{\mathbf{t}})}{\partial t_i}$$

We have not defined differentiable. Instead we quote a theorem which is the usual way to ensure a function is differentiable.

A function is differentiable if all of its partial derivatives exist and are continuous.

Implicit differentiation

Suppose we are given a "surface" $F(\vec{\mathbf{x}}) = 0$.

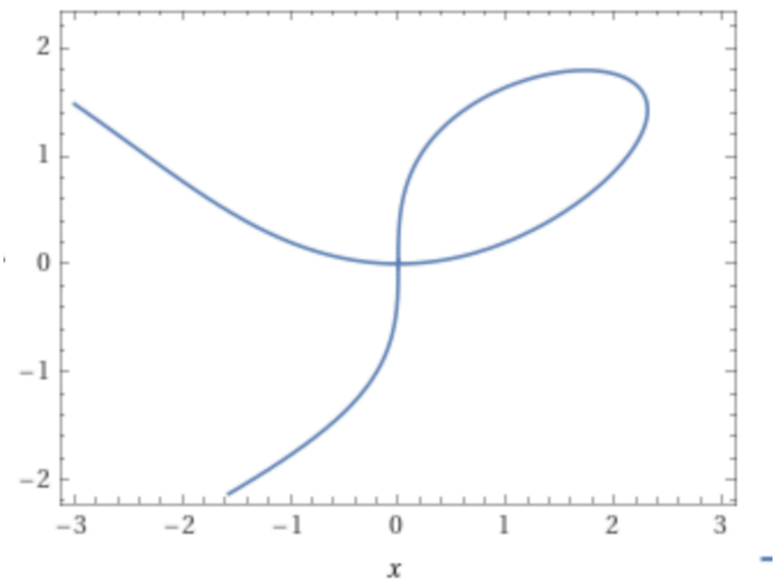
If $n = 2$ these are curves $F(x, y) = 0$ and if $n = 3$ they really are surfaces $F(x, y, z) = 0$.

We can pick one of the variables, say x_i, which becomes the dependent variable and then we say

> $F(\vec{\mathbf{x}}) = 0$ defines x_i *implicitly* in terms of the other variables x_k, which become the independent variables.

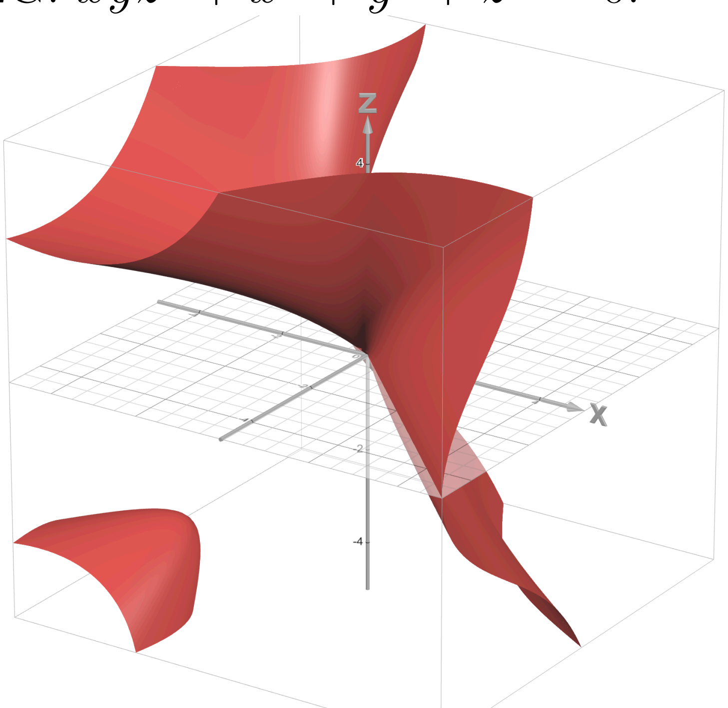
Using the Chain Rule we can work out the partial derivatives of the dependent variable x_i with respect to an independent variable x_k in terms of all the x_k.

EG: $x^3 - 5xy + y^4 = 0$ defines y as a function of x and x as a function of y. To find the value of y at $x = 1$ solve $1 - 5y + y^4 = 0$ and if there are several solutions, tell me which one you want.



Link: https://www.wolframalpha.com/widgets/view.jsp?id=d7cf9e4128713062211a710deaf28dc4

EG: $xyz^2 + x^3 + y^3 + z^3 = 0$.



Link: https://www.desmos.com/3d

Find $\dfrac{\partial y}{\partial x}$ when y is defined implicitly as a function of x by $x^3 - 5xy + y^4 = 0$. $F(x,y) = x^3 - 5xy + y^4$; $\vec{\mathbf{x}} = \langle x, y(x) \rangle$; $\nabla F = \langle 3x^2 - 5y, 4y^3 \rangle$; $\dfrac{\partial \vec{\mathbf{x}}}{\partial x} = \left\langle 1, \dfrac{\partial y(x)}{\partial x} \right\rangle$. Now $H(x) = F\big(x, y(x)\big) = 0$ so

$$\frac{\partial H}{\partial x} = 0 = \langle 3x^2 - 5y, 4y^3 \rangle \bullet \left\langle 1, \frac{\partial y(x)}{\partial x} \right\rangle$$

$$0 = 3x^2 - 5y + 4y^3 \frac{\partial y(x)}{\partial x} \quad \text{or} \quad \frac{\partial y}{\partial x} = -\frac{3x^2 - 5y}{4y^3}$$

In general,

$$\frac{\partial y}{\partial x} = -\frac{F_x(x,y)}{F_y(x,y)}$$

Of course the answer will be in terms of both x and y.

Find z_y if z is defined implicitly by the equation $xyz^x + x^3 + y^3 + z^3 = 0$.

$F(x, y, z) = xyz^2 + x^3 + y^3 + z^3$; $\vec{\mathbf{x}} = \langle x, y, z(x, y)\rangle$; $\nabla F = \left\langle yz^2 + 3x^2, xz^2 + 3y^2, 2xyz + 3z^2\right\rangle$; $\dfrac{\partial\,\vec{\mathbf{x}}(x, y)}{\partial\,y} = \left\langle 0, 1, \dfrac{\partial\,z}{\partial\,y}\right\rangle$.

$$0 = \left\langle yz^2 + 3x^2, xz^2 + 3y^2, 2xyz + 3z^2\right\rangle \bullet \left\langle 0, 1, \frac{\partial\,z}{\partial\,y}\right\rangle$$

$$0 = (xz^2 + 3y^2) + (2xyz + 3z^2)\frac{\partial\,z}{\partial\,y}$$

In general

$$\frac{\partial\,z}{\partial\,y} = -\frac{F_y(x, y, z)}{F_z(x, y, x)}$$

In general, general,

$$\frac{\partial\,x_i}{\partial\,x_k}(\vec{\mathbf{x}}) = -\frac{F_{x_k}(\vec{\mathbf{x}})}{F_{x_i}(\vec{\mathbf{x}})}$$

provided F is differentiable so we may use the Chain Rule. Then the Implicit Function Theorem says that as long as all the F_{x_k} are continuous and $F_{x_i}(\vec{\mathbf{x}}_0) \neq 0$ at a point $\vec{\mathbf{x}}_0$ with $F(\vec{\mathbf{x}}_0) = 0$, then x_i is differentiable around that point as the partial is given by the formula.

Directional Derivatives

The partial derivatives are just ordinary derivatives and so give information about how the function changes as the variable changes. In 1 dimension this there are only two directions (ie unit vectors, ± 1). In higher dimensions there are infinitely many more and sooner or later we are going to wonder about

$$\lim_{h \to 0} \frac{F(\vec{\mathbf{x}} + h\vec{\mathbf{a}}) - F(\vec{\mathbf{x}})}{h}$$

If $\vec{a}$ is a coordinate vector, then this limit if $F_{x_i}(\vec{\mathbf{x}})$.

If F is differentiable, the Chain Rule says

$$(1) \qquad \lim_{h \to 0} \frac{F(\vec{\mathbf{x}} + h\vec{\mathbf{a}}) - F(\vec{\mathbf{x}})}{h} = \nabla F \bullet \vec{\mathbf{a}}$$

The **definition** of differential is that equation (1) holds for all $\vec{\mathbf{a}}$.

- (1) holds if $\vec{\mathbf{a}} = \vec{0}$.
- If (1) holds for $\vec{\mathbf{a}}$, then (1) holds for $c\vec{\mathbf{a}}$.

Hence the issue is, if (1) holds for $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, does (1) hold for $\vec{\mathbf{a}} + \vec{\mathbf{b}}$?

If the partials of F are continuous then (1) holds for all $\vec{\mathbf{a}}$ although this is not obvious.

In the case of a unit vector we define

$$D_{\vec{\mathbf{u}}}(\vec{\mathbf{x}}) = \lim_{h\to 0} \frac{F(\vec{\mathbf{x}} + h\vec{\mathbf{u}}) - F(\vec{\mathbf{x}})}{h}$$

and say $D_{\vec{\mathbf{u}}}(\vec{\mathbf{x}})$ is the *directional derivative in the direction* $\vec{\mathbf{u}}$ *at* $\vec{\mathbf{x}}$.

By formula (1), if F is differentiable,

$$D_{\vec{\mathbf{u}}}(\vec{\mathbf{x}}) = \nabla F \bullet \vec{\mathbf{u}}$$

We sometimes say that we want to find the directional derivative in the direction $\vec{\mathbf{a}}$ at $\vec{\mathbf{x}}$, even if $\vec{\mathbf{a}}$ is not a unit vector. By this we mean that we want to find $D_{\vec{\mathbf{u}}}(\vec{\mathbf{x}})$ for the unit vector that points in the same direction as $\vec{\mathbf{a}}$ which means $\vec{\mathbf{a}} \neq \vec{0}$ and then $\vec{\mathbf{u}} = \dfrac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}$.

There is really nothing new here as far as doing problems goes: to compute the directional derivative of F in the direction $\vec{\mathbf{a}}$ at $\vec{\mathbf{x}}$ simply compute

$$\frac{\nabla F(\vec{\mathbf{x}}) \bullet \vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}$$

An important question that we can now discuss is the question of in which direction at a point on a surface $F(x, y, z) = c$ does the surface look steepest?

Rephrase as for what $\vec{\mathbf{u}}$ is $\nabla F \bullet \vec{\mathbf{u}}$ as large as possible?

Well $\nabla F \bullet \vec{\mathbf{u}} = |\nabla F| \cdot |\vec{\mathbf{u}}| \cos(\theta)$ where θ is the angle between the two vectors. Since the cos assumes it's maximal value of 1 at $\theta = 0$, the direction of steepest increase is the direction of the gradient vector, as long as it is non-zero.

Likewise, the direction of steepest decrease points in the opposite direction of the gradient when $\theta = \pi$.

Find the directional derivative of $z = x^3 + y^3 - 2xy$ at the point $(1, 2)$ in the direction of $\langle 3, 4\rangle$.

$\nabla z = \left\langle 3x^2 - 2y, 3y^2 - 2x \right\rangle$ so $\nabla z(1, 2) = \langle -1, 10\rangle$ and $\vec{\mathbf{u}} = \dfrac{1}{\sqrt{3^2 + 4^2}}\langle 3, 4\rangle = \dfrac{1}{5}\langle 3, 4\rangle$.

Hence $D_{\vec{\mathbf{u}}}(1, 2) = \langle -1, 10\rangle \cdot \left(\dfrac{1}{5}\langle 3, 4\rangle\right) = \dfrac{37}{5}$

Find the maximal rate of change of $F(x, y, z) = x^3 + y^3 - 2xy - z$ at $(1, 2, 5)$ and it what direction does it occur?

$$\nabla F(x, y, z) = \left\langle 3x^2 - 2y, 3y^2 - 2x, -1 \right\rangle$$

so

$$\nabla F(1, 2, 5) = \langle -1, 10, -1\rangle$$

The maximal rate of change at this point is

$$|\nabla F(1, 2, 5)| = \sqrt{102}$$

and the direction is

$$\vec{\mathbf{u}} = \frac{1}{\sqrt{102}}\langle -1, 10, -1\rangle.$$