Goals for today

Tangent Planes to Surfaces

Let a surface be given by $F(x, y, z) = k$ and the (a, b, c) be a point on the surface so $F(a, b, c) = k$. Suppose $\vec{r}(t) = \langle x(t), y(t)z(t) \rangle$ is a smooth curve in $\mathbb{R}^3$ such that $\vec{r}(0) = \langle a, b, c \rangle$ and $F\big(\vec{r}(t)\big) = k$.

Chain Rule: $\nabla F = \langle F_x, F_y, F_z \rangle$, $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

At $t = 0$, $\nabla F(a, b, c) \boldsymbol{\cdot} \vec{r}'(0) = 0$ so the tangent vector to the curve is orthogonal to the gradient vector of F at the point. Since this holds for every smooth curve through (a, b, c) it seems natural to **define** the *tangent plane* to the surface at the point (a, b, c) to be the plane with normal vector $\nabla F(a, b, c)$ and point (a, b, c).

The *normal line* to the surface at the point is the line with point (a, b, c) and tangent vector $\nabla F(a, b, c)$.

Find the tangent plane to the surface $x^3 + xy^2 + z^3 = 4$ at the point $(1, 2, -1)$.

$\nabla F = \left\langle 3x^2 + y^2, 2xy, 3z^2 \right\rangle$ so a normal vector at $(1, 2, -1)$ is $\langle 7, 2, 3 \rangle$ so

$$7x + 2y + 3z = 8 \ .$$

The normal line at $(1, 2, -1)$ has an equation

$$\vec{r}(t) = \langle 1, 2, -1 \rangle + t \langle 7, 2, 3 \rangle \ .$$

Find a point on the surface $x^2 + xz + yz + y^2 = 14$ whose tangent plane is parallel to $7x + 5y + 3z = 6$.

$\nabla F = \langle 2x + z, z + 2y, x + y\rangle$. $\vec{N} = \langle 7, 5, 3\rangle$ $\nabla F(x, y, z) = k\,\langle 7, 5, 3\rangle$. $2x + z = 7k$; $2y + z = 5k$; $x + y = 3k$;

$2x - 2y = 2k$; $x - y = k$; $x + y = 3k$; $2x = 4k$; $x = 2k$; $y = k$; $z = 3k$.

$(2k)^2 + (2k)(3k) + k(3k) + k^2 = (4+6+3+1)k^2 = 14k^2 = 14$ so $k = \pm 1$.

One point is $(2, 1, 3)$, the other is $(-2, -1, -3)$.

$$\vec{N} \times \nabla F = \left\langle \begin{vmatrix} 5 & 3 \\ z+2y & x+y \end{vmatrix}, -\begin{vmatrix} 7 & 3 \\ 2x+z & x+y \end{vmatrix}, \begin{vmatrix} 7 & 5 \\ 2x+z & z+2y \end{vmatrix} \right\rangle$$

$$\Big\langle (5x+5y)-(3z+6y), -\Big((7x+7y)-(6x+3z)\Big), (7z+14y)-(10x+5z) \Big\rangle$$

$$\langle 5x-y-3z, -x-7y+3z, -10x+14y+2z \rangle = \langle 0,0,0 \rangle$$

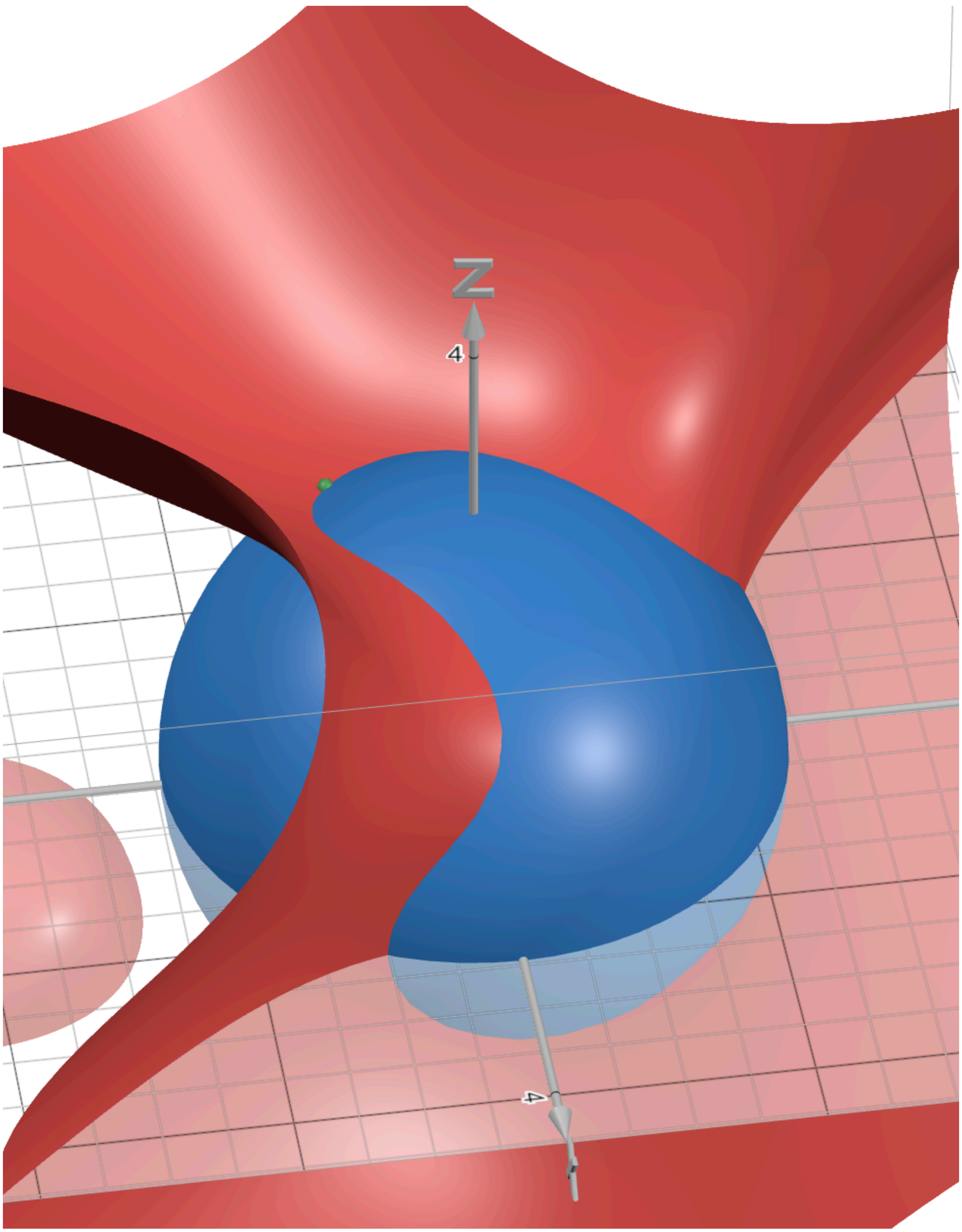
$$\langle 5x-y-3z, -x-7y+3z, -5x+7y+z \rangle = \langle 0,0,0 \rangle$$

$6y - 2z = 0$; $z = 3y$.

$5x - y - 3(3y) = 5x - 10y = 0$; $x = 2y$.

Hence $14 = x^2 + xz + yz + y^2 = (2y)^2 + (2y)(3y) + y(3y) + y^2 =$ $14y^2$ and finish as before.

Find an equation of the tangent line to the curve of intersection of the two surfaces $xyz^2 + x^3 + y^3 + z^3 = 10$ and $x^2 + y^2 + z^2 = 6$ at the point $(-1, -1, 2)$.



A tangent line to the intersection curve at a point is the intersection of the tangent planes at the point.

To find a vector for the line which is the intersection of two planes, take the cross product of the normal vectors to the two planes.

A normal vector to the tangent plane of a surface at a point is the gradient at that point.

$F(x, y, z) = xyz^2 + x^3 + y^3 + z^3 = 10$, $G(x, y, z) = x^2 + y^2 + z^2 = 6$ and the point is $(-1, -1, 2)$.

$$\nabla F = \left\langle yz^2 + 3x^2, xz^2 + 3y^2, 2xyz + 3z^2 \right\rangle; \nabla G = \langle 2x, 2y, 2z \rangle$$

.

$$\nabla F(-1, -1, 2) = \langle -1, -1, 16 \rangle;$$

$$\nabla G(-1, -1, 2) = \langle -2, -2, 4 \rangle$$

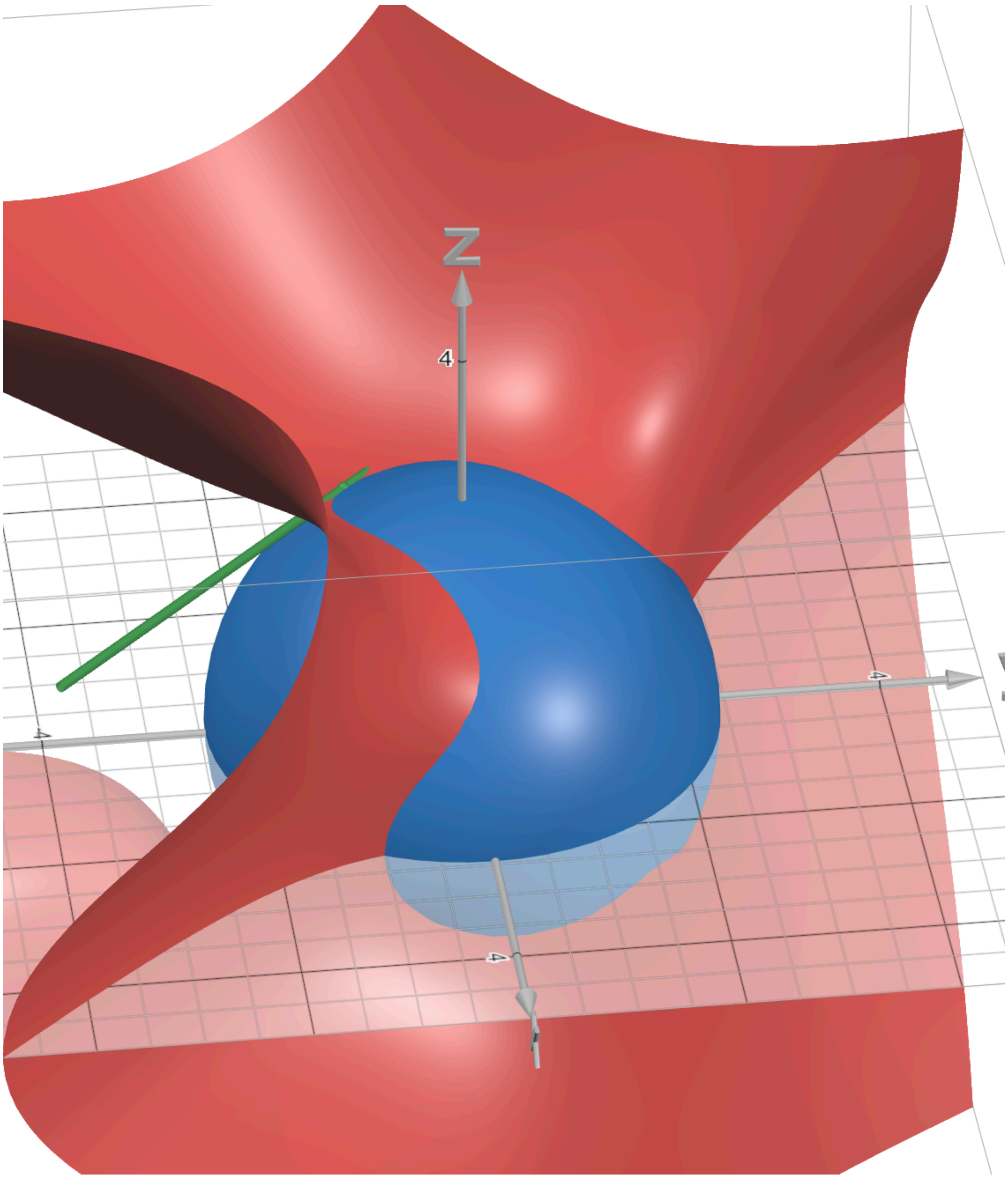
$$\nabla F(-1, -1, 2) \times \nabla G(-1, -1, 2) =$$

$$\left\langle \begin{vmatrix} -1 & 16 \\ -2 & 4 \end{vmatrix}, -\begin{vmatrix} -1 & 16 \\ -2 & 4 \end{vmatrix}, \begin{vmatrix} -1 & -1 \\ -2 & -2 \end{vmatrix} \right\rangle =$$

$$\langle 32 - 4, -(32 - 4), 0 \rangle = \langle 28, -28, 0 \rangle \ .$$

Hence we may use $\langle 1, -1, 0 \rangle$.

Hence the tangent line to the intersection curve is given by $\vec{r}(t) = \langle -1, -1, 2 \rangle + t \langle 1, -1, 0 \rangle$.



What does it mean for two surfaces to be tangent at a point?

Let's begin by agreeing that a surface and its tangent plane at a point are tangent at the point.

Let us also agree that if three surfaces $F(x, y, z) = k_F$, $G(x, y, z) = k_G$ and $H(x, y, a) = k_G$ all meet at a point (a, b, c) and if F is tangent to G and if G is tangent to H, then F is tangent to H.

It follows that if $F(x, y, z) = k_F$, $G(x, y, z) = k_G$ meet at a point (a, b, c), then F is tangent to G at that point if and only if their tangent planes are equal. This is equivalent to their normal vectors being parallel.

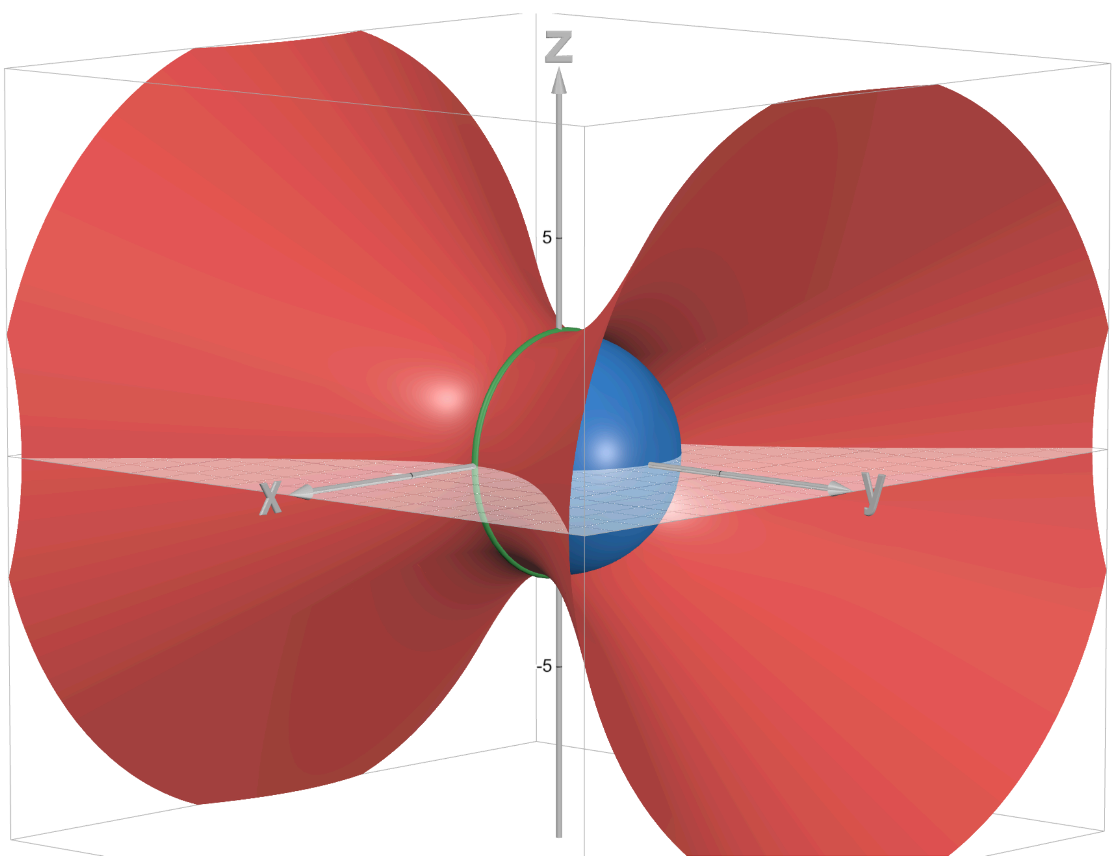
The Lecture Notes 13 works out an example.

Here is a second example. Let

$F(x, y, z)$ be the hyperboloid $x^2 - y^2 + z^2 = 8$ and let $G(x, y, z)$ be the sphere $x^2 + y^2 + z^2 = 8$.

In this example we can actually parametrize the intersection curve $\vec{r}(t) = \left\langle \sqrt{8}\cos(t), 0, \sqrt{8}\sin(t) \right\rangle$ for $0 \leqslant t \leqslant 2\pi$.

Then $\vec{r}'(t) = \left\langle -\sqrt{8}\sin(t), 0, \sqrt{8}\cos(t) \right\rangle$.



OR

$\nabla F = \langle 2x, -2y, 2z\rangle$; $\nabla G = \langle 2x, 2y, 2z\rangle$

$$\nabla F \times \nabla G = \left\langle \begin{vmatrix} -2y & 2z \\ 2y & 2z \end{vmatrix}, -\begin{vmatrix} 2x & 2z \\ 2x & 2z \end{vmatrix}, \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} \right\rangle =$$

$\langle -8yz, 0, 8xy\rangle$

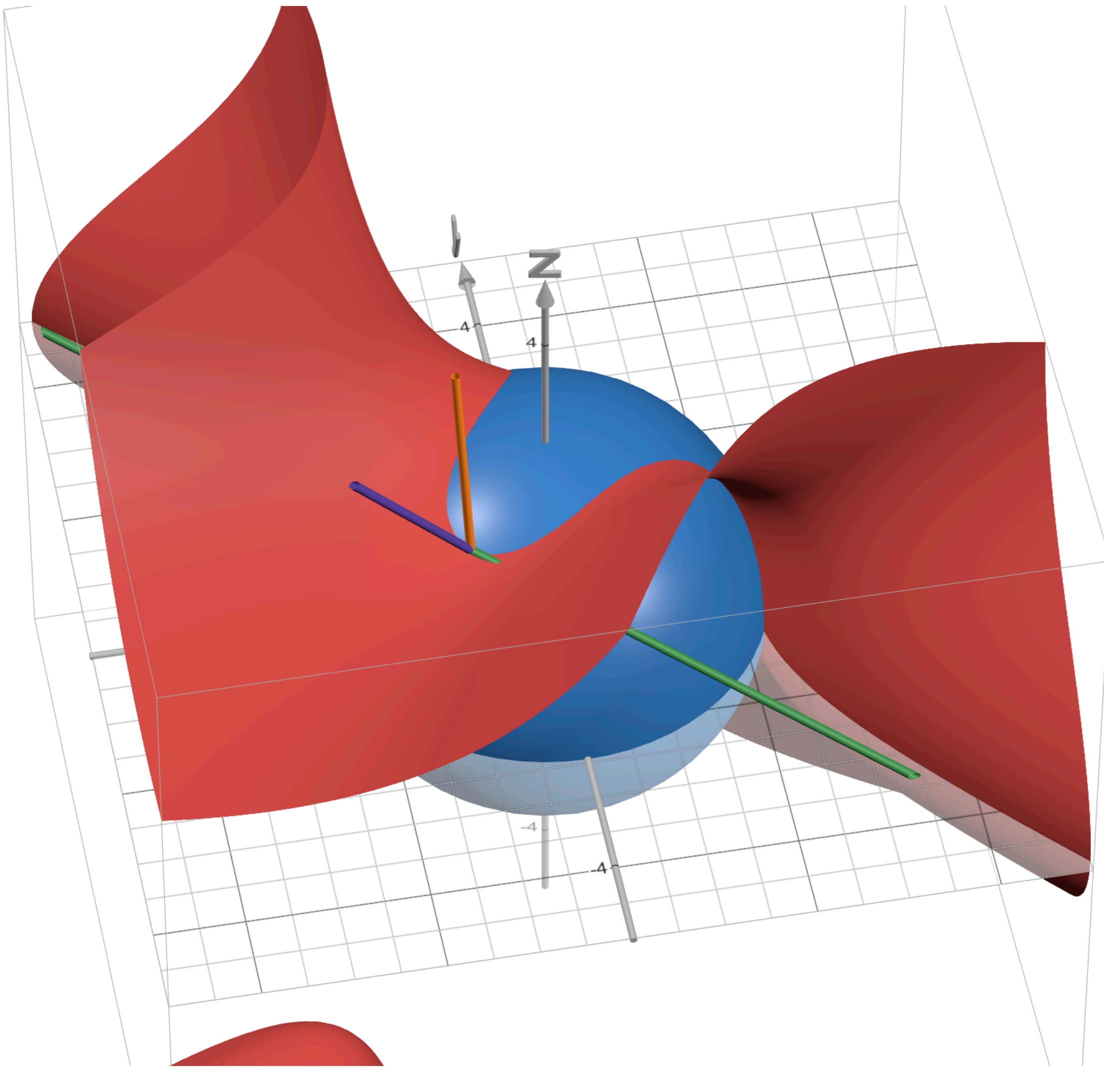
When $y = 0$ we see

$\nabla F \times \nabla G = \langle 0, 0, 0\rangle$ so ∇F and ∇G are parallel.

So $x^2 - y^2 + z^2 = 8$ and $x^2 + y^2 + z^2 = 8$ are tangent at every point along the intersection curve.

This example also shows that occasionally the cross product is the zero vector and then we don't get a tangent vector, but most of the time we will.

The first intersection example we did is the generic one. Here is the surface with the tangent line to the curve and the normal lines to the two surfaces.

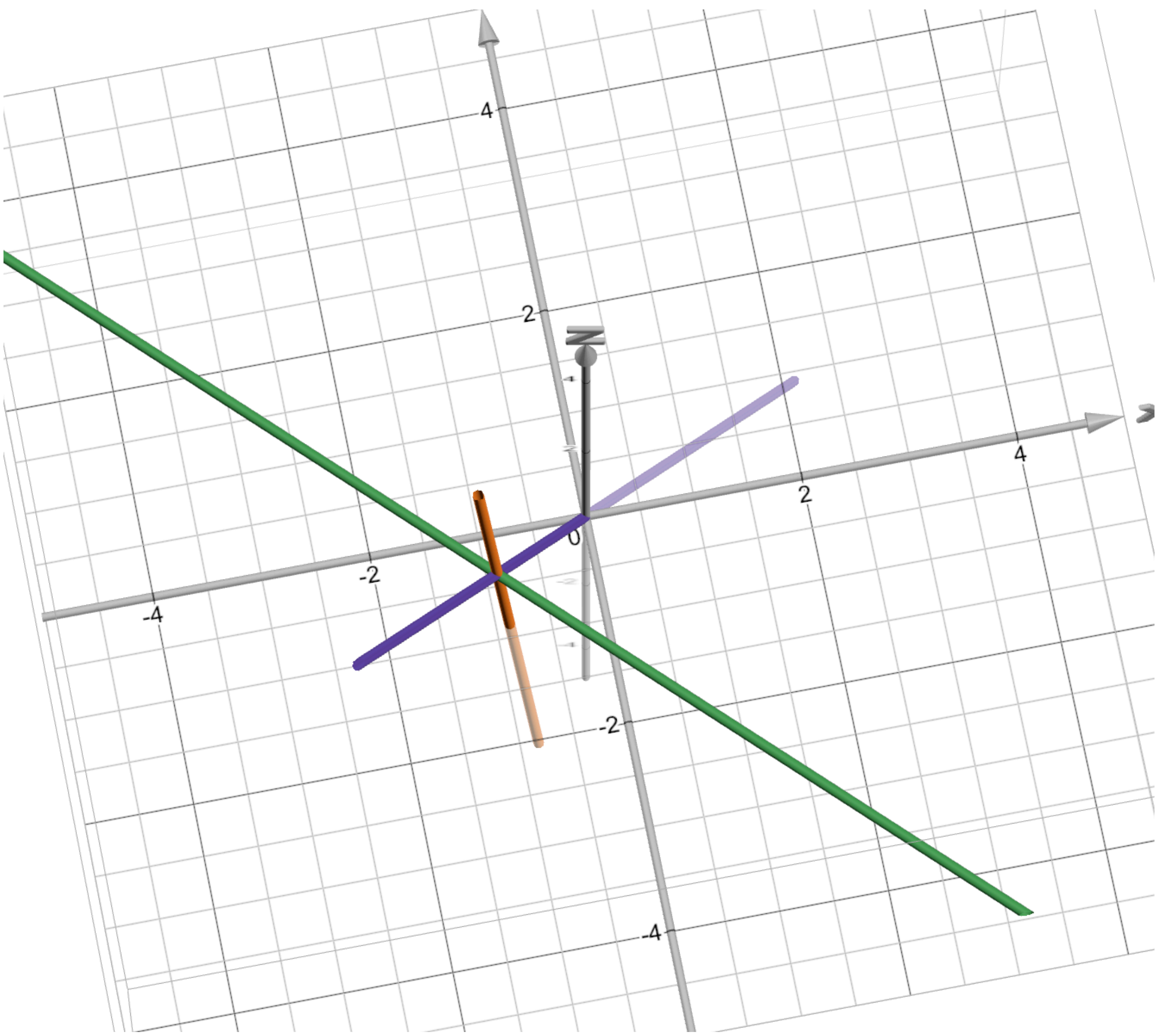


Here is the picture of just the lines.

Green is the tangent line to the intersection curve;

Orange is the normal line to $xyz^2 + x^3 + y^3 + z^3 = 10$;

Purple is the normal line to the sphere $x^2 + y^2 + z^2 = 6$.



Angles between intersetions.

Two surfaces may intersect; a line and a surface may intersect and two lines may intersect.

At any point of intersection we define the angle between the two objects as the angle between two vectors:

- Surfaces F and G intersect at (a, b, c) - angle between surfaces is angle between $\nabla F(a, b, c)$ and $\nabla G(a, b, c)$.
- Surface F and curve $\vec{r}(t)$ intersect at (a, b, c) and t_0 - angle between surface and curve is angle between $\nabla F(a, b, c)$ and $\vec{r}\,'(t_0)$.
- Two curves $\vec{r}_1(t)$ and $\vec{r}_2(s)$ intersect at t_0 and s_0 - angle between curves is angle between $\vec{r}_1'(t_0)$ and $\vec{r}_2'(s_0)$.

The angle between a surface and a curve lying in the surface is 90°.