Goals for today

Lagrange Multipliers with 1 constraint

Maximize/minimize the value of a function F subject to an implicit constraint $g = k$.

Maximize $F(x,y) = xy$ subject to $x + y = 5$. (Find the maximum area you can fence in with a rectangular fence given that you have 5 furlongs of fence.)

1st Semester Calculus: $y = 5 - x$, $F = x(5-x) = 5x - x^2$. $\frac{dF}{dx} = 5 - 2x$; solve $5 - 2x = 0$; $x = \frac{5}{2}$; $y = 5 - \frac{5}{2}$; $F\left(\frac{5}{2}, \frac{5}{2}\right) = \frac{25}{4}$ square furlongs.

Figure 1. Lagrange multipliers from the book

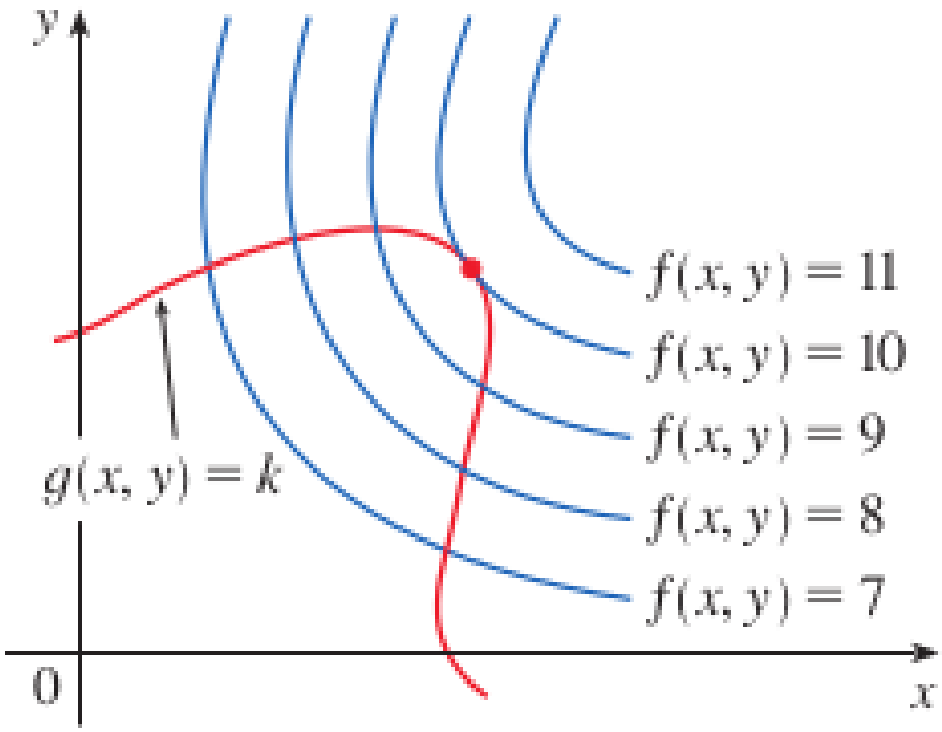
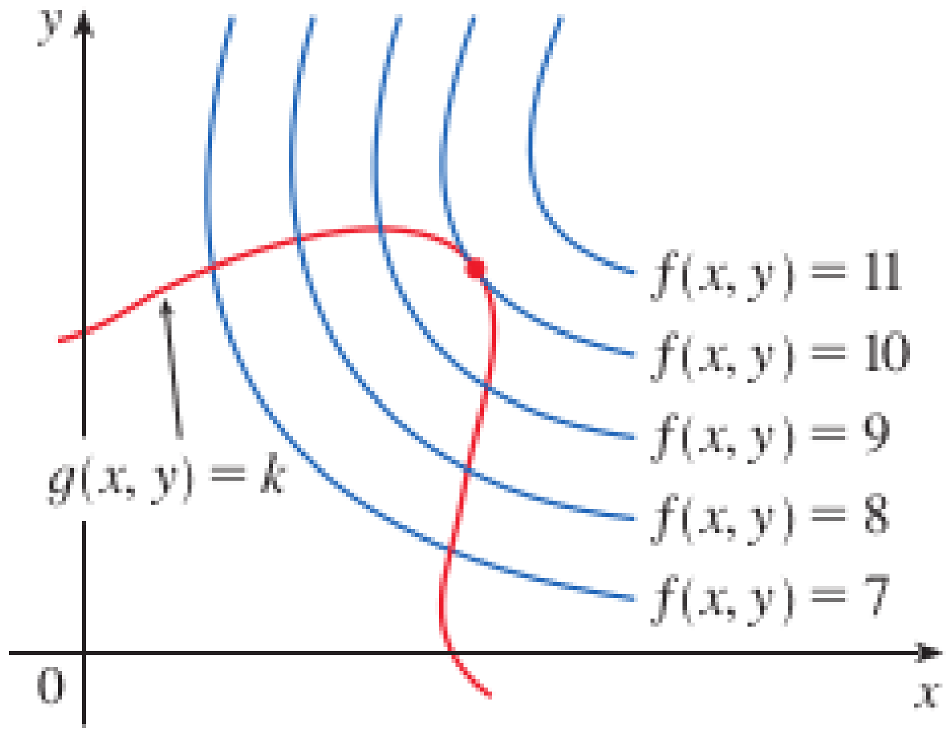


FIGURE 2. Lagrange multipliers from the book

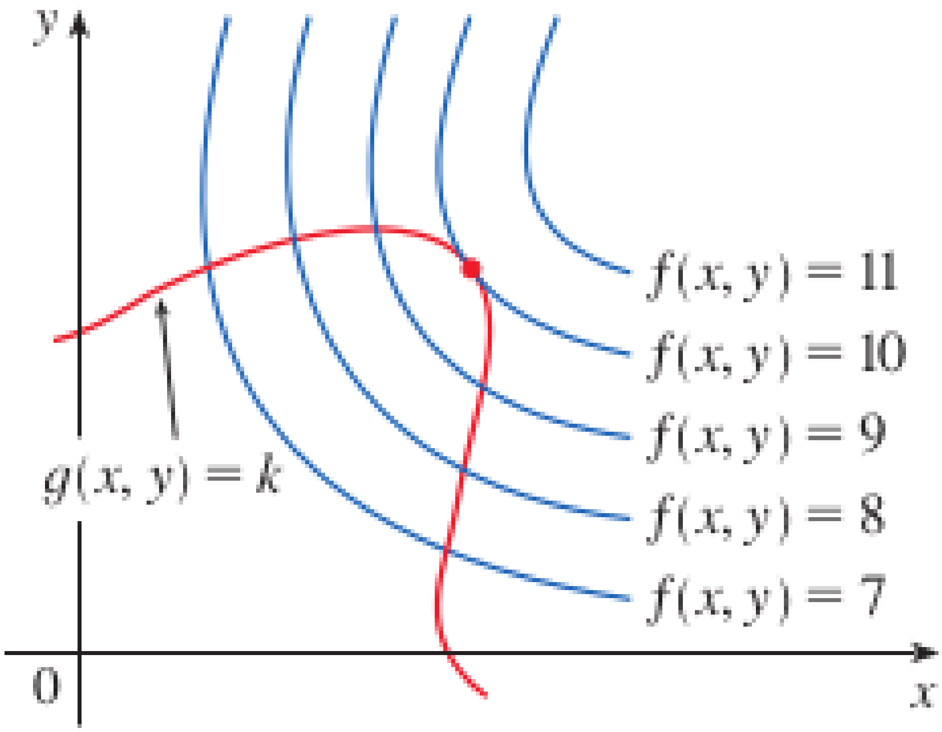


The red dot satisfies the equations

$$\nabla f = \lambda \nabla g$$

$$g(x, y) = k$$

FIGURE 3. Lagrange multipliers from the book



The red dot satisfies the equations

$$f_x = \lambda g_x \tag{1}$$

$$f_y = \lambda g_y \tag{2}$$

$$g(x, y) = k \tag{3}$$

Maximize $F(x,y) = xy$ subject to $x + y = 5$. (Find the maximum area you can fence in with a rectangular fence given that you have 5 furlongs of fence.)

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3rd Semester Calculus: $\nabla F = \langle y, x \rangle$; $g(x,y) = x + y$; $\nabla g = \langle 1, 1 \rangle$. Then $\lambda = x$; $\lambda = y$ and $g(x,y) = x + y = 5$.

Why did we find the maximum rather than the minimum? Or even just a local extrema? What happened to Fermat's theorem?

Find the maximum and minimum values of $5x^2 - 3y$ restricted to $x^2 + y^2 = 1$.

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$F(x, y) = 5x^2 - 3y$; $g(x, y) = x^2 + y^2$; $\nabla F = \langle 10x, -3 \rangle$; $\nabla g = \langle 2x, 2y \rangle$.

Equations: $10x = \lambda 2x$; $-3 = \lambda 2y$; $x^2 + y^2 = 1$.

$10x = \lambda 2x$ is satisfied whenever $x = 0$ OR $2\lambda = 10$.

If $x = 0$, $y = \pm 1$ and $\lambda = \mp \frac{3}{2}$.

If $2\lambda = 10$ and $-3 = 10y$, $y = -0.3$ and $x = \pm \frac{\sqrt{91}}{10}$.

Hence the points and values are $F(0, 1) = -3$; $F(0, -1) = 3$;
$F\left(\pm\sqrt{91}/10, -0.3\right) = 5 \cdot 91/100 + 0.9 = 5.45$

The maximum value is 5.45 at is is assumed at $\left(\pm\sqrt{91}/10, -0.3\right)$.
The minimum value is -3 and it occurs at $(0, 1)$.

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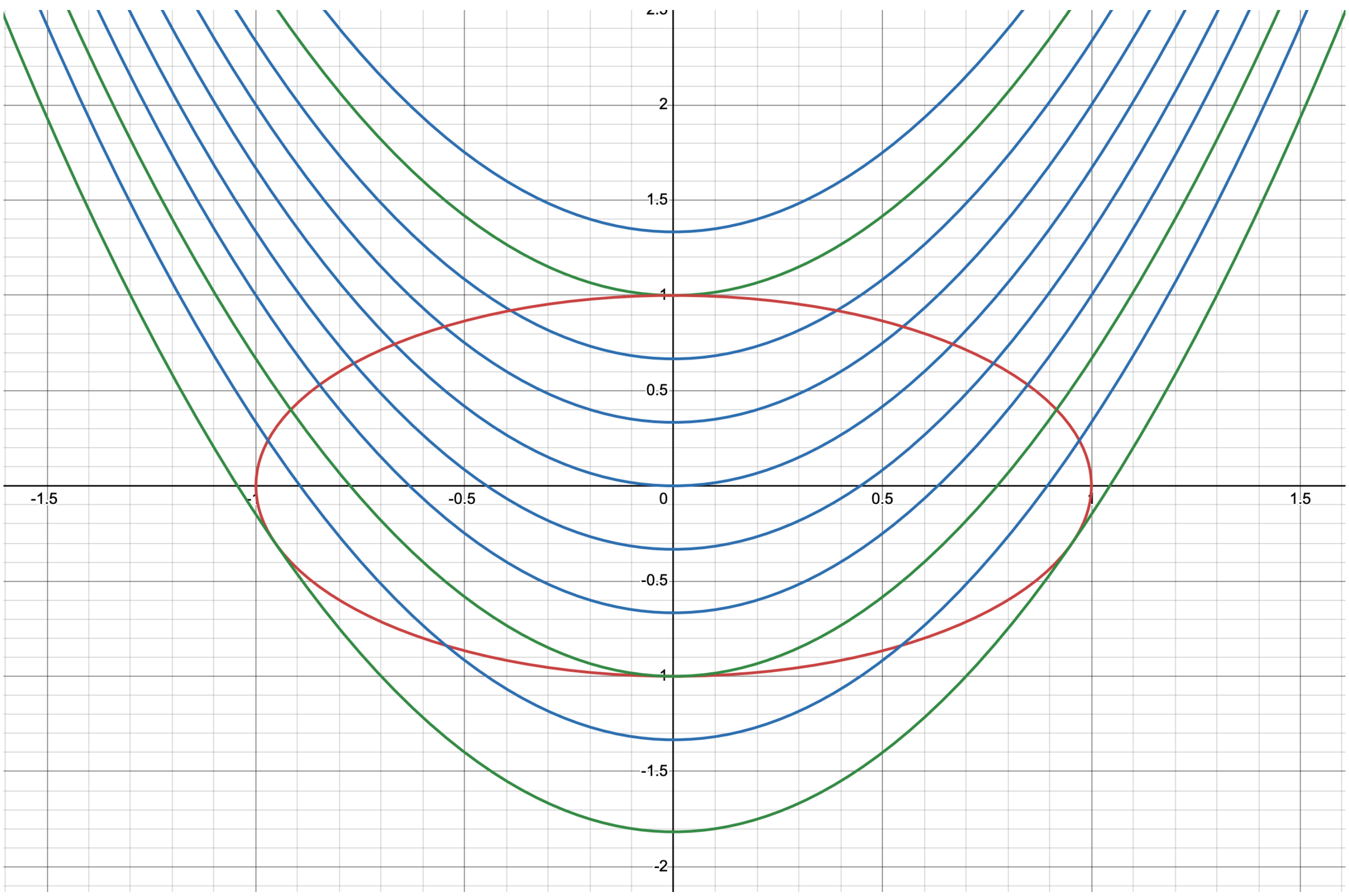


The constraint equation is closed and bounded. Implicit curves are closed if $g(x,y)$ is continuous. They may or may not be bounded but in this case $g = 1$ is obviously bounded. Hence Fermat says we have an absolute minimal value and an absolute maximal value.

Find the maximum and minimum values of $5x^2 - 3y$ restricted to $x + y = 1$.

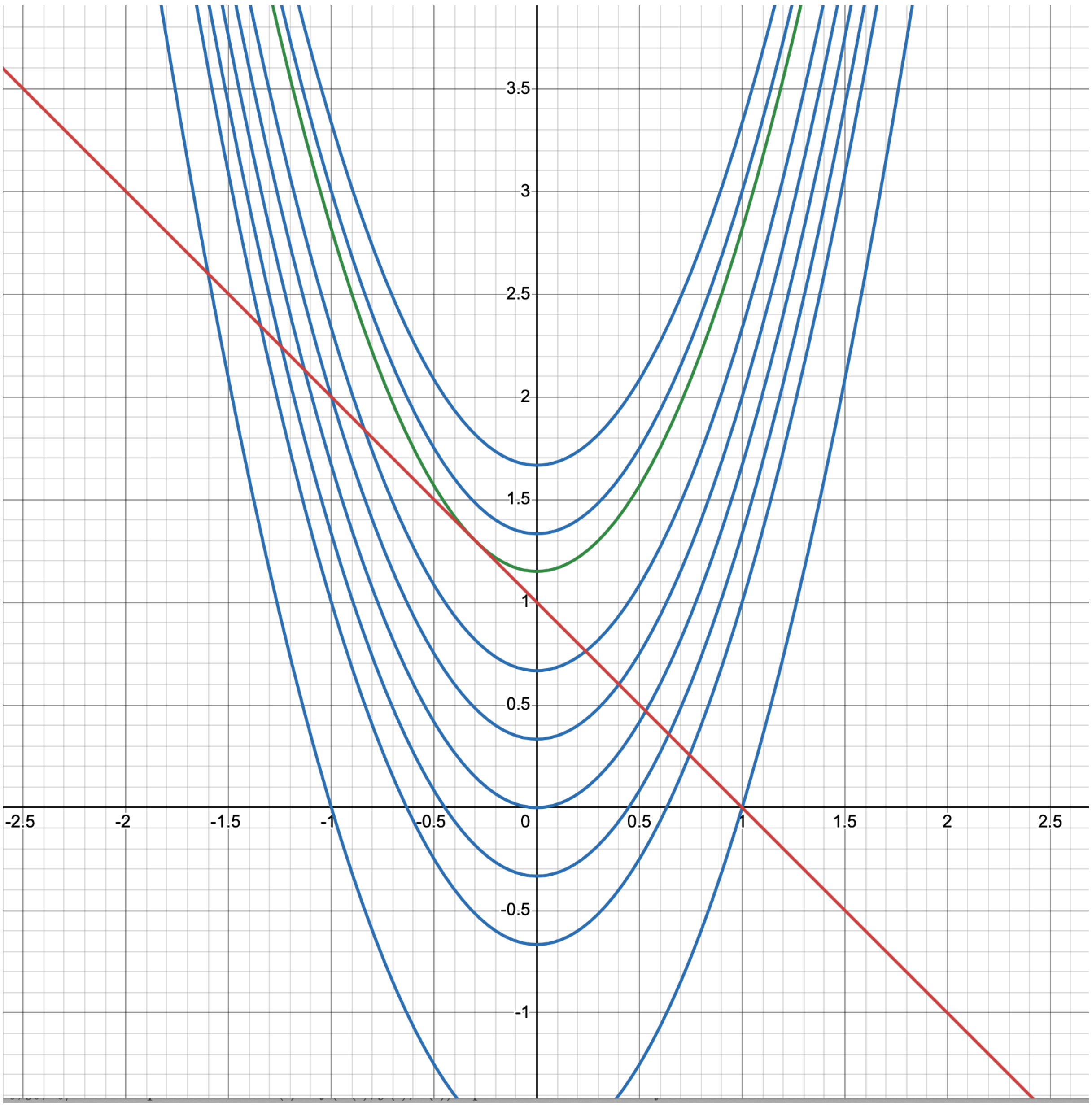
$F(x, y) = 5x^2 - 3y$; $g(x, y) = x + y$; $\nabla F = \langle 10x, -3 \rangle$; $\nabla g = \langle 1, 1 \rangle$.

Equations: $10x = \lambda$; $-3 = \lambda$; $x + y = 1$.

$\lambda = 10x$; $-3 = 10x$; $x = -0.3$; $y = 1.3$.
$F(-0.3, 1.3) = -3.45$

Find the maximum and minimum values of $5x^2 - 3y$ restricted to $x + y = 1$.

$F(-0.3, 1.3) = -3.45$ is a solution to the equations but is neither a minimum nor a maximum.



However the constraint equation in this case is not bounded, whereas it was in the previous example.