Goals for today

Lagrange Multipliers with 2 constraints

Maximize/minimize the value of a function

$$F(x, y, z)$$

subject to an implicit constraint $g(x, y, z) = k$

and an implicit constraint $h(x, y, z) = c$.

The two constraint surfaces intersect in a curve and we want to find all the points on that curve for which $F(x, y, z)$ is a maximum and all the points on the curve for which $F(x, y, z)$ is a minimum.

Pretend we have parametrized the intersection curve with $\vec{r}(t_0)$ being point which is an extrema. Since $\vec{r}(t)$ lies in $g(x, y, z) = k$, $\vec{r}'(t)$ lies in the tangent plane to g at the point and hence is orthogonal to $\nabla g = k$ at the point. A similar discussion shows $\vec{r}'(t)$ is orthogonal to ∇h.

Now look at a point where a level surface $F(x, y, z) = \ell$ intersects the curve. Recall ∇F is the direction in which the value of F is increasing the fastest. So if $\vec{r}'(t_0)$ is not orthogonal to ∇F, moving one way along the curve will increase the value of F whereas moving the other will decrease it. Hence you are not at a local max or a local min. Therefore, if you are at a local extrema, $\vec{r}'(t_0)$ will be orthogonal to ∇F.

We have now produced three vectors in the normal plane to the curve at our point, ∇g, ∇h and ∇F. We assume ∇g and ∇h are not parallel since if they are the tangent to the curve is not well defined.

But then we can write any vector in the plane uniquely as a multiple of ∇g plus a multiple of ∇h:

$$\nabla F = \lambda \nabla g + \mu \nabla h$$

Hence the two constraint version of Lagrange multipliers says solve the following system of equations in five unknowns

$$\begin{aligned}\nabla F(x,y,z) &= \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) &= k \\ h(x,y,z) &= c\end{aligned}$$

OR

$$\begin{aligned}
&(1) & F_x(x,y,z) &= \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\
&(2) & F_y(x,y,z) &= \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\
&(3) & F_z(x,y,z) &= \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\
&(4) & g(x,y,z) &= k \\
&(5) & h(x,y,z) &= c
\end{aligned}$$

Be sure to check that the intersection of your constraints is bounded or be prepared to think harder about what your answer means.

1. $F(x,y,z) = xy + z^2$, $g(x,y,z) = y - x = 0$, $h(x,y,z) = x^2 + y^2 + z^2 = 1$.

2. $F(x,y,z) = 5x^2 - 4z$; $g(x,y,z) = 3x - 5z = 10$; $h(x,y,z) = x^2 + y^2 = 25$.

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$$\langle 10x, 0, -4\rangle = \lambda\langle 3, 0, -5\rangle + \mu\langle 2x, 2y, 0\rangle$$

$\lambda = \dfrac{4}{5} = 0.8$; $0 = 2y\mu$ so

$y = 0$ OR $\mu = 0$.

$y = 0$: $x = \pm 5$; $5z = 3x - 10$ so $(5, 0, 1)$ and $(-5, 0, -5)$.

$\mu = 0$: $10x = 3\lambda = \dfrac{12}{5} = 2.4$; $x = \dfrac{12}{50} = \dfrac{6}{25} = 0.28$.

$5z = 3x - 10$; $z = \dfrac{3}{5}x - 2 = \dfrac{3}{5}\cdot\dfrac{6}{25} - \dfrac{250}{125} = \dfrac{18 - 250}{125} = -\dfrac{232}{125}$.

$\left(\dfrac{6}{25}\right)^2 + y^2 = \dfrac{25^2}{25}$ so $y^2 = \dfrac{25^2 - 6^2}{25^2} = \dfrac{589}{25^2}$ so $y = \pm\dfrac{\sqrt{589}}{25}$

2. $F(x, y, z) = 5x^2 - 4z$.

Critical points are:

$$(5, 0, 1); \ (-5, 0, -5); \ \left(\frac{6}{25}, \frac{\sqrt{589}}{25}, -\frac{232}{125}\right) \text{ and } \left(\frac{6}{25}, -\frac{\sqrt{589}}{25}, -\frac{232}{125}\right)$$

$$F(5, 0, 1) = 125 - 4 = 121$$

$$F(5, 0, -5) = 125 + 20 = 145$$

$$F\left(\frac{6}{25}, \frac{\sqrt{589}}{25}, -\frac{232}{125}\right) = 5 \cdot \frac{36}{25^2} + 4 \cdot \frac{232}{125} = \frac{928 + 36}{125} = \frac{964}{125}.$$

$$F\left(\frac{6}{25}, \frac{\sqrt{589}}{25}, \frac{232}{125}\right) = 5 \cdot \frac{36}{25^2} - 4 \cdot \frac{232}{125} = -\frac{928 - 36}{125} = -\frac{892}{125}.$$

$$-\frac{892}{125} < \frac{964}{125} < 8 < 121 < 145$$

3. $F(x,y,z) = x$; $g(x,y,z) = x + y - z = 0$; $h(x,y) = x^2 + 2y^2 + 2z^2 = 8$.

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$\nabla F = \langle 1, 0, 0 \rangle$; $\nabla g = \langle 1, 1, -1 \rangle$; $\nabla h = \langle 2x, 4y, 4z \rangle$.

$1 = \lambda + 2x\mu$; $0 = \lambda + 4y\mu$; $0 = -\lambda + 4z\mu$.

$4x\mu = 2 - 2\lambda$; $4y\mu = -\lambda$; $4z\mu = \lambda$;

$0 = 4x + 4y - 4z = 2 - 4\lambda$; $\lambda = \dfrac{1}{2}$.

$0 = 4y\mu + 4z\mu$ so $\mu = 0$ or $y + z = 0$.

If $\mu = 0$, $\lambda = 0$ which is false so we must have $y = -z$. It then follows that $x = 2z$.

If $x = 2z$, $y = -z$ we have $x^2 + 2y^2 + 2z^2 = 8z^2 = 8$ so $z = \pm 1$.

$(2, -1, 1)$ and $(-2, 1, -1$ are the critical points.

4. $F(x, y, z) = 4 - z$; $g(x, y, z) = x^2 + y^2 = 8$; $h(x, y) = x + y + z = 1$.

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$\nabla F = \langle 0, 0, -1 \rangle$; $\nabla g = \langle 2x, 2y, 0 \rangle$; $\nabla h = \langle 1, 1, 1 \rangle$.

$0 = 2x\lambda + \mu$; $0 = 2y\lambda + \mu$; $-1 = \mu$.

$2x\lambda = 1$; $2y\lambda = 1$; $\lambda \neq 0$ and $x = y$.

$2x^2 = 8$; $x = y = \pm 2$; $z = 1 - x - y = 1 - 2x$.

$(2, 2, -3)$ $(-2, -2, 5)$

5. $F(x, y, z) = x^2 + 2y^2 + z^2$; $g(x, y, z) = x + y + z = 4$; $h(x, y, z) = x - y + 2z = 12$.

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$\nabla F = \langle 2x, 4y, 2z \rangle$; $\nabla g = \langle 1, 1, 1 \rangle$; $\nabla h = \langle 1, -1, 2 \rangle$.

$2x = \lambda + \mu$; $2y = \lambda - \mu$; $2z = \lambda + 2\mu$.

$2x + 2y + 2z = 3\lambda + 2\mu = 8$

$2x - 2y + 4z = 2\lambda + 6\mu = 24$; $\lambda + 3\mu = 12$.

$3\lambda + 9\mu = 36$; $7\mu = 36 - 8 = 28$; $\mu = 4$; $\lambda = 0$.

$2x = 4$, $2y = -4$; $2z = 8$; so $(2, -2, 4)$ is the only critical point.

5. $G(x,y) = x^2 + 2y^2 + (4-x-y)^2$;
$g(x,y) = x-y+2(4-x-y) = 8-x-3y = 12$; $x+3y = -4$.

$\nabla G = \langle 2x + 2(4-x-y)(-1), 4y + 2(4-x-y)(-1)\rangle = \langle 4x+2y-8, 2x+6y-8\rangle$;
$\nabla g = \langle 1, 3\rangle$.

$4x + 2y - 8 = \lambda$; $2x + 6y - 8 = 3\lambda$.

$12x + 6y - 24 = 3\lambda = 2x + 6y - 8$; $10x = 16$; $5x + y = 8$.

$(2, -2)$.
$G(x,y) = x^2 + 2xy + y^2 - 8x - 8y + 16 + x^2 + 2y^2 = 2x^2 + 3y^2 + 2xy - 8x - 8y + 16$