Goals for today

Line integrals and conservative fields

Recall the line integral of a vector field $\vec{F}$ along a curve $\vec{r}(t)$, $t \in [a, b]$ is given by

$$\int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}\left(\vec{r}(t)\right) \cdot \vec{r}'(t) dt$$

Now suppose $\vec{F} = \nabla f$ so in this special case, the line integral is

$$\int_a^b \nabla f\left(\vec{r}(t)\right) \cdot \vec{r}'(t) dt = \int_a^b g(t) dt$$

where this last integral is just a first year calculus integral.

Now let $h(t) = f\left(\vec{r}(t)\right)$ and use the chain rule to compute

$$h'(t) = \nabla f\left(\vec{r}(t)\right) \cdot \vec{r}'(t)$$

so the line integral becomes

$$\int_a^b \nabla f\left(\vec{r}(t)\right) \cdot \vec{r}'(t) dt = \int_a^b h'(t) dt = h(b) - h(a)$$

and so if f is a potential function

$$\int_C \nabla f \cdot \vec{T} ds = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

Recall example from last time: $\int_C \langle y, x \rangle \cdot \vec{T} ds$ along 3 sides of a triangle.

A necessary condition for $\langle P, Q\rangle$ to be conservative.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

A necessary condition for $\langle P, Q, R\rangle$ to be conservative.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Not sufficient

$$\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

Necessary and sufficient

If $\int_C \vec{F} \bullet \vec{T} ds$ only depends on the beginning and end points of the path for all paths then the field is a gradient field.

If $\int_C \vec{F} \bullet \vec{T} ds = 0$ for all closed paths then the field is a gradient field. Sometimes you will see $\oint_C \vec{F} \bullet \vec{T} ds$ for $\int_C \vec{F} \bullet \vec{T} ds$ just to emphasize that the path is closed.

If the necessary conditions above are satisfied in a *simply connected* region D then the field is a gradient field.

Find a potential for a vector field

$$\langle y\cos(xy)+y, x\cos(xy)+x\rangle$$

$$\left\langle 2xe^{xy}+x^2ye^{xy}, x^3e^{xy}+2y\right\rangle$$

$$\left\langle 2xy^3z^5, 3x^2y^2z^5, 5x^2y^3z^5+3z^2\right\rangle$$

$$\left\langle -\frac{x}{(\sqrt{x^2+y^2+z^2}\,)^3}, -\frac{y}{(\sqrt{x^2+y^2+z^2}\,)^3}, -\frac{z}{(\sqrt{x^2+y^2+z^2}\,)^3}\right\rangle$$

$$\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$$