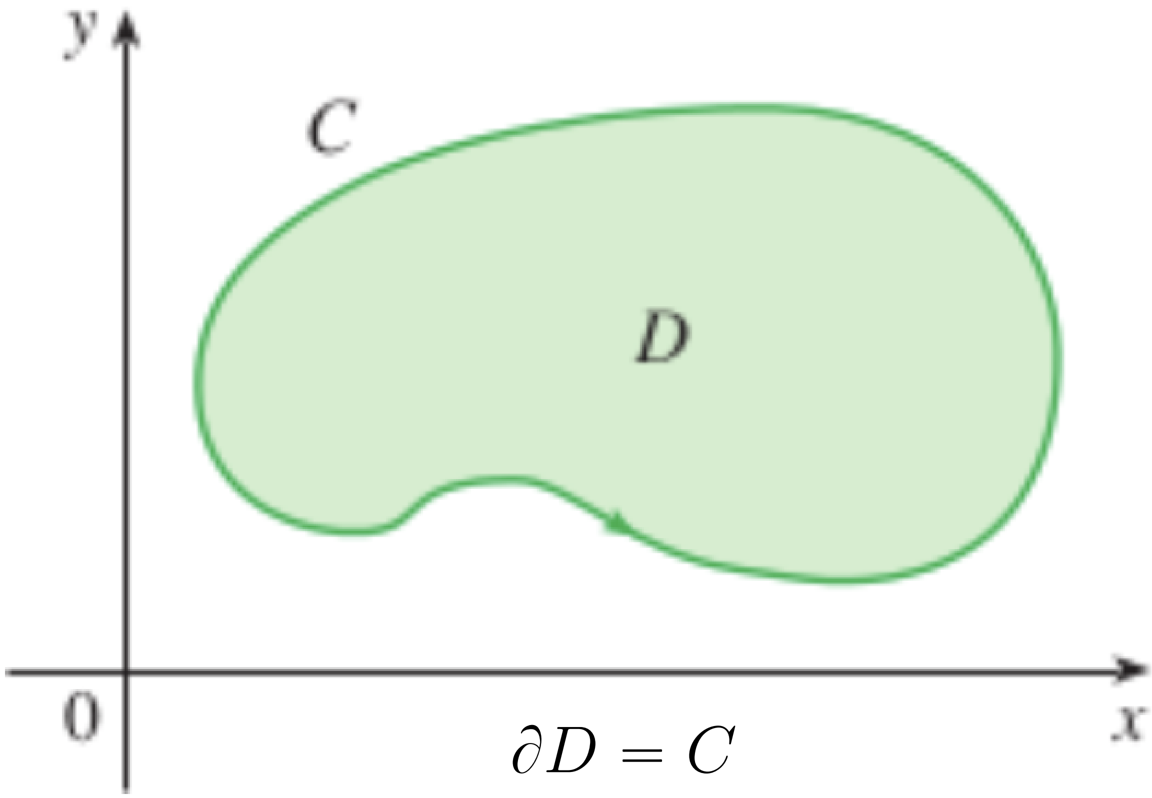
Goals for today

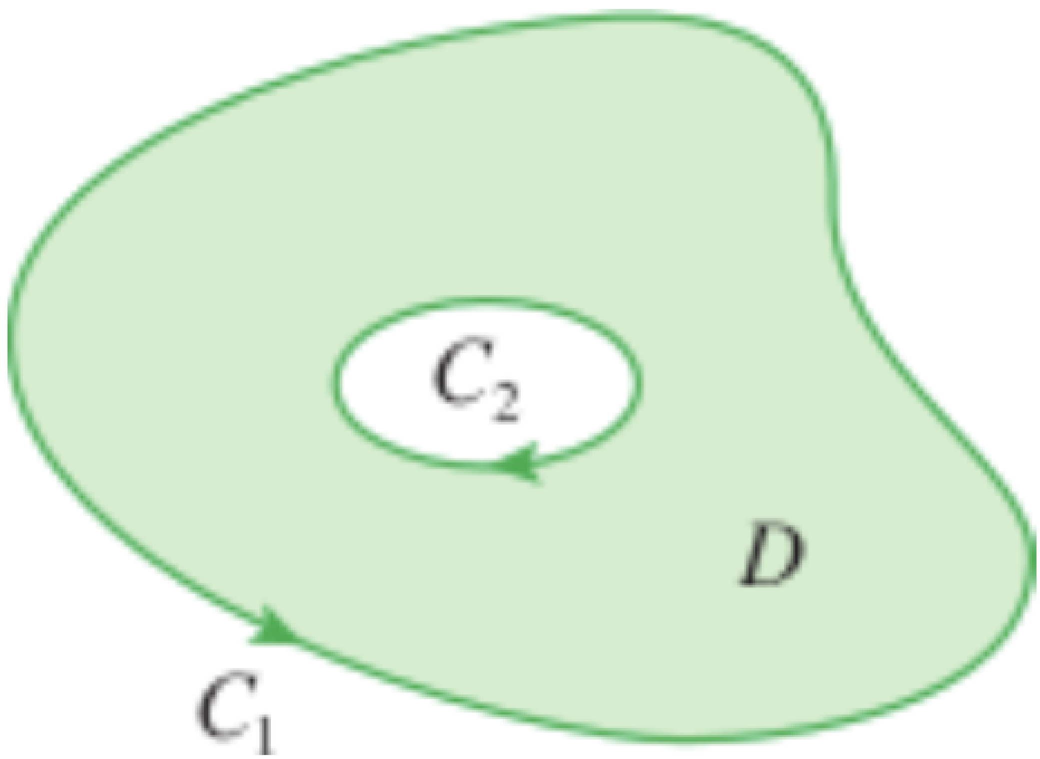
Green's Theorem

Let D be a region in the plane and let C be the curve which is the boundary of D. We will need an orientation on the curve which is the boundary of D to state Green's Theorem.

The *correct* orientation at a point on the boundary of D, is found by walking along CD in the direction which keeps D on your left. We will use ∂D to denote C with the correct orientation.

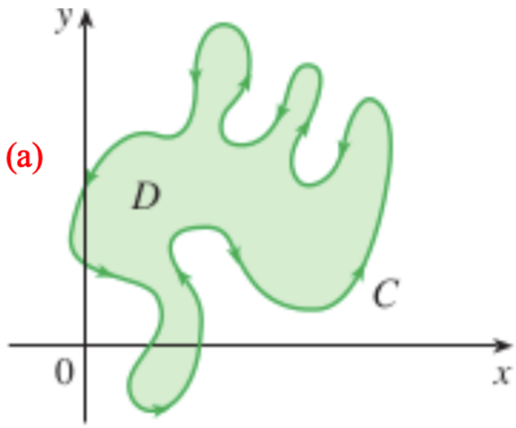


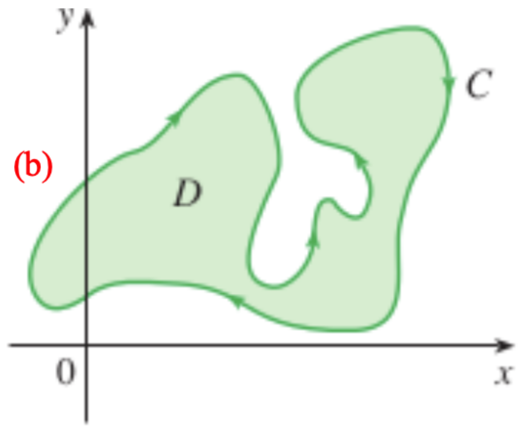
$\partial D = C$



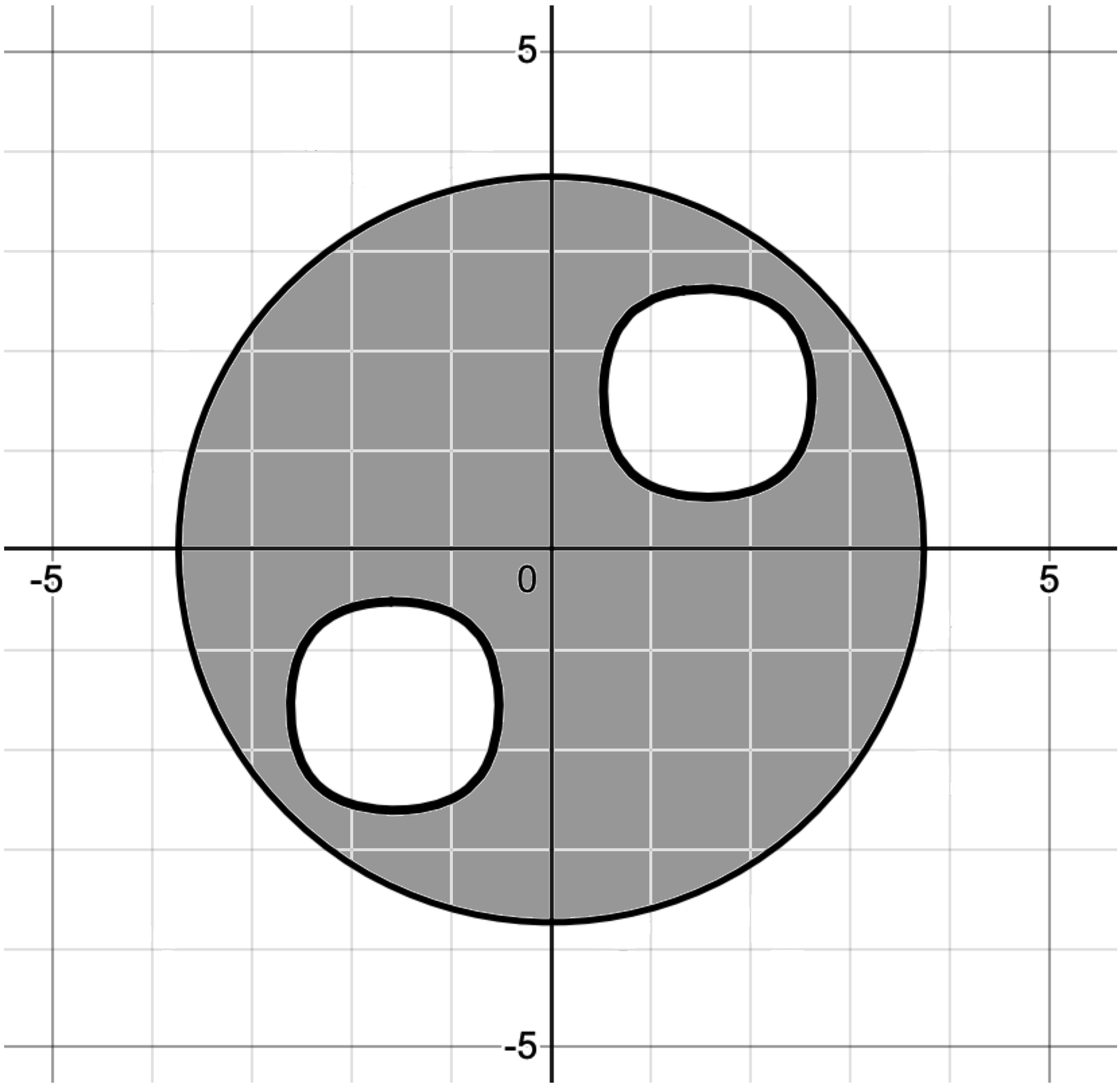
$\partial D = C_1 \amalg C_2$

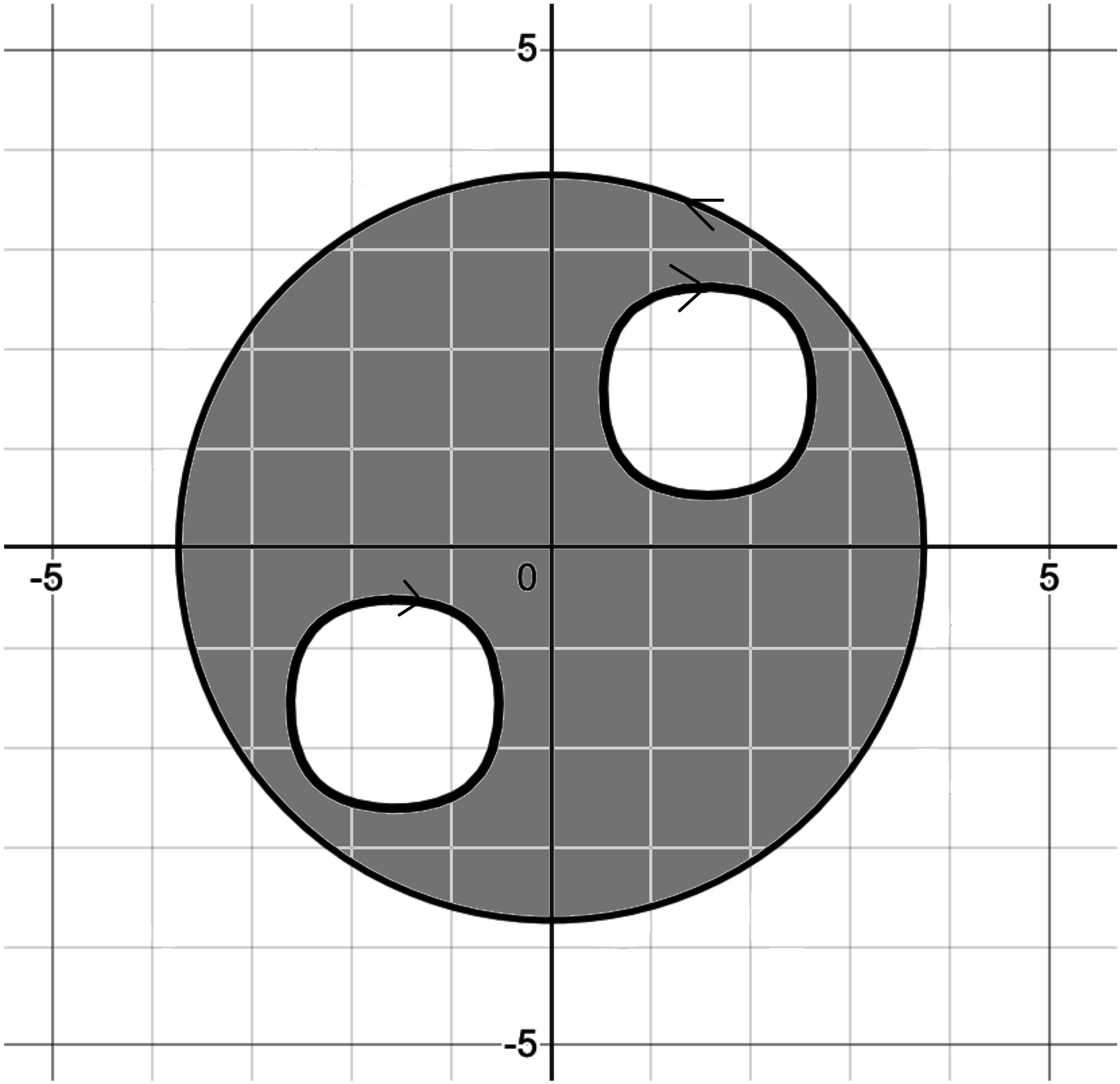
Which of (a) or (b) is the boundary of D with the correct orientation?





Work out the correct orientation for ∂D in the shaded region.





5
-5
0
5
-5

Green's Theorem

Let D be a region in the plane with ∂D a correctly oriented, piecewise smooth curve with field $\langle P, Q \rangle$ which has continuous first partials on a region containing D. Then

$$\int_{\partial D} P dx + Q dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Given $F(x,y)$, if we can find functions P and Q such that

$$(*) \qquad F(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

then

$$\iint_D F(x,y)dA = \int_{\partial D} Pdx + Qdy$$

It is usually difficult to do this but there are cases where it can be done.

Area: $F(x,y) = 1$, $\frac{1}{2}\langle -y, x\rangle$. Area of a region D,

$$Area = \frac{1}{2}\int_{\partial D} \langle -y, x\rangle \cdot d\vec{r} = -\int_{\partial D} ydx = \int_{\partial D} xdy$$

Moments of a region D about the axes,

$$M_x = \frac{1}{2}\int_{\partial D} \left\langle -xy, \frac{x^2}{2}\right\rangle \cdot d\vec{r} = -\int_{\partial D} xydx = \frac{1}{2}\int_{\partial D} x^2dy$$

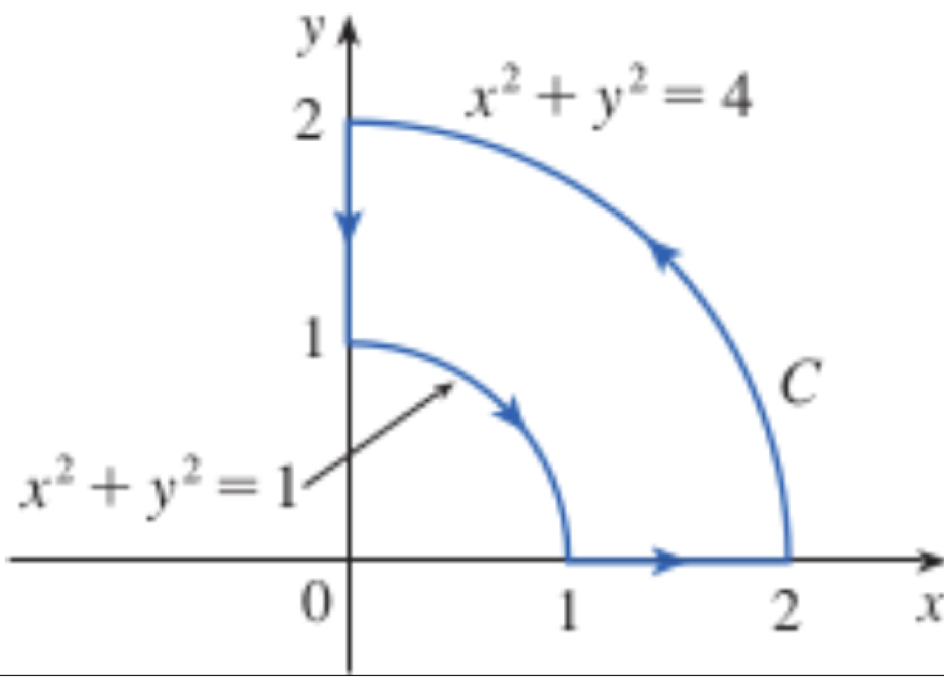
$$M_y = \frac{1}{2}\int_{\partial D} \left\langle -\frac{y^2}{2}, xy\right\rangle \cdot d\vec{r} = \int_{\partial D} xydy = -\frac{1}{2i}\int_{\partial D} y^2dx$$

Another observation which is sometimes useful is

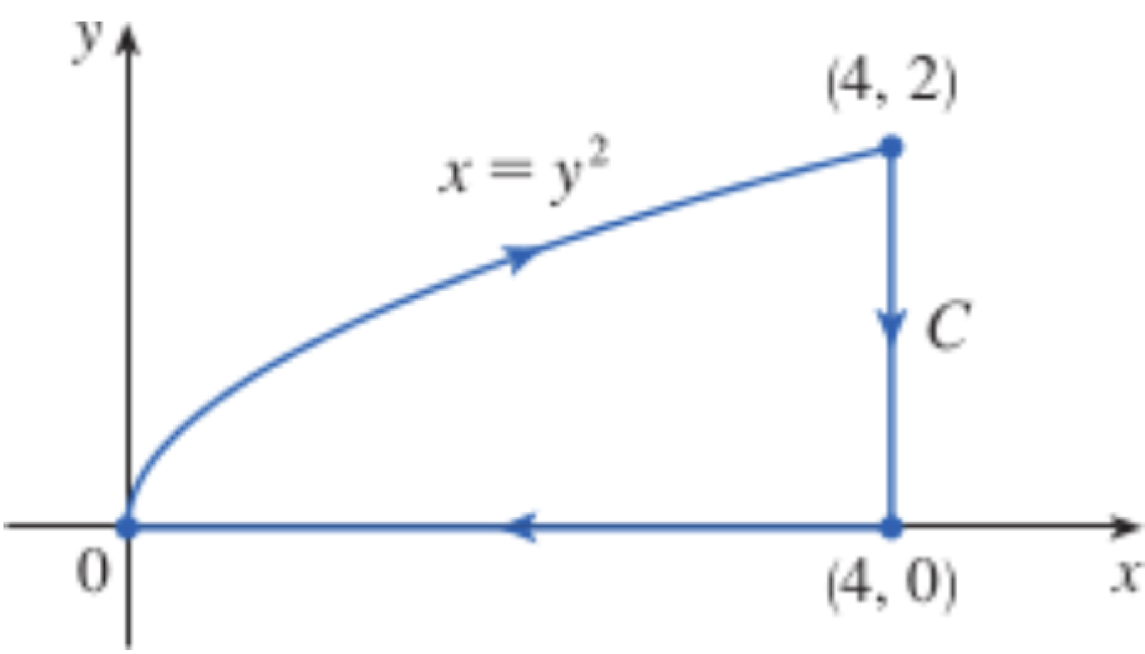
$$\int_{\partial D} Pdx+Qdy = \int_{\partial D} \big(P(x,y)+p(x)\big)dx+\big(Q(x,y)+q(y)\big)dy$$

for any functions $p(x)$ and $q(y)$.

$$\int_C (3 + e^{x^2})\, dx + (\tan^{-1} y + 3x^2)\, dy$$



$$\int_C (x^{2/3} + y^2)\, dx + (y^{4/3} - x^2)\, dy$$



$$\oint_C \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \bullet d\vec{r}$$

where C is any piecewise smooth, simple, closed curve which contains the origin.