Curl and Divergence

Let $\vec{F} = \langle P, Q, R\rangle$ be a 3D vector field. Define the *curl* of $\vec{F}$ using the following mnemonics.

$$\operatorname{curl}(F) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\operatorname{curl}(F) = \left\langle \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right\rangle$$

$$\operatorname{curl}(F) = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

The curl of a 3D field is a 3D field.

If

$$\operatorname{curl}\vec{F} = \vec{0}$$

we say $\vec{F}$ is *irrotational.*

$$\operatorname{curl}\left(\nabla f\right) = \vec{0}\ .$$

The *divergence* of $\vec{F}$ is easier to define.

$$\text{div}(\vec{f}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence of a 3D field is a scalar function.

If

$$\text{div}\vec{F} = 0$$

we say $\vec{F}$ is *incompressible*.

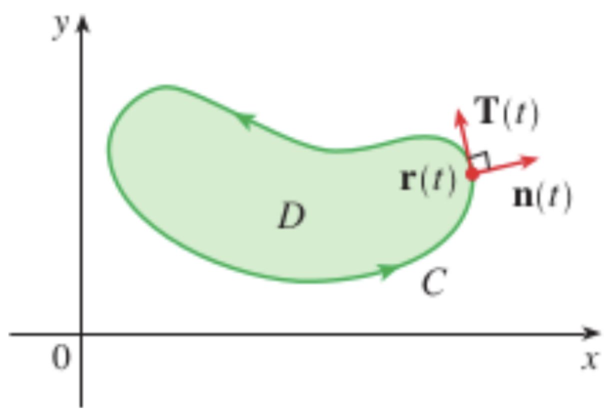
$$\text{div}\big(\text{curl}(\vec{F})\big) = 0$$

Given any 2D field $\langle P, Q\rangle$ construct a 3D field $\langle P, Q, 0\rangle$.

With the usual hypotheses and notation Green's Theorem becomes

$$\oint_{\partial D} \langle P, Q\rangle \bullet \vec{T}\, ds = \iint_D \operatorname{curl}\langle P, Q, 0\rangle \bullet \mathbf{k}\, dA = \iint_D \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\, dA$$

There is also unit vector orthogonal to $\vec{T}$ in the plane which which points *outwards*.



$$\oint_{\partial D} \langle P, Q\rangle \bullet \vec{N}\, ds = \iint_D \operatorname{div}\langle P, Q, 0\rangle\; dA = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\, dA$$

Note that if $\vec{r}(t) = \langle x(t), y(t)\rangle$

$$\vec{T}\, ds = \langle x'(t), y'(t)\rangle\; dt$$

$$\vec{N}\, ds = \langle y'(t), -x'(t)\rangle\; dt$$

Alphabetical right hand rule: Normal, Tangent, Up .

Maxwell's Equations

$$\mathrm{div}\vec{E} = 0 \qquad \mathrm{div}\vec{H} = 0$$

$$\mathrm{curl}\vec{E} = -\frac{1}{c}\frac{\partial H}{\partial t} \qquad \mathrm{curl}\vec{H} = \frac{1}{c}\frac{\partial E}{\partial t}$$