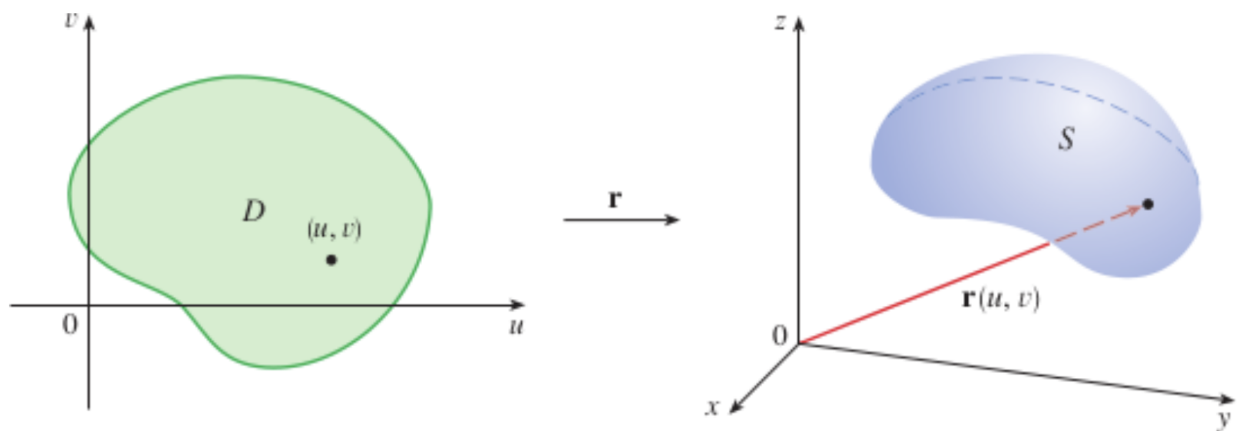
Parametrized surfaces

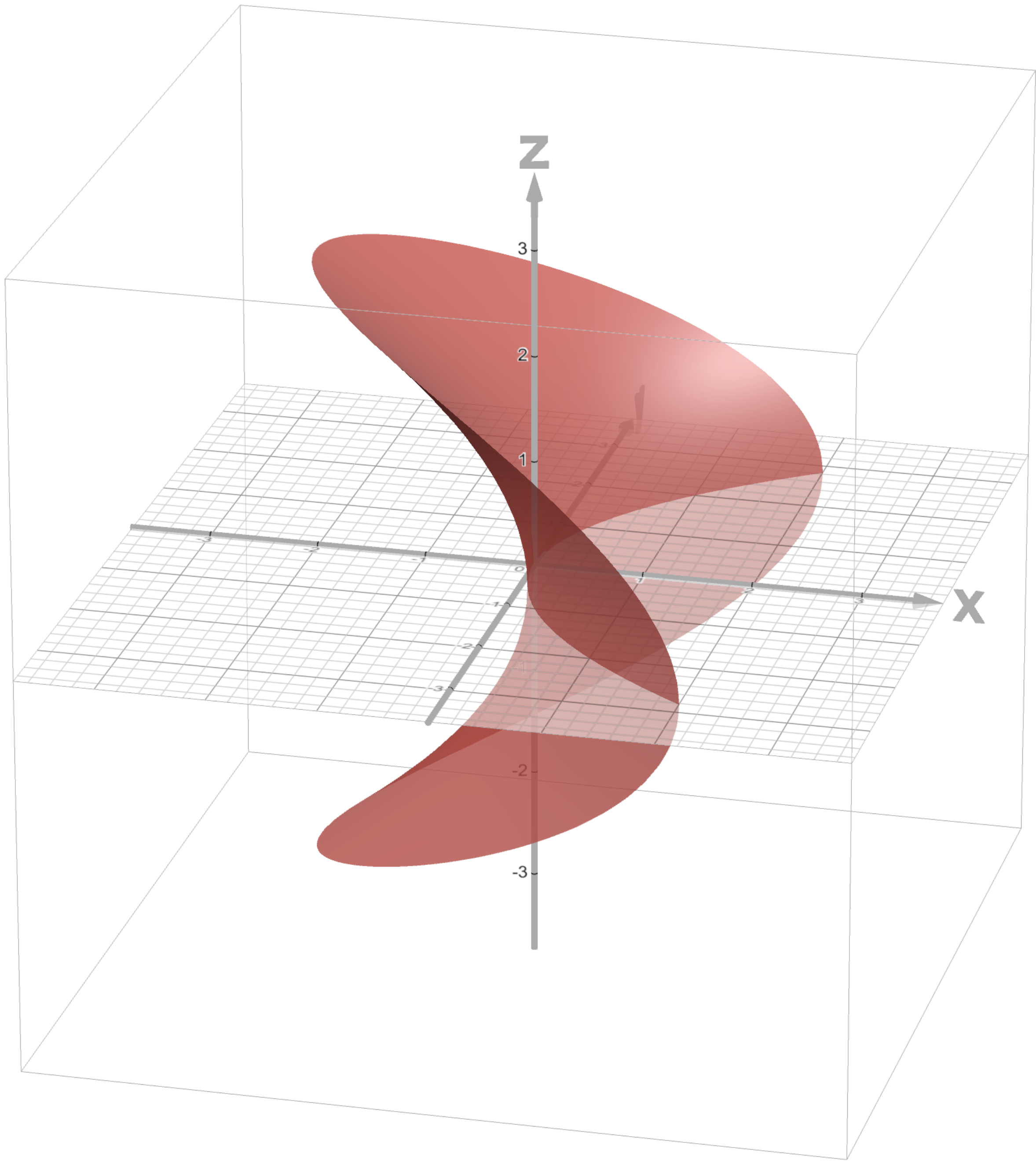
A *parametrized surface* consists of a domain D in $u\,v$ space and a vector valued function

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

for all $(u,v) \in D$.



Example: $\vec{r}(u,v) = \langle uv, u+v, u-v\rangle$ and $D : u^2 + v^2 \leqslant 4$.



$$\vec{r}(u,v) = \langle uv, u+v, u-v \rangle \text{ and } D : u^2 + v^2 \leqslant 4.$$

Figure 1.

Standard examples.

A graph $z = f(x, y)$: $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$ and D is the domain of f unless you further restrict it.

There are similar formulas for graphs $x = g(y, z)$ and $y = h(x, z)$.

A sphere centered at the origin with radius r.

$$\vec{r}(\theta, \rho) = \langle r\cos(\theta)\sin(\rho), r\sin(\theta)\sin(\rho), \cos(\rho)\rangle$$

and $D = [0, 2\pi] \times [0, \pi]$ or equvalently
D: $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \rho \leqslant \pi$.

Let $\vec{p}(t) = \langle x(t), y(t)\rangle$, $a \leqslant t \leqslant b$ be a parametrized curve in the xy plane. Then

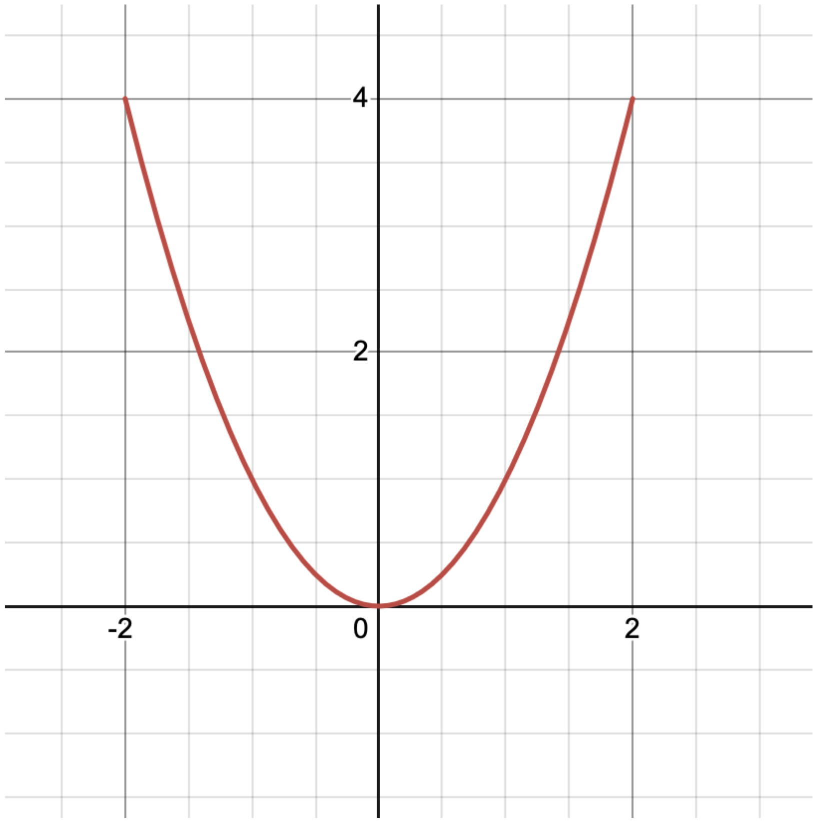
$$\vec{r}(t, z) = \langle x(t), y(t), z\rangle, \quad D : a \leqslant t \leqslant b, z \in (-\infty, \infty)$$

is the cylinder perpendicular to the xy plane along the curve.

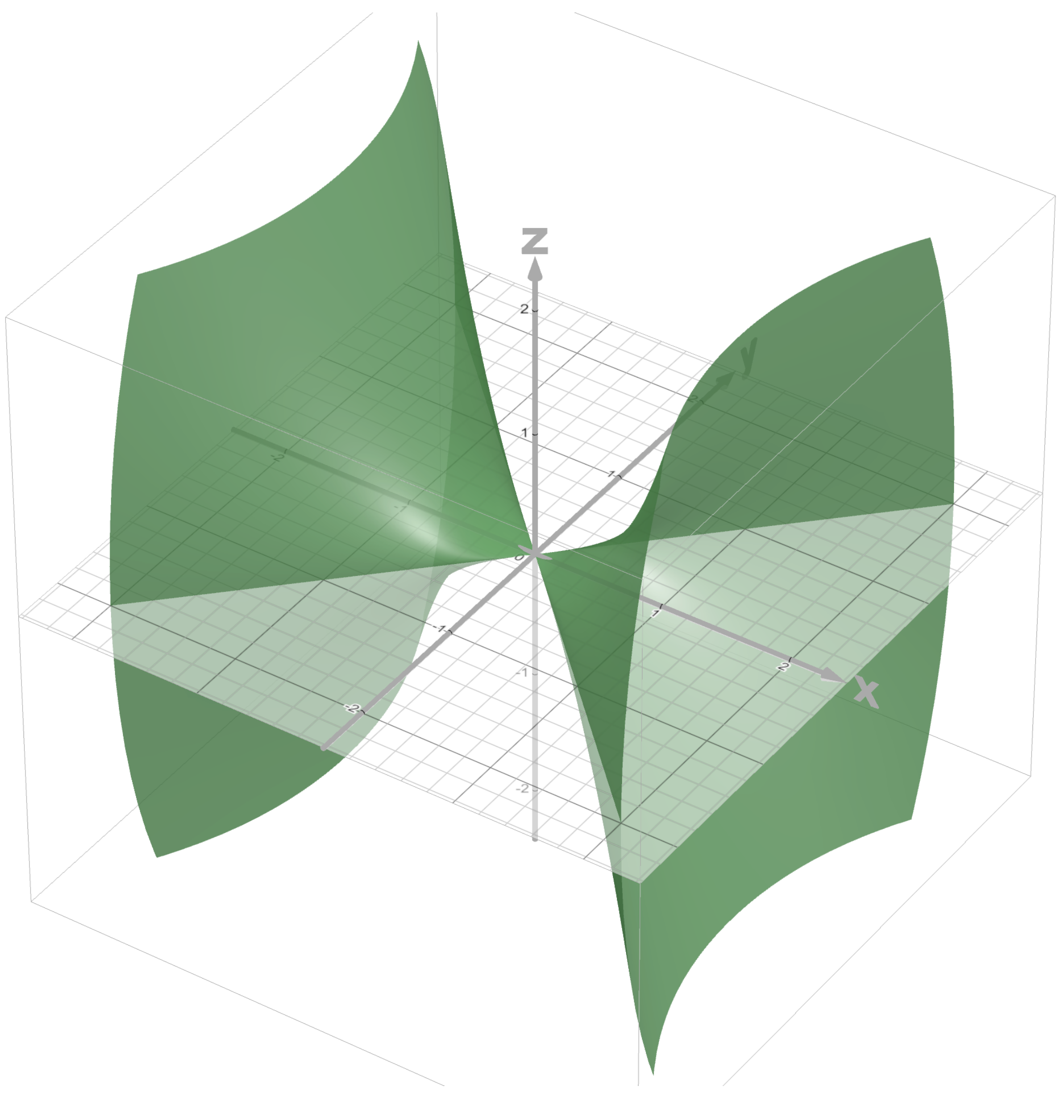
Let $\vec{p}(t) = \langle x(t), y(t)\rangle$, $a \leqslant t \leqslant b$ be a parametrized curve in the xy plane lying above the x-axis. Then

$$\vec{r}(t, \theta) = \langle x(t), y(t)\cos(\theta), y(t)\sin(\theta)\rangle, \quad D : t \in [a, b], \theta \in [0, 2\pi]$$

is the surface of revolution of the curve about the x-axis.



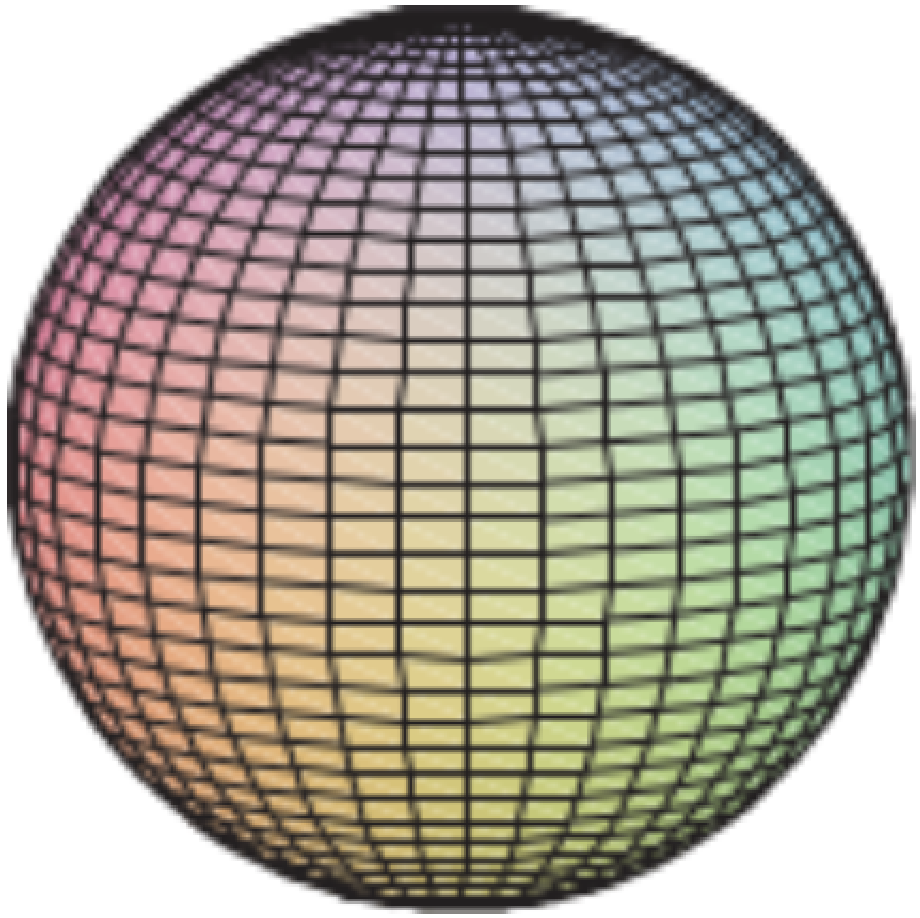
4
2
-2
0
2



z
y
x

Even with all these families of examples, you will be unable to identify a generic parametrized surface as anything else.

Grid lines



Given a parametrized surface $\vec{r}(u, v)$, D, there are families of parametrized curves, called *grid lines*, obtained as follows.

Any value u_0 determines a set, I_{u_0} of v so that $(u_0, v) \in D$ Then

$$\vec{r}(u_0, v) \; v \in I_{u_0}$$

is a parametrized curve in 3D.

Any value v_0 determines a set, J_{v_0} of u so that $(u, v_0) \in D$ Then

$$\vec{r}(u, v_o) \; u \in J_{v_0}$$

is a parametrized curve in 3D.

Tangent Planes

Since each grid line is a curve, it has a tangent vector (and a normal vector and a binormal vector which we will not be discussing). At a particular point $\vec{r}(u_0, v_0)$, there are two such tangent vectors

$$\vec{r}_u(u_0, v_0) = \left\langle \frac{\partial x(u, v_0)}{\partial u}, \frac{\partial y(u, v_0)}{\partial u}, \frac{\partial z(u, v_0)}{\partial u} \right\rangle \bigg|_{u=u_0}$$

$$\vec{r}_v(u_0, v_0) = \left\langle \frac{\partial x(u_0, v)}{\partial v}, \frac{\partial y(u_0, v)}{\partial v}, \frac{\partial z(u_0, v)}{\partial v} \right\rangle \bigg|_{v=v_0}$$

The normal vector to the surface at (u_0, v_0) is

$$\vec{N}(u_0, v_0) = \vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0)$$

A parametrization is called *smooth* if the partial derivatives are continuous and $|\vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0)| > 0$ for all $(u_0, v_0) \in D$ except for maybe points in the boundary of D.

Here is an example.