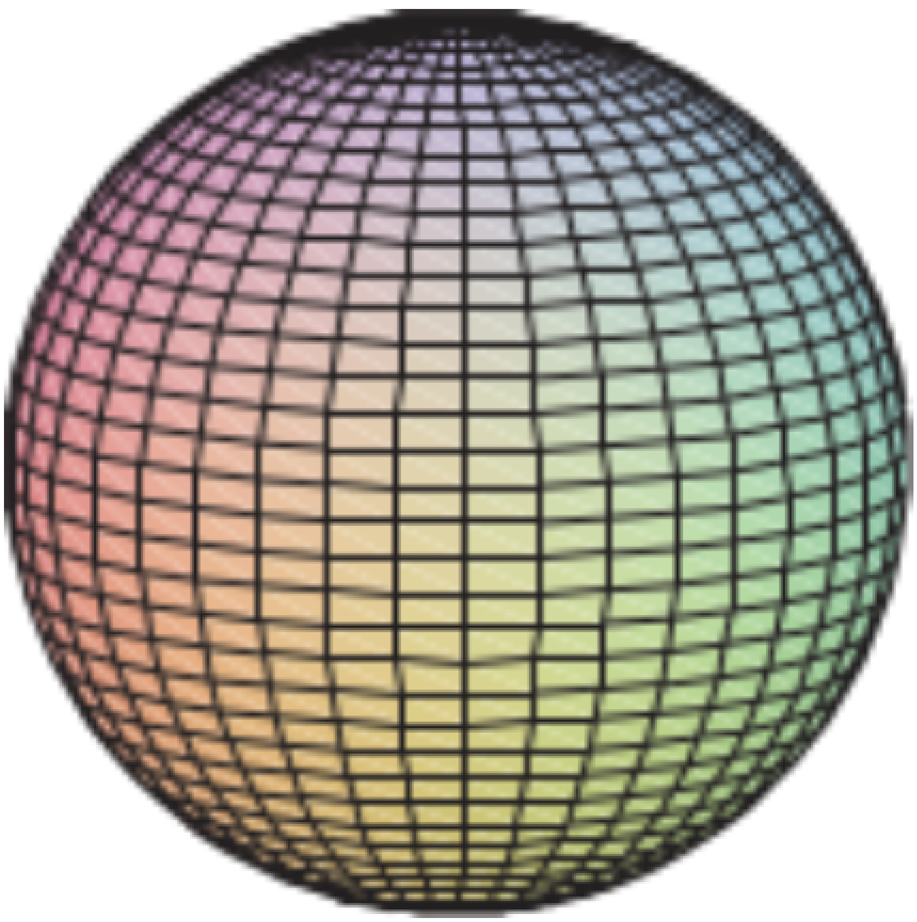
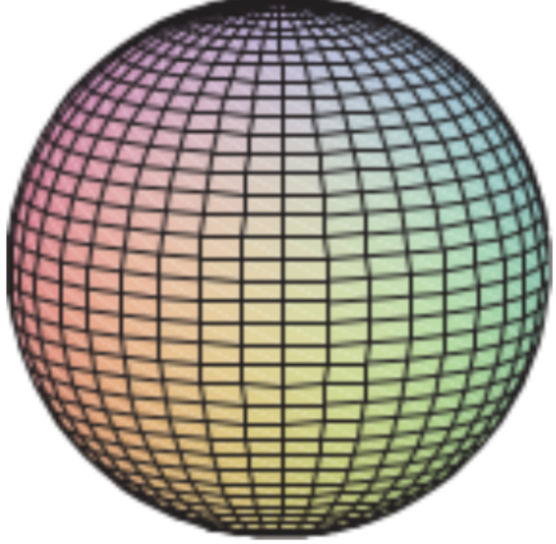
Parametrized surfaces, especially surface area

A *parametrized surface* consists of a domain D in $u\,v$ space and a vector valued function

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

for all $(u,v) \in D$.



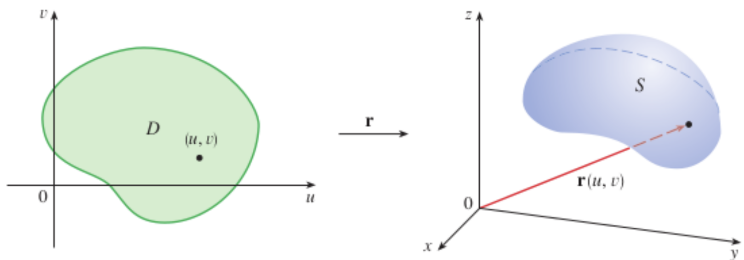


We want the area of such a little piece. The two orange lines are meant to be $\vec{r}_v(u_0, v_0)$ and $\vec{r}_u(u_0, v_0)$ and our approximation to the area of the piece will be the area of the parallelogram spanned by $\vec{r}_v(u_0, v_0)\Delta v$ and $\vec{r}_u(u_0, v_0)\Delta u$. This is

$$|\vec{r}_v(u_0, v_0) \times \vec{r}_u(u_0, v_0)|\Delta u \Delta v$$

In the limit we get

$$\text{Area}(S) = \iint_D |\vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0)| dA$$



Graphs: $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$:

$$\langle 1, 0, f_u \rangle$$
$$\langle 0, 1, f_v \rangle$$
$$\left\langle \begin{vmatrix} 0 & f_u \\ 1 & f_v \end{vmatrix}, - \begin{vmatrix} 1 & f_u \\ 0 & f_v \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right\rangle$$
$$\langle -f_u, -f_v, 1 \rangle$$

$$\text{Area}(S) = \iint_D \sqrt{1 + f_u^2 + f_v^2}\, dA$$

for any D over which f is smooth.

A sphere centered at the origin with radius r.

$$\vec{r}(\theta,\rho) = \langle r\cos(\theta)\sin(\rho), r\sin(\theta)\sin(\rho), r\cos(\rho)\rangle$$

$$\langle -r\sin(\theta)\sin(\rho), r\cos(\theta)\sin(\rho), 0\rangle$$

$$\langle r\cos(\theta)\cos(\rho), r\sin(\theta)\cos(\rho), -r\sin(\rho)\rangle$$

$$\left\langle \begin{vmatrix} r\cos(\theta)\sin(\rho) & 0 \\ r\sin(\theta)\cos(\rho) & -r\sin(\rho)\end{vmatrix} - \begin{vmatrix} -r\sin(\theta)\sin(\rho) & 0 \\ r\cos(\theta)\cos(\rho) & -r\sin(\rho)\end{vmatrix}, \begin{vmatrix} -r\sin(\theta)\sin(\rho) & r\cos(\theta)\sin(\rho) \\ r\cos(\theta)\cos(\rho) & r\sin(\theta)\cos(\rho)\end{vmatrix} \right\rangle$$

$$\langle -r^2\cos(\theta)\sin^2(\rho), -r^2\sin(\theta)\sin^2(\rho), -r^2\sin(\rho)\cos(\rho)\rangle$$

Then

$$|\vec{r}_\theta \times \vec{r}_\rho|^2 =$$

$$r^4\cos^2(\theta)\sin^4(\rho) + r^4\sin^2(\theta)\sin^4(\rho) + r^4\sin^2(\rho)\cos^2(\rho) =$$

$$r^4\sin^4(\rho) + r^4\sin^2(\rho)\cos^2(\rho) = r^4\sin^2(\rho)\big(\sin^2(\rho) + \cos^2(\rho)\big)$$

$$r^4\sin^2(\rho)$$

and

$$\mathrm{S} = \iint_D r^2\sin(\rho)\,dA$$

for any $D \subset [0,2\pi]\times[0,\pi]$.

If your ρ is outside the interval $[0,\pi]$ then you need an absolute value around the $\sin(\rho)$.

Let $\vec{p}(t) = \langle x(t), y(t) \rangle$, $a \leqslant t \leqslant b$ be a parametrized curve in the xy plane. Then

$$\vec{r}(t, z) = \langle x(t), y(t), z \rangle, \quad a \leqslant t \leqslant b, z \in (-\infty, \infty)$$

is the cylinder perpendicular to the xy plane along the curve.

$$\langle x', y', 0 \rangle$$
$$\langle 0, 0, 1 \rangle$$
$$\langle y', -x', 0 \rangle$$

$$\text{S} = \iint\limits_D \sqrt{(x')^2 + (y')^2}\, dA$$

for any $D \subset [a, b] \times (-\infty, \infty)$.

Let $\vec{p}(t) = \langle x(t), y(t) \rangle$, $a \leqslant t \leqslant b$ be a parametrized curve in the xy plane lying above the x-axis. Then

$$\vec{r}(t,\theta) = \langle x(t), y(t)\cos(\theta), y(t)\sin(\theta) \rangle, \quad t \in [a,b], \theta \in [0,2\pi]$$

is the surface of revolution of the curve about the x-axis.

$$\langle x'(t), y'(t)\cos(\theta), y'(t)\sin(\theta) \rangle$$

$$\langle 0, -y(t)\sin(\theta), y(t)\cos(\theta) \rangle$$

$$\left\langle \begin{vmatrix} y'(t)\cos(\theta) & y'(t)\sin(\theta) \\ -y(t)\sin(\theta) & y(t)\cos(\theta) \end{vmatrix}, - \begin{vmatrix} x'(t) & y'(t)\sin(\theta) \\ 0 & y(t)\cos(\theta) \end{vmatrix}, \begin{vmatrix} x'(t) & y'(t)\cos(\theta) \\ 0 & -y(t)\sin(\theta) \end{vmatrix} \right\rangle$$

$$\left\langle \big(y(t)y'(t)\big), -x'(t)y(t)\cos(\theta), -x'(t)y(t)\sin(\theta) \right\rangle$$

$$\mathrm{S} = \iint\limits_D |y(t)| \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\, dA$$

for any $D \subset [a,b] \times [0,2\pi]$.

Sphere.

Other.