Flux integrals

From last lecture: surface parametrized by $\vec{r}(u, v)$ for $(u, v) \in D$.

$$\iint\limits_S f(x, y, z)dS = \iint\limits_D f\big(\vec{r}(u, v)\big)\Big|\vec{r}_u \times \vec{r}_v\Big|dA$$

The answer is independent of the parametrization as long as the parametrization is smooth.

If $\vec{F}$ is a vector field and the surface is oriented by $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ the flux integral is defined by

$$\iint_S \vec{F}(x, y, z) \bullet \vec{n} \, dS$$

We often write $d\vec{S} = \vec{n} \, dS$ (In analogy with the line integral case.) so

$$\iint_S \vec{F}(x, y, z) \bullet \vec{n} \, dS = \iint_S \vec{F} \bullet d\vec{S}$$

Also as in the line integral case

$$d\vec{S} = \vec{r}_u \times \vec{r}_v \, dA$$

with the usual parametrization notation.

For this to work well, the surface S needs to be oriented, which means it is possible to pick a well defined, continuous unit vector to the surface.

Here is the classic example of a surface where this is not possible.

Mobius band.

Mobius bands are really the only examples occurring in 3D.

The following examples cover all you will need for this course.

- If S is orientable, so is any piece of S.
- Graphs are orientable.
- Surfaces of revolution are orientable.
- Any surface which is the boundary of a solid is orientable.

Last example from class.

Cylinder: $y^2 + z^2 = 2^2$ between $x = 0$ and $x = 3 - z$;

Field: $\vec{F}(x, y, z) = \langle x^2, 2z, -3y \rangle$.

Parametrize by $\vec{r}(x, \theta) = \langle x, 2\sin(\theta), 2\cos(\theta) \rangle$; $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant x \leqslant 3 - 2\cos(\theta)$.

$\vec{r}_x = \langle 1, 0, 0 \rangle$

$\vec{r}_\theta = \langle 0, 2\cos(\theta), -2\sin(\theta) \rangle$

$$\vec{r}_x \times \vec{r}_\theta = \left\langle \begin{vmatrix} 0 & 0 \\ * & * \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & -2\sin(\theta) \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 2\cos(\theta) \end{vmatrix} \right\rangle$$

$\vec{r}_x \times \vec{r}_\theta = \langle 0, 2\sin(\theta), 2\cos(\theta) \rangle$.

$\vec{F}\big(\vec{r}(x, \theta)\big) = \langle x, 4\cos(\theta), -6\sin(\theta) \rangle$.

Need to see which way the normal vector points.

The point $\vec{r}(1, \pi/2) = (1, 2, 0)$ is on the surface.

With these values of x and θ the normal vector point in the direction $\langle 0, 3, 0 \rangle$ Putting the tail of the vector at $(1, 2, 0)$, the end of the vector is at $(1, 5, 0)$ and since $1^2 + 5^2 > 4$, the normal vector points out of the cylinder.