Divergence Theorem

$$\iint\limits_{\partial E} \vec{F} \cdot d\vec{S} = \iiint\limits_{E} (\nabla \cdot \vec{F})\ dV$$

Evaluate $\displaystyle\iint_{\partial B} \vec{F} \bullet d\vec{S}$ where $\vec{F} = \langle \sin(\pi x), zy^3, z^2 + 4x \rangle$ and B is the box $-1 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 1$, $1 \leqslant z \leqslant 4$.

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Instead do $\iiint\limits_{B} \nabla \bullet \vec{F} dV$.

$$\nabla \bullet \vec{F} = \frac{\partial \sin(\pi x)}{\partial x} + \frac{\partial zy^3}{\partial y} + \frac{\partial (z^2 + 4x)}{\partial z} = \pi\cos(\pi x) + 3zy^2 + 2z$$

$$\iint\limits_{\partial B} \vec{F} \bullet d\vec{S} = \int_{-1}^{2}\int_{0}^{1}\int_{1}^{4} \pi\cos(\pi x) + 3zy^2 + 2z \; dz \; dy \; dx =$$

$$\int_{-1}^{2}\int_{0}^{1} z\pi\cos(\pi x) + 3y^2\frac{z^2}{2} + z^2 \Big|_{1}^{4} dy \; dx =$$

$$\int_{-1}^{2}\int_{0}^{1} 3\pi\cos(\pi x) + 3\frac{15y^2}{2} + 15 \; dy \; dx =$$

$$\int_{-1}^{2} 3y\pi\cos(\pi x) + 3\frac{5y^3}{2} + 15y \;\Big|_{0}^{1} dy \; dx =$$

$$\int_{-1}^{2} 3\pi\cos(\pi x) + \frac{15}{2} + 15 \; dx =$$

$$3\sin(\pi x) + \frac{45}{2}x\Big|_{-1}^{2} = \frac{135}{2}$$

Let $\vec{F} = \langle x, y, z \rangle$ and let E be the ball of radius R centered at $(0, 0, 0)$. Let $\vec{F}_0 = \dfrac{1}{R}\vec{F}$.

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Evaluate $\displaystyle\iint_{\partial E} \vec{F}_0 \bullet d\vec{S}$.

On the sphere of radius R, $\vec{F}_0$ is the outward pointing unit normal so

$$\iint_{\partial E} \vec{F}_0 \bullet d\vec{S} = \iint_{\partial E} 1 \, dS$$

which is the surface area of the sphere of radius R.

Evaluate $\displaystyle\iiint_{E} \nabla \bullet \vec{F}_0 \, dV$.

$\nabla \bullet \vec{F} = 3$ so $\nabla \bullet \vec{F}_0 = \dfrac{3}{R}$. Hence

$$\iiint_{E} \nabla \bullet \vec{F}_0 \, dV = \iiint_{E} \frac{3}{R} \, dV = \frac{3}{R} \iiint_{E} 1 \, dV$$

which is $\dfrac{3}{R}$ times the volume of the ball.

Surface area: $4\pi R^2$

Volume: $\dfrac{4}{3}\pi R^3$.

Let $\vec{F} = \left\langle y^2 + ye^{z^2}, z^2 - x^2, y^2 \right\rangle$.

Find $\displaystyle\iint_S \vec{F} \bullet d\vec{S}$ over the upper unit hemisphere using the upward normal.

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In spherical coordinates
$d\vec{S} = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle \; d\phi \; d\theta$

$$\vec{F}(\phi,\theta) = \Big\langle \big(\sin(\phi)\sin(\theta)\big)^2 + \big(\sin(\phi)\sin(\theta)\big)^2 e^{\cos^2(\phi)}, \\ \cos^2(\phi) - \big(\sin(\phi)\cos(\theta)\big)^2, \\ \big(\sin(\phi)\sin(\theta)\big)^2 \Big\rangle$$

$$\iint\limits_S \vec{F} \bullet d\vec{S} = \int_0^{2\pi} \int_0^{\pi/2} \big(\sin(\phi)\cos(\theta)\big)\Big(\big(\sin(\phi)\sin(\theta)\big)^2 + e^{\cos^2(\phi)}\Big) + \\ \big(\sin(\phi)\sin(\theta)\big)\Big(\cos^2(\phi) - \big(\sin(\phi)\cos(\theta)\big)^2\Big) + \\ \cos(\phi)\Big(\big(\sin(\phi)\sin(\theta)\big)^2\Big) \; d\phi \; d\theta \text{ which is}$$

$$\int_0^{2\pi} \int_0^{\pi/2} \text{function of trigs} + \sin(\phi)\sin(\theta) e^{\cos^2(\phi)} + \text{function of trigs} \, d\phi \, d\theta$$

Stuck.

Check $\nabla \cdot \vec{F} = \dfrac{\partial(y^2 + e^{z^2})}{\partial x} + \dfrac{\partial(z^2 - x^2)}{\partial y} + \dfrac{\partial y^2}{\partial z} = 0$ so, if E is the solid bounded above by the upper unit hemisphere and below by the unit disk in the xy plane, then

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_{E} (\nabla \cdot \vec{F})\ dV = 0$$

Now ∂E is the union of the upper unit hemisphere and the disk D so

$$0 = \iint_{\partial E} \vec{F} \cdot d\vec{S} = \iint_{S} \vec{F} \cdot d\vec{S} + \iint_{D} \vec{F} \cdot d\vec{S}$$

Parametrize D by $\langle \rho\cos(\theta), \rho\sin(\theta), 0\rangle$ $0 \leqslant \rho \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$.

The outward normal on D is $\langle 0, 0, -1\rangle$ so $d\vec{S} = \langle 0, 0, -1\rangle\, \rho\, d\rho\, d\theta$ and

$$\vec{F}(\rho, \theta) = \left\langle \rho^2\sin^2(\theta) + e^{0^2}, 0^2 - \rho^2\cos^2(\theta), \rho^2\sin^2(\theta) \right\rangle$$

so

$$\iint_{D} \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^1 -\rho^3\sin^2(\theta)\ d\rho\ d\theta = -\frac{1}{4}\int_0^{2\pi} \sin^2(\theta)\ d\theta =$$

$$-\frac{1}{4}\int_0^{2\pi} \frac{1 - \cos(2\theta)}{2} d\theta = -\frac{\pi}{4}\ .$$

Hence

$$\int_S \vec{F} \cdot d\vec{S} - \frac{\pi}{4} = 0 \qquad \text{so} \qquad \int_S \vec{F} \cdot d\vec{S} = \frac{\pi}{4}$$

Evaluate

$$\iint_S \vec{F} \cdot d\vec{S}$$

where $\vec{F} = \left\langle \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$

where S is any closed surface with $(0,0,0)$ not on the surface.

Evaluate

$$\iint\limits_S \vec{F} \bullet d\vec{S}$$

where $\vec{F} = \left\langle \dfrac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \dfrac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \dfrac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$

where S is any closed surface with $(0,0,0)$ not on the surface.

There are two cases: $(0,0,0)$ inside S or $(0,0,0)$ outside S. More precisely, S is the boundary of a bounded solid E and either $(0,0,0) \in E$ or it is not.

Check $\nabla \bullet \vec{F} = 0$.

If $(0,0,0) \notin E$, $\vec{F}$ is defined on all of E and so

$$\iint\limits_S \vec{F} \bullet d\vec{S} = \iint\limits_{\partial} \vec{F} \bullet d\vec{S} = \iiint\limits_E \nabla \bullet \vec{F} \underset{\cdot}{\mathrm{V}} = 0$$

If $(0,0,0) \in E$, $\vec{F}$ is not defined on all of E. Since E is bounded, there exists $R > 0$ so that E is contained in the ball of radius R centered at the origin.

Let E_0 denote the ball with the interior of E removed. Then ∂E_0 is the sphere of radius R disjoint union ∂E.

Now $\vec{F}$ is defined on all of E_0 so

$$\iint_{E_0} \vec{F} \bullet d\vec{S} = 0 = \iint_{Sphere} \vec{F} \bullet d\vec{S} + \iint_{S} \vec{F} \bullet d\vec{S}$$

and hence

$$\iint_{S} \vec{F} \bullet d\vec{S} = -\iint_{Sphere} \vec{F} \bullet d\vec{S}$$

In spherical coordinates

$$\vec{F} = \frac{1}{R^3} \langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) \rangle$$

The outward unit normal to the sphere is

$$\vec{n} = R \langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) \rangle$$

but we need the inward one here so

$$\vec{F} \bullet \vec{n} = -\frac{1}{R^2}$$

and thus

$$\iint_{S} \vec{F} \bullet d\vec{S} = \frac{1}{R^2} \text{surface area of sphere} = 4\pi$$

Evaluate

$$\iint_S \vec{F}_{(a,b,c)} \bullet d\vec{S}$$

where $\vec{F}_{(a,b,c)} = \left\langle \frac{x-a}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{\frac{3}{2}}}, \frac{y-b}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{\frac{3}{2}}}, \frac{z-c}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{\frac{3}{2}}} \right\rangle$

where S is any closed surface with (a, b, c) not on the surface.