Divergence et al. in other coordinate systems.

Maxwell's Equations

$$\begin{aligned} \operatorname{div} \mathbf{E} &= 0 & \qquad \operatorname{div} \mathbf{H} &= 0 \\ \operatorname{curl} \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} & \qquad \operatorname{curl} \mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Cartesian (x,y,z): Scalar function F; Vector field $\mathbf{f} = f_1\,\mathbf{i} + f_2\,\mathbf{j} + f_3\,\mathbf{k}$

$$\text{gradient}: \quad \nabla F = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}$$

$$\text{divergence}: \quad \nabla\cdot\mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\text{curl}: \nabla\times\mathbf{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}$$

$$\text{Laplacian}: \quad \Delta F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

Cylindrical (r,θ,z): Scalar function F; Vector field $\mathbf{f} = f_r\,\mathbf{e}_r + f_\theta\,\mathbf{e}_\theta + f_z\,\mathbf{e}_z$

$$\text{gradient}: \quad \nabla F = \frac{\partial F}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial F}{\partial \theta}\mathbf{e}_\theta + \frac{\partial F}{\partial z}\mathbf{e}_z$$

$$\text{divergence}: \quad \nabla\cdot\mathbf{f} = \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{\partial f_z}{\partial z}$$

$$\text{curl}: \nabla\times\mathbf{f} = \left(\frac{1}{r}\frac{\partial f_z}{\partial \theta} - \frac{\partial f_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r f_\theta) - \frac{\partial f_r}{\partial \theta}\right)\mathbf{e}_z$$

$$\text{Laplacian}: \quad \Delta F = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \frac{\partial^2 F}{\partial z^2}$$

Spherical (ρ,θ,ϕ): Scalar function F; Vector field $\mathbf{f} = f_\rho\,\mathbf{e}_\rho + f_\theta\,\mathbf{e}_\theta + f_\phi\,\mathbf{e}_\phi$

$$\text{gradient}: \quad \nabla F = \frac{\partial F}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho\sin\phi}\frac{\partial F}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial F}{\partial \phi}\mathbf{e}_\phi$$

$$\text{divergence}: \quad \nabla\cdot\mathbf{f} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 f_\rho) + \frac{1}{\rho\sin\phi}\frac{\partial f_\theta}{\partial \theta} + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, f_\phi)$$

$$\text{curl}: \nabla\times\mathbf{f} = \frac{1}{\rho\sin\phi}\left(\frac{\partial}{\partial \phi}(\sin\phi\, f_\theta) - \frac{\partial f_\phi}{\partial \theta}\right)\mathbf{e}_\rho + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho f_\phi) - \frac{\partial f_\rho}{\partial \phi}\right)\mathbf{e}_\theta + \left(\frac{1}{\rho\sin\phi}\frac{\partial f_\rho}{\partial \theta} - \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho f_\theta)\right)\mathbf{e}_\phi$$

$$\text{Laplacian}: \quad \Delta F = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial F}{\partial \rho}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 F}{\partial \theta^2} + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial F}{\partial \phi}\right)$$

The derivation of the above formulas for cylindrical and spherical coordinates is straightforward but extremely tedious. The basic idea is to take the Cartesian equivalent of the quantity in question and to substitute into that formula using the appropriate coordinate transformation. As an example, we will derive the formula for the gradient in spherical coordinates.

From: https://www.mecmath.net/VectorCalculus.pdf

Δ here is the Laplace operator which in our book is written ∇^2 or $\nabla\cdot\nabla$.

The Cartesian equations are the definitions and the others are how to compute them in other coordinate systems.

Good summary of the similarities among the various integrals you have been studying the last year and a half.

Vector Calculus —MA2VC/MA3VC 2016–17— Summary and comparison of the different integrals of fields

The seven types of integrals we have considered in the lectures are in the red boxes.
Those in blue are the formulas we use to compute integrals over curvilinear domains of integration (paths Γ and surfaces S) as integrals over flat domains (intervals (t_I, t_F) and regions R) of the same dimension, by using the parametrisations $\vec{\mathbf{a}}:(t_I,t_F)$ and $\vec{\mathbf{X}}: R \to S$.
The most important theorems relating these kinds of integrals are mentioned in green.
$\hat{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{a}}}{dt} / \left|\frac{d\vec{\mathbf{a}}}{dt}\right|$ and $\hat{\boldsymbol{n}} = \frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w} / \left|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right|$ are the *orientations* of paths and (parametric) surfaces, namely unit tangent and normal vector fields, respectively.

Integrals of real functions $\int_{(t_I,t_F)} G(t)\,\mathrm{d}t$ Fundamental theorem of calculus applies here.	**Line integrals of scalar fields** $\int_\Gamma f\,\mathrm{d}s = \int_{(t_I,t_F)} f(\vec{\mathbf{a}}) \left\|\frac{d\vec{\mathbf{a}}}{dt}\right\| \mathrm{d}t$	**Line integrals of vector fields** $\int_\Gamma \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = \int_\Gamma (\vec{\mathbf{F}}\cdot\hat{\boldsymbol{\tau}})\,\mathrm{d}s = \int_{(t_I,t_F)} \vec{\mathbf{F}}(\vec{\mathbf{a}})\cdot \frac{d\vec{\mathbf{a}}}{dt}\,\mathrm{d}t$ Fundamental theorem of vector calculus applies here.	1D sets
Double integrals $\iint_R f\,\mathrm{d}A$ (Polar coordinates can be used.)	**Surface integrals of scalar fields** $\iint_S f\,\mathrm{d}S = \iint_R f(\vec{\mathbf{X}}) \left\|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right\| \mathrm{d}A$	**Fluxes = surface integrals of vector fields** $\iint_S \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}})\,\mathrm{d}S = \iint_R \vec{\mathbf{F}}(\vec{\mathbf{X}})\cdot \frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\,\mathrm{d}A$	2D sets
Triple integrals $\iiint_D f\,\mathrm{d}V$ (Cylindrical or spherical coordinates can be used.)			3D sets

Green's th. (line integrals of vector fields ↔ double integrals)
Stokes' th. (line integrals of vector fields ↔ fluxes)
Divergence th. (fluxes ↔ triple integrals)

Flat domains of integration: defined by boundary only.

Curvilinear domains of integration: parametrisation (either $\vec{\mathbf{a}}$ or $\vec{\mathbf{X}}$) is needed. Parametrisation allows to compute integrals over flat domains of integration: (t_I,t_F) in place of Γ, R in place of S.

Scalar integrands f, G: no need of orientation.

Vector integrands $\vec{\mathbf{F}}$: orientation ($\hat{\boldsymbol{\tau}}$ or $\hat{\boldsymbol{n}}$) of path/surface needed.

- $G:\mathbb{R}\to\mathbb{R}$ is a real function
- $f:\mathbb{R}^3\to\mathbb{R}$ is a scalar field
- $\vec{\mathbf{F}}:\mathbb{R}^3\to\mathbb{R}^3$ is a vector field

These are the *integrands*, namely the functions/fields *of* which we compute integrals.

- $(t_I,t_F)\subset\mathbb{R}$ is an **interval** (1D)
- $R\subset\mathbb{R}^2$ is a planar **region** (2D)
- $D\subset\mathbb{R}^3$ is a **domain** (3D)
- $\Gamma\subset\mathbb{R}^3$ is a **path** (1D), parametrised by $\vec{\mathbf{a}}:(t_I,t_F)\to\Gamma$
- $S\subset\mathbb{R}^3$ is a **surface** (2D), parametrised by $\vec{\mathbf{X}}: R\to S$

These are the *domains of integration*, namely the geometric objects *on* which we compute integrals.

Vector Calculus —MA2VC/MA3VC 2016–17— Summary and comparison of main vector calculus theorems

	The integral on a(n)	*of*	*of a*	*is equal to the*	*of the*	*on/at the*	*Equivalently, in formulae,*
Fundamental theorem of calculus	interval (t_I,t_F)	the derivative	real function G	difference of the values	function G	endpoints	$\int_{t_I}^{t_F} G'(t)\,\mathrm{d}t = G(t_F) - f(t_I)$
Fundamental theorem of vector calculus	oriented path Γ from $\vec{\mathbf{p}}$ to $\vec{\mathbf{q}}$	$\hat{\boldsymbol{\tau}}\cdot$ gradient	scalar field f	difference of the values	field f	endpoints $\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$	$\int_\Gamma \vec{\nabla} f\cdot \mathrm{d}\vec{\mathbf{r}} = f(\vec{\mathbf{q}}) - f(\vec{\mathbf{p}})$
Green's theorem	two-dimensional region R	$\hat{\boldsymbol{k}}\cdot$ curl	vector field $\vec{\mathbf{F}}$	circulation	field $\vec{\mathbf{F}}$	boundary ∂R	$\iint_R \hat{\boldsymbol{k}}\cdot(\vec{\nabla}\times\vec{\mathbf{F}})\,\mathrm{d}A = \oint_{\partial R} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}}$
Stokes' theorem	oriented surface $(S,\hat{\boldsymbol{n}})$	$\hat{\boldsymbol{n}}\cdot$ curl	vector field $\vec{\mathbf{F}}$	circulation	field $\vec{\mathbf{F}}$	boundary ∂S	$\iint_S (\vec{\nabla}\times\vec{\mathbf{F}})\cdot \mathrm{d}\vec{\mathbf{S}} = \int_{\partial S} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}}$
Divergence theorem	3D domain D	divergence	vector field $\vec{\mathbf{F}}$	flux	field $\vec{\mathbf{F}}$	boundary ∂D	$\iiint_D (\vec{\nabla}\cdot\vec{\mathbf{F}})\,\mathrm{d}V = \oiint_{\partial D} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{S}}$
	1st domain of integration	differential operator	function or field	integral type or evaluation	function or field	2nd domain of integration	formula