Multiple Choice

1.(5pts) Suppose the vector-valued function $\vec{r}(t)$ satisfies $\vec{r}'(t) = \langle -6t^2, 2t+1, 8t^3 \rangle$ and $\vec{r}(0) = \langle 1, 2, 3 \rangle$. Find $\vec{r}(1)$.

(a) $\langle -1, 4, 5 \rangle$ (b) $\langle -2, 2, 2 \rangle$ (c) $\langle -6, 3, 8 \rangle$ (d) $\langle -12, 2, 24 \rangle$ (e) $\langle -11, 3, 27 \rangle$

Solution. First find the general antiderivative of $\vec{r}'(t)$:

$$\vec{r}(t) = \left\langle -2t^3 + c_1, t^2 + t + c_2, 2t^4 + c_3 \right\rangle$$

Then $\vec{r}(0) = \langle c_1, c_2, c_3 \rangle$, and so $c_1 = 1$, $c_2 = 2$, and $c_3 = 3$. So

$$\vec{r}(t) = \left\langle -2t^3 + 1, t^2 + t + 2, 2t^4 + 3 \right\rangle$$

Finally: $\vec{r}(1) = \langle -1, 4, 5 \rangle$.

2.(5pts) In how many points does the twisted cubic $\vec{r}(t) = \langle t^3, t, t^2 \rangle$ intersect the plane $x + y + z = 0$?

(a) 1 (b) 0 (c) 2

(d) 3 (e) infinitely many points

Solution. Solve $t^3 + t + t^2 = 0$ or $t(t^2 + t + 1) = 0$. The equation $t^2 + t + 1 = 0$ has no real solutions so $t = 0$ is the only point of intersection. So there is just one intersection point.

3.(5pts) Which vector below is the vector from $P = (1, 2, -5)$ to $Q = (2, 1, 5)$.

(a) $\langle 1, -1, 10 \rangle$ (b) $\langle 3, 3, 0 \rangle$ (c) $\langle 1, 1, 10 \rangle$ (d) $\langle -1, -1, -10 \rangle$

(e) $\langle 1, 1, 0 \rangle$

Solution. The vector is $\langle a, b, c \rangle$ where $a + 1 = 2$, $b + 2 = 1$ and $c + (-5) = 5$. Hence $a = 1$, $b = -1$ and $c = 10$.

4.(5pts) Compute the tangential component of the acceleration of a particle at $t = \pi$ whose motion is given by $\vec{r}(t) = \left\langle 4\cos(t), 4\sin(t), \dfrac{3}{2\pi}t^2 \right\rangle$.

(a) $\dfrac{9}{5\pi}$ (b) 0 (c) $\dfrac{4}{5}\sqrt{25 + \dfrac{9}{\pi^2}}$ (d) $\dfrac{16}{5\pi}$ (e) $\dfrac{5}{\pi}$

Solution. Recall

$$a_T(t) = \frac{\vec{r}'(t) \bullet \vec{r}''(t)}{|\vec{r}'(t)|^2}\ .$$

Now $\vec{r}'(t) = \left\langle -4\sin(t), 4\cos(t), \frac{3}{\pi}t \right\rangle$ and $\vec{r}''(t) = \left\langle -4\cos(t), -4\sin(t), \frac{3}{\pi} \right\rangle$. Thus

$$a_T(\pi) = \frac{\langle 0, 4, 3\rangle \bullet \left\langle -4, 0, \frac{3}{\pi} \right\rangle}{\sqrt{0^2 + 4^2 + 3^2}} = \frac{9}{5\pi}\ .$$

5.(5pts) Determine which of the following expressions gives the length of the curve defined by $\vec{r}(t) = 2t\vec{\imath} + \cos t\vec{\jmath} + 2\sin t\vec{k}$ between the points $(0, 1, 0)$ and $(2\pi, -1, 0)$.

(a) $\displaystyle\int_0^{\pi} \sqrt{4 + \sin^2 t + 4\cos^2 t}\, dt$

(b) $\displaystyle\int_0^{\pi} \sqrt{4t^2 + \cos^2 t + 4\sin^2 t}\, dt$

(c) $\displaystyle\int_0^{2\pi} \sqrt{4t^2 + \cos^2 t + 4\sin^2 t}\, dt$

(d) $\displaystyle\int_0^{2\pi} \sqrt{4 + \sin^2 t + 4\cos^2 t}\, dt$

(e) $\displaystyle\int_0^{\pi} (2\vec{\imath} - \sin t\,\vec{\jmath} + 2\cos t\vec{k})\, dt$

Solution. $\vec{r}'(t) = \langle 2, -\sin t, 2\cos t\rangle$ so $\left|\vec{r}'(t)\right| = \sqrt{4 + \sin^2 t + 4\cos^2}$ and so $\left|\vec{r}'(t)\right|$ is never 0 and this is a smooth parametrization. Also $\vec{r}(0) = \langle 0, 1, 0\rangle$ and $\vec{r}(\pi) = \langle 2\pi, -1, 0\rangle$ so the length is $\displaystyle\int_0^{\pi} \sqrt{4 + \sin^2 t + 4\cos^2 t}\, dt$.

6.(5pts) Find the projection of the vector $\langle 1, 2, -1\rangle$ onto the vector $\langle 3, -1, 0\rangle$.

(a) $\frac{3}{10}\vec{\imath} - \frac{1}{10}\vec{\jmath}$

(b) $4\vec{\imath} + \vec{\jmath} - \vec{k}$

(c) $2\vec{\imath} - 3\vec{\jmath} + \vec{k}$

(d) $\frac{2}{\sqrt{10}}\vec{\jmath} + \frac{3}{\sqrt{10}}\vec{k}$

(e) $\frac{3}{\sqrt{6}}\vec{\imath} - \frac{1}{\sqrt{6}}\vec{\jmath}$

Solution. $\text{proj}_{\langle 3,-1,0\rangle}\big(\langle 1, 2, -1\rangle\big) = \dfrac{\langle 3, -1, 0\rangle \bullet \langle 1, 2, -1\rangle}{\langle 3, -1, 0\rangle \bullet \langle 3, -1, 0\rangle}\langle 3, -1, 0\rangle = \dfrac{1}{10}\langle 3, -1, 0\rangle$

7.(5pts) Find the limit of the vector function

$$\vec{r}(t) = \left\langle \sqrt{1+t}, \frac{\sin t}{2t}, (t+2)e^{-t} \right\rangle$$

at $t = 0$.

(a) $\left\langle 1, \frac{1}{2}, 2 \right\rangle$

(b) $\langle 0, 1, 2\rangle$

(c) $\langle 1, 1, 2\rangle$

(d) $\left\langle \frac{1}{\sqrt{2}}, 1, 2 \right\rangle$

(e) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{2}, 2 \right\rangle$

Solution. Since $\lim_{t\to 0}\vec{r}(t) = \left\langle \lim_{t\to 0} x(t), \lim_{t\to 0} y(t), \lim_{t\to 0} z(t)\right\rangle$. Here

(1) $\lim_{t\to 0}\sqrt{1+t} = \sqrt{1} = 1.$

(2) $\lim_{t\to 0}\dfrac{\sin(t)}{2t} = \dfrac{1}{2}$ by l'Hôpital's Rule.

(3) $\lim_{t\to 0}(t+2)e^{-t} = 2\cdot e^0 = 2.$

Hence $\lim_{t\to 0}\vec{r}(t) = \left\langle 1, \dfrac{1}{2}, 2\right\rangle$.

8.(5pts) Suppose the magnitude of the vector $\vec{a}$ is 2 and the magnitude of the vector $\vec{b}$ is 3 and the angle between them is 60°. Then the magnitude of the cross product $\vec{a}\times\vec{b}$ is

(a) $3\sqrt{3}$ (b) 5 (c) $2\sqrt{3}$ (d) $\sqrt{3}$ (e) $\sqrt{2}$

Solution. The magnitude is $|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\sin(\theta) = 2\cdot 3\sin\left(\dfrac{\pi}{3}\right) = 6\dfrac{\sqrt{3}}{2} = 3\sqrt{3}.$

Partial Credit

9.(10pts) Find the curvature of the function $y = 2\cos x$ at $x = \dfrac{\pi}{4}$.

Solution. $\vec{r}(x) = \langle x, 2\cos x, 0\rangle$; $\vec{r}'(x) = \langle 1, -2\sin x, 0\rangle$; $\vec{r}''(x) = \langle 0, -2\cos x, 0\rangle$. Then

$$\vec{r}'(x)\times\vec{r}''(x) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -2\sin x & 0 \\ 0 & -2\cos x & 0 \end{vmatrix} = \begin{vmatrix} -2\sin x & 0 \\ -2\cos x & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 1 & -2\sin x \\ 0 & -2\cos x \end{vmatrix}\vec{k} = (-2\cos x)\vec{k}.$$

Since $\cos\left(\dfrac{\pi}{4}\right) = \sin\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$, $\left|\vec{r}'\left(\dfrac{\pi}{4}\right)\times\vec{r}''\left(\dfrac{\pi}{4}\right)\right| = |-\sqrt{2}| = \sqrt{2}$ and $\left|\vec{r}'\left(\dfrac{\pi}{4}\right)\right| = |\langle 1, -\sqrt{2}, 0\rangle| = \sqrt{1+2} = \sqrt{3}$

$$\kappa\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{(\sqrt{3})^3} = \frac{\sqrt{2}}{3\sqrt{3}}$$

10.(10pts) A variable force of $30\cos(t)$ newtons acts on mass of 6 kilograms, beginning at $t = 0$ in the vertical direction: $\vec{F} = 30\cos t\vec{k}$. If the mass is initially moving with a velocity of $\langle 1, 0, 0\rangle$, find the **speed** at $t = \dfrac{\pi}{2}$.

Solution. $6\vec{a} = 30\cos t\vec{k}$ so $\vec{a} = 5\cos t\vec{k}$. Then $\vec{v}(t) = 5\sin(t)\vec{k} + \vec{C}$. Since $\vec{v}(0) = \langle 1, 0, 0\rangle$, $\vec{v}(t) = \vec{\imath} + 5\sin(t)\vec{k}$ and $\vec{v}\left(\dfrac{\pi}{2}\right) = \vec{\imath} + 5\sin\left(\dfrac{\pi}{2}\right)\vec{k} = \vec{\imath} + 5\vec{k}$ Hence the speed is $\sqrt{26}$.

11.(10pts) (i) Find a vector equation of the line through the two points $(1, 1, 1)$ and $(2, 2, 0)$.

(ii) Find an equation of the plane which contains the point $(1, 1, 1)$ and is perpendicular to the line in part (i).

(iii) Find the equation of the plane containing the line you found in part (i) and the point $(0, 0, -1)$.

Solution. A vector lying the line is $\vec{N} = \langle 2, 2, 0\rangle - \langle 1, 1, 1\rangle = \langle 1, 1, -1\rangle$. Hence an equation for this line is $\vec{L}(t) = t\,\langle 1, 1, -1\rangle + \langle 1, 1, 1\rangle$.

The vector $\vec{N} = \langle 1, 1, -1\rangle$ is perpendicular to the plane and $(1, 1, 1)$ is on the plane so

$$\vec{N} \bullet \langle x, y, z\rangle = \vec{N} \bullet \langle 1, 1, 1\rangle = 1$$

OR

$$x + y - z = 1$$

The vector $\langle 1, 1, 1\rangle - \langle 0, 0, -1\rangle = \langle 1, 1, 2\rangle$ is also in the plane so

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} \vec{\imath} - \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \vec{k} = \langle 3, -3, 0\rangle$$

is a normal vector to the plane and an equation is

$$\langle 3, -3, 0\rangle \bullet \langle x, y, z\rangle = \langle 3, -3, 0\rangle \bullet \langle 1, 1, 1\rangle = 0$$

or

$$x - y = 0$$

12.(10pts) Let P be the plane given by the equation

$$2x + y - z = 3,$$

and L be the line whose symmetric equation is

$$\frac{x}{1} = \frac{y-1}{-1} = \frac{z}{1}.$$

Do the line L and the plane P intersect? Justify your answer. If they do not intersect, then find the distance between them.

Solution. The vector equation for the line is

$$\langle 0, 1, 0\rangle + t\,\langle 1, -1, 1\rangle$$

If we plug this equation into the plane for the equation we get $2(t) - (t - 1) - (t) = 0\,t + 1$ which is never 3 so the line is parallel to the plane.

Pick a point on the line, say $\langle 0, 1, 0\rangle$. The distance from the line to the plane is the distance from the any point on this line and this is

$$\frac{|\,\langle 2, 1, -1\rangle \bullet \langle 0, 1, 0\rangle - 3|}{|\,\langle 2, 1, -1\rangle\,|} = \frac{|1 - 3|}{|\,\langle 2, 1, -1\rangle\,|} = \frac{|1 - 3|}{\sqrt{2^2 + 1^2 + (-1)^2}} = \frac{2}{\sqrt{6}}$$

13.(10pts) Given the curve $\vec{r}(t) = \left\langle \frac{1}{t}, t, \frac{t^2}{2} \right\rangle$, find the unit tangent vector, unit normal vector and unit binormal vector at the point $\left(1, 1, \frac{1}{2}\right)$.

Solution. The particle is at the point $\left(1, 1, \frac{1}{2}\right)$ when and only when $t = 1$.

$\vec{r}'(t) = \left\langle -\frac{1}{t^2}, 1, t \right\rangle$; $\vec{r}'(1) = \langle -1, 1, 1 \rangle$.

$\vec{r}''(t) = \left\langle 2\frac{1}{t^3}, 0, 1 \right\rangle$; $\vec{r}''(1) = \langle 2, 0, 1 \rangle$

Then $\vec{r}'(1) \times \vec{r}''(1) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \vec{\imath} - \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} \vec{k} = \langle 1, 3, -2 \rangle$.

Then $\vec{T}(1) = \frac{1}{\sqrt{3}} < -1, 1, 1 >$; $\vec{B}(1) = \frac{1}{\sqrt{14}} \langle 1, 3, -2 \rangle$.

A vector pointing in the same direction as $\vec{N} = \vec{B} \times \vec{T}$ is

$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 3 & -2 \\ -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \vec{\imath} - \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} \vec{k} = \langle 5, 1, 4 \rangle$ so $\vec{N}(1) = \frac{1}{\sqrt{42}} \langle 5, 1, 4 \rangle$.

14.(10pts) Give parametric equations for the tangent line to the curve $\vec{r}(t) = \langle -\sin t, \cos t, t^2 \rangle$ at the point where $t = \frac{\pi}{2}$.

Solution. The derivative of the curve is

$$\vec{r}'(t) = \langle -\cos t, -\sin t, 2t \rangle .$$

So the tangent vector at $t = \frac{\pi}{2}$ is $\langle 0, -1, \pi \rangle$. This is a direction vector for the tangent line. The point on the curve corresponding to $t = \frac{\pi}{2}$ is $\left(-1, 0, \frac{\pi^2}{4}\right)$. So a set of parametric equations of the line are

$$x = -1$$
$$y = -t$$
$$z = \frac{\pi^2}{4} + \pi t$$