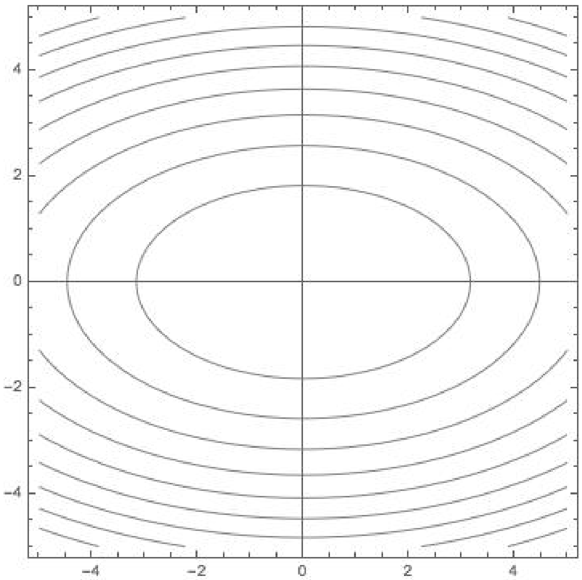
1.(5pts) Which function has the following level curves?



(a) $z = x^2 + 3y^2$ (b) $z = 3x^2 - y^2$ (c) $z = x + 3y^2$

(d) $z = 3x - 2y$ (e) $z = 3ye^x$

Solution. The only answer choice with ellipses as level curves is $z = x^2 + 3y^2$. The level curves of $z = 3x^2 - y^2$ are hyperbolas. The level curves of $z = x + 3y^2$ are parabolas. The level curves of $z = 3x - 2y$ are lines. The level curves of $z = 3ye^x$ are exponential functions.

2.(5pts) Let $H = xe^{y-z^2}$, $x = 2uv$, $y = u - v$ and $z = u + v$. Find $\dfrac{\partial H}{\partial u}$ when $u = 3$ and $v = -1$.

(a) 16 (b) 2 (c) 3

(d) -1 (e) 36

Solution. By the Chain Rule

$$\frac{\partial H}{\partial u} = \nabla H\Big|_{\langle 2uv, u-v, u+v\rangle} \bullet \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

Now $\left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u} \right\rangle = \langle 2v, 1, 1\rangle$ and $\nabla H = \left\langle e^{y-z^2}, xe^{y-z^2}, xe^{y-z^2}(-2z)\right\rangle$. When $u = 3$, $v = -1$, $\langle x, y, z\rangle = \langle -6, 4, 2\rangle$ so

$$\frac{\partial H}{\partial u}(3, -1) = \langle 1, -6, 24\rangle \bullet \langle -2, 1, 1\rangle = 16$$

3.(5pts) Compute $\dfrac{\partial z}{\partial x}$ at $(e, 1, 1)$ where z is implicitly defined by

$$e^{2z} = x^2 yz$$

(a) $\dfrac{2}{e}$ (b) $\dfrac{2}{e(e-4)}$ (c) $\dfrac{e(e-4)}{2}$

(d) $\dfrac{2}{e^2-4}$ (e) $\dfrac{e^2-4}{2}$

Solution. Taking the partial with respect to x on both sides gives

$$2e^{2z}\frac{\partial z}{\partial x} = 2xyz + x^2 y\frac{\partial z}{\partial x}\ ,$$

remembering to use the product rule on the RHS. Solving for $\dfrac{\partial z}{\partial x}$, we have

$$\frac{\partial z}{\partial x} = \frac{2xyz}{2e^{2z} - x^2 y}\ .$$

Plugging in $(e, 2, 1)$ we have

$$\frac{\partial z}{\partial x} = \frac{2e}{2e^2 - e^2} = \frac{2}{e}\ .$$

4.(5pts) Find the rate of change of the function $f(x, y) = 5y^2 \sin x$ at the point $\left(\frac{\pi}{2}, 1\right)$ in the direction of $\langle 4, -3\rangle$.

(a) -6 (b) -30 (c) 0

(d) 4 (e) 20

Solution. Calculate the gradient $\nabla f(x, y) = \langle 5y^2 \cos x, 10y \sin x\rangle$. So $\nabla f\left(\frac{\pi}{2}, 1\right) = \langle 0, 10\rangle$. The magnitude of the direction vector $\langle 4, -3\rangle$ is $\sqrt{16+9} = 5$. So the corresponding unit vector is $\vec{u} = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle$. Altogether,

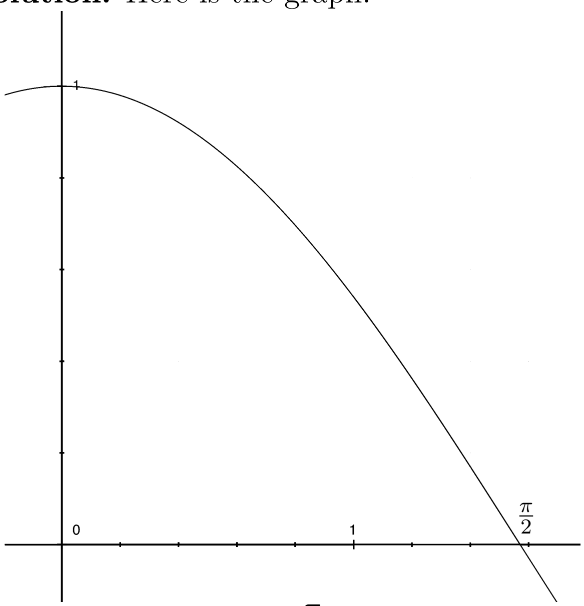
$$D_{\vec{u}} f\left(\frac{\pi}{2}, 1\right) = \langle 0, 10\rangle \bullet \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle = -6$$

5.(5pts) Which iterated integral below is equal to the double integral $\iint_D xe^y dA$ if D is the region in the first quadrant below the graph $y = \cos(x)$ and to the left of $x = \pi$?
Warning: Draw a graph of $y = \cos(x)$ to determine the region D.

(a) $\int_0^{\pi/2} \int_0^{\cos(x)} xe^y \, dy \, dx$ (b) $\int_0^{\pi} \int_0^{\cos(x)} xe^y \, dy \, dx$ (c) $\int_0^{\pi/2} \int_{\cos(x)}^{0} xe^y \, dy \, dx$

(d) $\int_0^{\pi} \int_{\cos(x)}^{0} xe^y \, dy \, dx$ (e) $\int_{-\pi/2}^{\pi/2} \int_0^{\cos(x)} xe^y \, dy \, dx$

Solution. Here is the graph:



The x limits are 0 to $\frac{\pi}{2}$. For a fixed x in this range, y goes from 0 to $\cos(x)$. Hence

$$\int_0^{\pi/2} \int_0^{\cos(x)} xe^y \, dy \, dx$$

6.(5pts) Evaluate

$$\int_0^1 \int_{-y}^{y^2} 24xy \, dx \, dy$$

(a) -1 (b) 0 (c) $-\frac{1}{4}$

(d) $\frac{1}{3}$ (e) -6

Solution.

$$12\int_0^1 x^2y\Big|_{-y}^{y^2} dy = 12\int_0^1 \left(y^4 - (-y)^2\right)y \, dy = 12\int_0^1 (y^5 - y^3)dy = 12\left(\frac{y^6}{6} - \frac{y^4}{4}\right)\Big|_0^1 = 2 - 3 = -1$$

7.(5pts) The limit $\lim_{(x,y)\to(0,0)} \dfrac{-4x^4 + x^3 + 3x^2 - 4x^2y^4 + xy^4 + xy^5 + 3y^4}{x^2 + y^4}$ exists. Which number below is the value of this limit?

(a) 3 (b) 4 (c) -4

(d) -3 (e) 0

Solution. Since you are told that the limit exists, you can evaluate it along any path to the origin. The path $x = 0$, $y = t$ looks easiest:

$$\lim_{(x,y)\to(0,0)} \frac{-4x^4 + x^3 + 3x^2 - 4x^2y^4 + xy^4 + xy^5 + 3y^4}{x^2 + y^4} = \lim_{t\to 0} \frac{3t^4}{t^4} = 3$$

but any path to the origin will do.

8.(5pts) Suppose the temperature at a point (x, y, z) in space is given by

$$x^2y^2 + e^{-z} .$$

At the point $(1, -1, 1)$, in what direction is the temperature increasing the fastest?

(a) In the direction of $\langle 2, -2, -e^{-1}\rangle$ (b) In the direction of $\langle 2, 2, -2e^{-2}\rangle$

(c) In the direction of $\langle -2e^{-1}, -2, 1\rangle$ (d) In the direction of $\langle 1, -1, 1\rangle$

(e) In the direction of $\langle 1, -1, 1\rangle \times \langle 2, 2, -2e^{-2}\rangle$

Solution. The gradient function is $\nabla f(x, y, z) = \langle 2xy^2, 2x^2y, -e^{-z}\rangle$. So the gradient at $(1, -1, 1)$ is $\langle 2, -2, -e^{-1}\rangle$, which gives the direction of the fastest rate of increase. The rate is the magnitude of this vector: $\sqrt{4 + 4 + e^{-2}} = \sqrt{8 + e^{-2}}$.

9.(10pts) At what point on the elliptic paraboloid

$$z = x^2 + y^2 - 4x - 6y + 16$$

is the tangent plane parallel to the plane $4x + 2y + z = 3$?

Solution. If $f(x, y, z) = x^2 + y^2 - 4x - 6y + 16 - z$, $\nabla f(x, y, z) = \langle 2x - 4, 2y - 6, -1\rangle$. A normal vector to the plane $4x + 2y + z = 3$ is $\langle 4, 2, 1\rangle$.

The two plane are parallel if and only if the two normal vectors are multiples of each other. If $\langle 2x - 4, 2y - 6, -1\rangle$ is a multiple of $\langle 4, 2, 1\rangle$ the multiple must be -1 from looking at the third coordinate. Then $2x - 4 = -4$ and $2y - 6 = -2$. Hence $x = 0$ and $y = 2$ are the x and y coordinates of this point on the surface. But then $z = 8$ so

$$(0, 2, 8)$$

is the point on the surface.

OR

As above $\langle 2x - 4, 2y - 6, -1\rangle$ and $\langle 4, 2, 1\rangle$ are parallel so $\langle 2x - 4, 2y - 6, -1\rangle \times \langle 4, 2, 1\rangle = \langle 0, 0, 0\rangle$. But

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2x-4 & 2y-6 & -1 \\ 4 & 2 & 1 \end{vmatrix} = \langle 2y - 6 + 2, -(2x - 4 + 4), 4x - 8 - (8y - 24)\rangle =$$

$$\langle 2y - 4, -2x, 4x - 8y + 16\rangle = \langle 0, 0, 0\rangle$$

Hence $y = 2$ and $x = 0$ from the first two equations and the third is also satisfied. Compute $z = 8$ as above.

10.(10pts) Find an equation for the tangent line to the curve of intersection of $x^2 + y^2 + z^2 = 25$ and $y = 2\sqrt{3}$ at $(3, 2\sqrt{3}, 2)$?

Solution. We already have a point on the line, namely $(3, 2\sqrt{3}, 2)$. Thus all we need is a direction vector, which will be of the form $\left\langle 1, 0, \frac{\partial z}{\partial x}\right\rangle$. So we must compute $\frac{\partial z}{\partial x}$. We begin by taking the partial with respect to x of the first equation treating z as a function of x,

$$2x + 2z\frac{\partial z}{\partial x} = 0\ .$$

Therefore $\frac{\partial z}{\partial x} = -\frac{x}{z}$ and thus $\frac{\partial z}{\partial x}(3, 2\sqrt{3}, 2) = -\frac{3}{2}$. So the tangent line is given by

$$l(t) = \left\langle 3, 2\sqrt{3}, 2\right\rangle + t\left\langle 1, 0, -\frac{3}{2}\right\rangle = \left\langle 3 + t, 2\sqrt{3}, 2 - \frac{3}{2}t\right\rangle$$

OR

There are two implicit surfaces, $f(x, y, z) = x^2 + y^2 + z^2 = 25$ and $g(x, y, z) = y = 2\sqrt{3}$. A vector in the tangent line is

$$\nabla f \times \nabla g = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2x & 2y & 2z \\ 0 & 1 & 0 \end{vmatrix} = \langle -2z, 0, 2x\rangle$$

evaluated at $\left\langle 3, 2\sqrt{3}, 2\right\rangle$ which is $\langle -4, 0, 6\rangle$. Hence

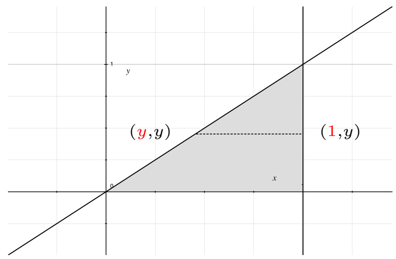
$$l(t) = \left\langle 3, 2\sqrt{3}, 2\right\rangle + t\,\langle -4, 0, 6\rangle$$

11.(10pts) Find the volume of the solid below the graph of $z = 12x^2 + 12y$ and above the region T, where T is the triangle in the xy-plane bounded by the lines $y = x$, $y = 0$, and $x = 1$.

(a) Write the volume as an iterated integral $dx\,dy$.

Solution:

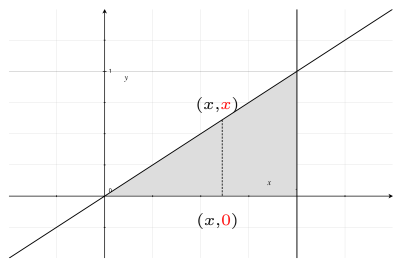
$$\int_0^1 \int_y^1 (12x^2 + 12y)dx\,dy$$



(b) Write the volume as an iterated integral $dy\,dx$.

Solution:

$$\int_0^1 \int_0^x (12x^2 + 12y)dy\,dx$$



(c) Evaluate the iterated integral in part (a) or the iterated integral in part (b). You should only do one of them.

Solution:

$$\int_0^1 \int_y^1 (12x^2 + 12y)dx\,dy = \int_0^1 \left(4x^3 + 12xy\right)\Big|_y^1\, dy = \int_0^1 (4 + 12y - 4y^3 - 12y^2)\,dy =$$

$$4y + 6y^2 - y^4 - 4y^2\Big|_0^1 = 4 + 6 - 1 - 4 = 5$$

OR

$$\int_0^1 \int_0^x (12x^2+12y)dy\,dx = \int_0^1 \left(12x^2y + 6y^2\right)\Big|_0^x\, dy = \int_0^1 \left(12x^3 + 6x^2\right)\,dx = 3x^4 + 2x^3\Big|_0^1 = 5$$

12.(10pts) Define a function $f(x, y) = 3xy - x^3 - y^3$.

(a) Find all the critical points of $f(x, y)$.

Solution:

$$\nabla f = \left\langle 3y - 3x^2, 3x - 3y^2 \right\rangle = \langle 0, 0\rangle$$

so $3y - 3x^2 = 0$, and $3x - 3y^2 = 0$. Hence $y = x^2$ and $x - x^4 = 0$. Hence $x = 0$ or $x = 1$. So the critical points are

$$(0,0) \quad \text{and} \quad (1,1)$$

(b) Use the Second Derivative Test to identify the local minima, local maxima and saddle points of f.

Solution:

$$\mathcal{H}_f(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -6x & 3 \\ 3 & -6y \end{vmatrix}$$

$\mathcal{H}_f(0,0) = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 < 0$ so $(0,0)$ is a saddle point.

$\mathcal{H}_f(1,1) = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 27 > 0$. Since the diagonal elements are negative, $(1,1)$ is a local maximum.

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13. (10pts) Find the maximum value of the function x^2y along the ellipse $3x^2 + y^2 = 27$, *using the method of Lagrange multipliers.*

Solution. Let $f(x, y) = x^2y$ and let the constraint equation be $g(x, y) = 3x^2 + y^2 = 27$.
The Lagrange multiplier equations are

$$\begin{aligned} \nabla f =& \lambda \nabla g \\ g(x, y) =& 1 \end{aligned}$$

or

$$2xy = \lambda 6x \tag{1}$$

$$x^2 = \lambda 2y \tag{2}$$

$$3x^2 + y^2 = 27 \tag{3}$$

If $x = 0$, then $y = \pm\sqrt{27}$ and $\lambda = 0$ solve the equations.
If $y = 0$, (2) implies $x = 0$ and $(0, 0)$ is not on the constraint curve.
Any other solutions must have $x \neq 0$ and $y \neq 0$. Then equation (1) above becomes (4) below.

$$y = 3\lambda \tag{4}$$

Equations (4) and (2) imply $3x^2 = 2y^2$ and then (3) implies $3y^2 = 27$ or $y = \pm 3$.
If $y = 3$, then $\lambda = 1$, $x = \pm\sqrt{6}$ satisfy all three equations.
If $y = -3$, then $\lambda = -1$, $x = \pm\sqrt{6}$ satisfy all three equations.
Evalating $f(x, y) = x^2y$ at points, $f(0, \pm\sqrt{27}) = 0$; $f(\pm\sqrt{6}, 3) = 6 \cdot 3 = 18$ and $f(\pm\sqrt{6}, -3) = 6 \cdot (-3) = -18$. Hence the maximum value is 18.