Multiple Choice

1.(5pts) Compute the Jacobian, $\dfrac{\partial(x,y)}{\partial(u,v)}$, of the coordinate transformation $x = u^2 - v^4$, $y = uv$.

(a) $2u^2 + 4v^4$ (b) $xu - yv$ (c) $3u^2 + 7v^6$

(d) $2u^2$ (e) $u^2v - uv^4$

Solution. $\begin{vmatrix} 2u & -4v^3 \\ v & u \end{vmatrix} = 2u^2 + 4v^4$.

2.(5pts) Find $\iint_D \sin(x^2 + y^2)\, dA$ where D is the disk centered at the origin of radius a.

(a) $\pi\big(1 - \cos(a^2)\big)$ (b) $2\pi\cos(a^2)$ (c) $\pi\big(\sin(a^2) - \cos(a^2)\big)$

(d) $2\pi \arcsin(a)$ (e) $2\pi \arccos(a)$

Solution. Convert to iterated polar integral: $\iint_D \sin(x^2+y^2)\, dA = \int_0^{2\pi}\int_0^a \sin(r^2) r\, dr\, d\theta = \int_0^{2\pi} -\frac{\cos(r^2)}{2}\Big|_0^a d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{\cos(a^2)}{2} d\theta = \pi\big(1 - \cos(a^2)\big)$

3.(5pts) Let C be a thin wire lying as the straight line $x+y=2$ in the first quadrant. Suppose the charge density on the wire is given by $q(x,y)=x-3y$. Then the total charge on the wire is $\int_C q\,ds$. Which number below is the total charge?

(a) $-4\sqrt{2}$ (b) 0 (c) $-2\sqrt{2}$

(d) 6 (e) $3\sqrt{2}$

Solution. To parametrize the wire we can use $\vec{r}(t)=\langle 2-t,t\rangle$, $0\leqslant t\leqslant 2$. charge $=\displaystyle\int_C q\,ds$. Furthermore $ds=|\langle -1,1\rangle|\,dt=\sqrt{2}\,dt$ so

$$\text{charge}=\int_0^2 \big((2-t)-3t\big)\sqrt{2}\,dt=\int_0^2(2-4t)\sqrt{2}\,dt=\sqrt{2}(2t-2t^2)\Big|_0^2=\sqrt{2}(-4-0)=-4\sqrt{2}$$

4.(5pts) The density function of the quarter of the disk of radius 3 in the first quadrant is given by the formula

$$\rho(r,\theta)=\cos(\theta).$$

Which iterated integral below gives the moment about the x-axis of the quarter of this disk in the first quadrant.

(a) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^3 r^2\sin(\theta)\cos(\theta)\,dr\,d\theta$ (b) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^3 r\sin(\theta)\cos(\theta)\,dr\,d\theta$ (c) $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^3 r^2\sin(\theta)\cos(\phi)\,dr\,d\theta\,d\phi$

(d) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^3 r^2\sin(\theta)\cos^2(\theta)\,dr\,d\theta$ (e) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^3 r^3\sin(\theta)\cos^2(\theta)\,dr\,d\theta$

Solution.

$$\iint_D y\rho\,dA=\int_0^{\frac{\pi}{2}}\int_0^3 r\sin(\theta)\cos(\theta)r\,dr\,d\theta$$

The answer, if you care, is

$$\int_0^{\frac{\pi}{2}}\frac{r^3}{3}\Big|_0^3\sin(\theta)\cos(\theta)\,d\theta=\int_0^{\frac{\pi}{2}}9\sin(\theta)\cos(\theta)\,d\theta=9\frac{\sin^2(\theta)}{2}\Big|_0^{\frac{\pi}{2}}=\frac{9}{2}$$

5.(5pts) Let A be a thin plate in the xy plane. The precise shape is irrelevant. Suppose the mass of A is 5 and that the center of mass of A is at $(3, 2)$. What is the moment of A about the x-axis?

(a) 10 (b) 15 (c) $\frac{2}{5}$

(d) $\frac{3}{5}$ (e) $\frac{6}{25}$

Solution. $\bar{y} = \dfrac{M_x}{mass}$ so $2 = \dfrac{M_x}{5}$.

6.(5pts) Let $\vec{F}(x, y) = \langle y, 4x\rangle$. Calculate $\int_C \vec{F} \bullet d\vec{r}$ where C is the curve $\vec{r}(t) = \langle t^2, t\rangle$, $0 \leqslant t \leqslant 1$.

(a) 2 (b) 4 (c) -2

(d) -4 (e) $\frac{4}{3}$

Solution. $dr = \langle 2t, 1\rangle\, dt$ and $\vec{F}(t) = \langle t, 4t^2\rangle$ and $\int_C \vec{F} \bullet dr = \int_0^1 \left\langle t, 4t^2 \right\rangle \bullet \langle 2t, 1\rangle\, dt = \int_0^1 (2t^2 + 4t^2)\, dt = \int_0^1 6t^2\, dt = 2t^3\Big|_0^1 = 2.$

7.(5pts) Let E be the tetrahedron bounded by the planes $2x+y+z=6$, $x-y=0$, $x=0$, and $z=0$. Write a 2-fold iterated integral with respect to $dy\,dx$, which computes the volume of E.

(a) $\int_0^2 \int_x^{6-2x} (6-2x-y)\, dy\, dx$

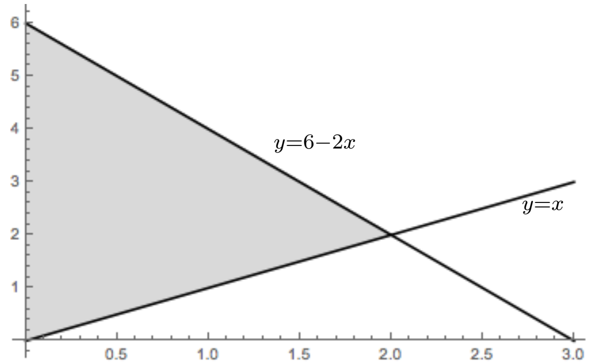
(b) $\int_0^6 \int_x^6 (6-2x-y)\, dy\, dx$

(c) $\int_0^6 \int_x^{6-2x} (6-2x-y)\, dy\, dx$

(d) $\int_0^6 \int_x^{6-2x} (6-2x-z)\, dy\, dx$

(e) $\int_0^6 \int_x^2 (6-2x-y)\, dy\, dx$

Solution. The region of integration for the double integral is the triangle determined by the lines obtained by intersecting $2x+y+z=6$, $x=y=0$, and $x=0$ with the xy-plane. These lines are $y=x$ and $y=6-2x$. The region of integration is



The point of intersection of the two lines is $(2,2)$. The tetrahedron is the solid under the plane $z=6-2x-y$ and above $z=0$. So the iterated integral is

$$\int_0^2 \int_x^{6-2x} 6-2x-y\ dy\, dx$$

8.(5pts) Evaluate $\iint_R \sqrt{4x^2+9y^2}\, dA$, where R is the region bounded by the ellipse $4x^2+9y^2=36$.

(a) 24π

(b) $\frac{1}{6}\pi$

(c) 0

(d) $6\pi^2$

(e) 18

Solution.

If we write $u = 2x$ and $w = 3y$, the region becomes the interior of $u^2 + w^2 = 6^2$, in other words, the disk centered at the origin of radius 6, say D.

Since $\det \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$,

$$\iint_R (4x^2 + 9y^2)\, dA = \frac{1}{6} \iint_D \sqrt{u^2 + w^2}\, dA = \frac{1}{6} \int_0^6 \int_0^{2\pi} (r) r \, d\theta \, dr = \frac{1}{6} \int_0^6 2\pi r^2 \, dr =$$

$$\frac{\pi}{3} \frac{r^3}{3} \bigg|_0^6 = \frac{\pi}{3} \frac{6^3}{3} = \pi \cdot 3 \cdot 2^3 = 24\pi$$

Partial Credit

9.(10pts) Find the mass of a thin wire in the shape of a parabola $x = 1 - y^2$, $0 \leqslant y \leqslant \sqrt{2}$, if the density is given by $\rho(x, y) = 6y$.

Solution. The mass is $\displaystyle\int_C \rho\, ds$ where the curve C can be parametrized by $\vec{r}(y) = \langle 1 - y^2, y\rangle$, $0 \leqslant y \leqslant \sqrt{2}$. $ds = |\vec{r}\,'(y)|\, dy = |\langle -2y, 1\rangle|\, dy = \sqrt{1 + 4y^2}\, dy$. Hence the mass is

$$\int_0^{\sqrt{2}} 6y\sqrt{1+4y^2}\, dy \overset{u=1+4y^2}{=} \frac{6}{8}\int_1^9 \sqrt{u}\, du = \frac{3}{4}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}}\Big|_1^9 = \frac{1}{2}(27-1) = \frac{26}{2} = \boxed{13}$$

10.(10pts) Rewrite the following integral

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_{\sqrt{\frac{x^2+y^2}{3}}}^{\sqrt{4-x^2-y^2}} \frac{y\sqrt{x^2+y^2+z^2}}{z}\, dz\, dy\, dx\ .$$

(a) (6pts) Using cylindrical coordinates.

Solution:
Notice the domain of integration is a quarter of a cone in the first octant and the octant with negative x and positive y and z.
Because we are in these octants $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant r \leqslant 2\sqrt{3}$. Now for cylindrical coordinates z remains unchanged. Therefore

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_{\sqrt{\frac{x^2+y^2}{3}}}^{\sqrt{4-x^2-y^2}} \frac{y\sqrt{x^2+y^2+z^2}}{z}\, dz\, dy\, dx$$
$$= \int_0^{\pi} \int_0^{\sqrt{3}} \int_{\frac{r}{\sqrt{3}}}^{\sqrt{4-r^2}} r\sin(\theta)\, \frac{\sqrt{r^2+z^2}}{z} r\, dz\, dr\, d\theta\ .$$

(b) (6pts) Using spherical coordinates.

Solution:

Now $z = \sqrt{4 - x^2 - y^2}$ is a sphere of radius 2 so $0 \leqslant \rho \leqslant 2$. $z = \sqrt{\frac{x^2+y^2}{3}}$ is a cone at an angle of $\frac{\pi}{3}$ from the positive z-axis. So

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_{\sqrt{\frac{x^2+y^2}{3}}}^{\sqrt{4-x^2-y^2}} \frac{y\sqrt{x^2+y^2+z^2}}{z}\, dz\, dy\, dx$$
$$= \int_0^{\frac{\pi}{3}} \int_0^{\pi} \int_0^2 \frac{\rho^2\sin(\theta)\sin(\phi)}{\rho\cos(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$
$$= \int_0^{\frac{\pi}{3}} \int_0^{\pi} \int_0^2 \rho^3\, \sin(\theta)\frac{\sin^2(\phi)}{\cos(\phi)}\, d\rho\, d\theta\, d\phi$$

11.(10pts) Evaluate $\displaystyle\iint_R \frac{2x-3y}{4x-5y}\,dA$, where R is the parallelogram in the xy-plane bounded by $2x-3y=0$, $2x-3y=5$, $4x-5y=1$, $4x-5y=8$.

Solution. The form of the region and the integrand suggests trying $u = 2x-3y$ and $w = 4x-5y$. The boundary in uw space becomes $u=0$, $u=5$, $w=1$, $w=8$. Let W be this region.

The Jacobian calculation is

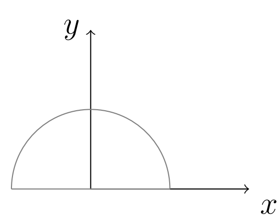
$$\frac{\partial(u,w)}{\partial(x,y)} = \left|\det\begin{vmatrix} 2 & -3 \\ 4 & -5 \end{vmatrix}\right| = 2 \quad\text{and}\quad \frac{\partial(x,y)}{\partial(u,w)} = \frac{1}{2}$$

so

$$\iint_R \frac{2x-3y}{4x-5y}\,dA = \frac{1}{2}\iint_W \frac{u}{w}\,dA = \frac{1}{2}\int_0^5\int_1^8 \frac{u}{w}\,dw\,du = \frac{1}{2}\int_0^5 u\ln(w)\Big|_1^8\,du =$$

$$\frac{\ln(8)}{2}\frac{u^2}{2}\Big|_0^5 = \boxed{\frac{25}{4}\ln(8)}$$

12.(10pts) The density of the semi-disk



of radius 2 at (x,y) is $\rho(x,y) = x^2+y^2$. Find the center of mass given that the mass is 4π.

Solution. By symmetry, the density and the region are symmetric with respect to the y-axis so $\overline{x} = 0$. Next mass $= \displaystyle\iint_D (x^2+y^2)\,dA$ and $M_x = \displaystyle\iint_D y(x^2+y^2)\,dA$.

Polar seems a good way to go so

$$\text{mass} = \int_0^\pi\int_0^2 r^2 r\,dr\,d\theta = \pi\frac{r^4}{4}\Big|_0^2 = 4\pi \quad \text{as given.}$$

$$M_x = \int_0^\pi\int_0^2 r\sin(\theta)r^2 r\,dr\,d\theta = \int_0^\pi \frac{r^5}{5}\Big|_0^2 d\theta = \frac{32}{5}\int_0^\pi \sin(\theta)\,d\theta = \frac{32}{5}\left(-\cos(\theta)\Big|_0^\pi\right) = \frac{64}{5}$$

Then $\overline{y} = \dfrac{\frac{64}{5}}{4\pi} = \dfrac{64}{20\pi} = \dfrac{16}{5\pi}$

13.(10pts) What is the work done by the force field $\vec{F}(x,y) = x\vec{\imath} + 2y\vec{\jmath}$ to move a particle from $(0,0)$ to $\left(\frac{\pi}{2}, 1\right)$ along the curve $\vec{r}(t) = \langle t, \sin t\rangle$?
HINT: Work done = Force • Displacement.

Solution. $\displaystyle\int_C \vec{F} \bullet d\vec{r} = \int_0^{\frac{\pi}{2}} \langle t, 2\sin(t)\rangle \bullet \langle 1, \cos(t)\rangle\ dt = \int_0^{\frac{\pi}{2}} t + 2\sin(t)\cos(t)\, dt =$
$\displaystyle\frac{t^2}{2} + \sin^2(t)\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{8} + 1.$