Here is a statement of Green's Theorem. It involves regions and their boundaries. In order have any hope of doing calculations, you must see the region as the set of points described by inequalities involving at least level curves and often by graphs. A *smooth region* is a region given by a finite collection of inequalities of differentiable functions. With a little algebra one can always assume that these inequalities are of the form $f_i(x, y) \geqslant 0$. The boundary of D is the set of points where at least one of the inequalities is an equality.

The precise generality in which Green's Theorem holds is still an area of theoretical research but you have no hope of doing any calculations except for smooth bounded regions.

Let D be a smooth bounded region in the plane. Let $\langle M, N\rangle$ be a field defined on D such that $\dfrac{\partial N}{\partial x}$ and $\dfrac{\partial M}{\partial y}$ are continuous on D. Then

$$(GT) \qquad \oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

> The left hand side of (GT) depends on an orientation of the curve; the right hand side does not. Hence for the result to be true, ∂D must be oriented correctly. The correct orientation is the usual one: if you stand at a point on the boundary with the region on your left, you are facing in the preferred direction. Hereafter, ∂D denotes the boundary curve with this orientation.

Remark. It is important that the needed partials exist on all of D.

For the field $\mathbf{v} = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right\rangle$, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 0$ on the punctured plane but not on all of the plane, even though the right hand side is a perfectly good function on all of $\mathbb{R}^2$. If you forget this issue you may think $2\pi = \oint_{\partial D} \mathbf{v} \bullet \mathbf{r} = \iint_D 0\,dA = 0$ which is just wrong.

Here is one way to remember the function in the double integral. Look at the determinant $(*)$ below. It makes no sense without further interpretation. Write $\det\begin{vmatrix} a & b \\ c & d\end{vmatrix} = ad - bc$, being careful with the order of the terms. Then $\dfrac{\partial}{\partial x}N$ is defined to be $\dfrac{\partial N}{\partial x}$ and $\dfrac{\partial}{\partial y}M$ is defined to be $\dfrac{\partial M}{\partial y}$. Then we have

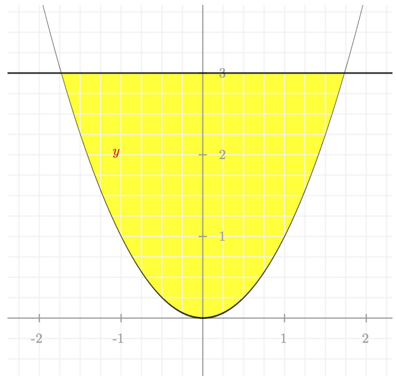
$$(*) \qquad \det\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ M & N \end{vmatrix} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

so Green's Theorem says

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\det\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ M & N \end{vmatrix}\right) dA$$

This is not particularly useful for calculations but if you are suddenly not sure what goes under the double integral sign, $(*)$ may help you remember (especially after we get to Stokes Theorem).

Example. Replace a line integral by a double integral: $\langle xy, x^2\rangle$ over D the region above the parabola $y = x^2$ and below $y = 3$.



Note the yellow region can be described as $y - x^2 \geqslant 0$ and $3 - y \geqslant 0$ so we have a smooth region which is clearly bounded and the boundary is the part of $y - x^2 = 0$ in D union the part of $3 - y = 0$ in D.

Then $\oint_{\partial D} xy\,dx + x^2\,dy = \iint_D (2x - x)\,dA = \iint_D x\,dA = 0$. You should be able to explain why I know that the double integral is 0 without doing any work. Click here for why.

To check Green's Theorem, let us do two line integrals $\int_{C_1} xy\,dx + x^2\,dy$ and $\int_{C_2} xy\,dx + x^2\,dy$, where C_1 is the line segment along the top and C_2 is the parabola. Then C_1 is parametrized by $\mathbf{r}_1(t) = \langle t, 3\rangle$, $-\sqrt{3} \leqslant t \leqslant \sqrt{3}$ and C_2 is parametrized by $\mathbf{r}_2(t) = \langle t, t^2\rangle$, $-\sqrt{3} \leqslant t \leqslant \sqrt{3}$

$$\int_{C_1} xy\,dx + x^2\,dy = \int_{-\sqrt{3}}^{\sqrt{3}} \left\langle 3t, t^2\right\rangle \bullet \langle 1, 0\rangle\ dt = \int_{-\sqrt{3}}^{\sqrt{3}} 3t\,dt = \left.\frac{3t^2}{2}\right|_{-\sqrt{3}}^{\sqrt{3}} = \frac{9}{2} - \frac{9}{2} = 0$$

$$\int_{C_2} xy\,dx + x^2\,dy = \int_{-\sqrt{3}}^{\sqrt{3}} \left\langle t^3, t^2\right\rangle \bullet \langle 1, 2t\rangle\ dt = \int_{-\sqrt{3}}^{\sqrt{3}} 3t^3\,dt = \left.\frac{3t^4}{4}\right|_{-\sqrt{3}}^{\sqrt{3}} = \frac{81}{4} - \frac{81}{4} = 0$$

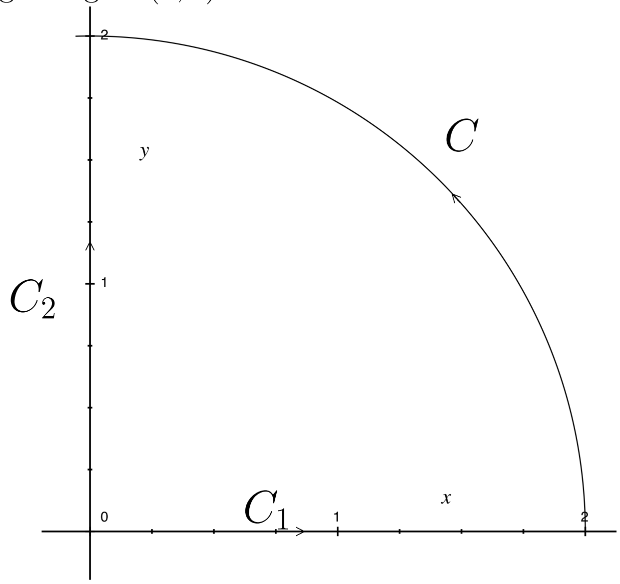
Strictly speaking,

$$\oint_{\partial D} xy\,dx + x^2\,dy = \int_{-C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy = -\int_{C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy$$

but since both line integrals are 0 you can't really tell in this example.

Another way to apply Green's Theorem is to do a line integral along a non-closed path, say C. The trick is to find another path C_1 such that $C \cup C_1 = \partial D$ for some region D and then $\int_C \langle P, Q\rangle \bullet d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA - \int_{C_2} \langle P, Q\rangle \bullet d\mathbf{r}$ and sometimes the two integrals on the right are easier than the original.

Example. Integrate $\langle x^5 + y, 2x - y^3\rangle$ along the quarter circle of radius 2 centered at the origin and lying in the first quadrant, beginning at $(2, 0)$.



Here C is our quarter circle, C_1 goes from the origin to $(2, 0)$ and C_2 goes from the origin to $(0, 2)$. Let D be the quarter disk so $\partial D = C_1 \cup C \cup -C_2$.

You can set up

$$\int_C \left\langle x^5 + y, 2x - y^3 \right\rangle \bullet d\mathbf{r} = \int_0^{\pi/2} \left\langle 32\cos^5(t) + 2\sin(t), 4\cos(t) - 8\sin^3(t) \right\rangle \bullet \langle -2\sin(t), 2\cos(t)\rangle \, dt$$

which looks messy.

Now $\dfrac{\partial\,(2x - y^3)}{\partial\, x} - \dfrac{\partial\,(x^5 + y)}{\partial\, y} = 2 - 1 = 1$, so $\displaystyle\iint_D \left(\frac{\partial\,(2x - y^3)}{\partial\, x} - \frac{\partial\,(x^5 + y)}{\partial\, y} \right) dA = \pi$.

$$\int_{C_1} \left\langle x^5 + y, 2x - y^3 \right\rangle \bullet d\mathbf{r} = \int_0^2 \left\langle t^5 + 0, 2t - 0^3 \right\rangle \bullet \langle 1, 0\rangle \, dt = \int_0^2 t^5 dt = \frac{64}{6} = \frac{32}{3}$$

$$\int_{C_2} \left\langle x^5 + y, 2x - y^3 \right\rangle \bullet d\mathbf{r} = \int_0^2 \left\langle 0 + t, 2 \cdot 0 - t^3 \right\rangle \bullet \langle 0, 1\rangle \, dt = \int_0^2 -t^3 dt = -\frac{16}{4} = -4$$

Hence

$$\int_C \left\langle x^5 + y, 2x - y^3 \right\rangle \bullet d\mathbf{r} = \pi - \frac{32}{3} + (-4) = \pi - \frac{44}{3}$$

1. Area

The other obvious direction is to replace a double integral with a line integral. Given $\iint_D F(x,y)\,dA$ if you can find a field $\langle M, N\rangle$ such that $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = F(x,y)$ for every point in D then you can do $\oint_{\partial D} \langle M, N\rangle \bullet d\mathbf{r}$ instead.

Example. Find $\iint_D \sin(x)\,dA$ where D is the region below $y = \sin x$ and above the x axis between $x = 0$ and $x = \pi$. Take $\mathbf{v} = \langle y\sin(x), 0\rangle$.

$\iint_D \sin(x)\,dA = -\int_{-C_1} \langle y\sin(x), 0\rangle \bullet \langle 1, \cos x\rangle\,dx + \int_{C_2} \langle y\sin(x), 0\rangle \bullet \langle 1, 0\rangle\,dx$ where C_1 is the curve $\langle x, \sin(x)\rangle$, $0 \leqslant x \leqslant \pi$ and C_2 is the line segment $\langle x, 0\rangle$, $0 \leqslant x \leqslant \pi$.

$$\iint_D \sin(x)\,dA = -\int_0^\pi \sin^2(t)\,dt + \int_0^\pi 0\,dt = -\left.\frac{t - \frac{\cos(2t)}{2}}{2}\right|_0^\pi = -\left(\frac{\pi - \frac{1}{2}}{2}\right) - \left(-\frac{0 - \frac{1}{2}}{2}\right) = \frac{\pi}{2}$$

The granddaddy of this sort of application is to find area (and less often centroids). Recall that the centroid of a region is the center of mass of that region assuming uniform density.

If $\mathbf{v} = \frac{1}{2}\langle -y, x\rangle$ or $\langle -y, 0\rangle$ or $\langle 0, x\rangle$,

$$\frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy = \oint_{\partial D} -y\,dx = \oint_{\partial D} x\,dy$$

is the area of D.

Note $\frac{1}{2}\oint_{\partial D} \left\langle 0, x^2\right\rangle \bullet d\mathbf{r}$ is the moment about the y axis and $\frac{1}{2}\oint_{\partial D} \left\langle -y^2, 0\right\rangle \bullet d\mathbf{r}$ is the moment about the x axis. Other choices for the field will occur to you.

Example. Find the area of the polygon, $\langle x_0, y_0\rangle$, $\langle x_1, y_1\rangle$, $\ldots$, $\langle x_n, y_n\rangle$, where we list the points in order starting somewhere and traversing the polygon counterclockwise. Assume that the points are distinct and that the curve formed by connecting the points with line segments in the order given and then returning from the last listed point to the starting point is a polygon. Equivalently, none of the line segments intersect except possible in one point which is an end point for each segment. Call the area A. Let C_i be the line segment from $\langle x_{i-1}, y_{i-1}\rangle$ to $\langle x_i, y_i\rangle$, $1 \leqslant i \leqslant n$ and let C_{n+1} be the line segment from $\langle x_n, y_n\rangle$ to $\langle x_0, y_0\rangle$ needed to closeup the polygon. The notation works out better if we define $\langle x_{n+1}, y_{n+1}\rangle = \langle x_0, y_0\rangle$ and then C_{n+1} has the same definition as all the others.

By Green's Theorem $A = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n+1}\int_{C_i} \langle -y, x\rangle \bullet d\mathbf{r}$

Next let us work out the line integral for an arbitrary line segment.

$$C = (1-t)\langle a, b\rangle + t\langle c, d\rangle = \langle a - at + ct, b - bt + dt\rangle$$

$$\int_C \langle -y, x\rangle \bullet d\mathbf{r} = \int_0^1 \langle bt - dt - b, a + ct - at\rangle \bullet \langle -a + c, -b + d\rangle\, dt =$$

$$\int_0^1 \big((b-d)(c-a)t - b(c-a) + (c-a)(d-b)t + (a(d-b)\big)\, dt =$$

$$\int_0^1 \big((b-d)(c-a) + (c-a)(d-b)\big)t - b(c-a) + (a(d-b)\big)\, dt =$$

$$\int_0^1 \big((b-d)(c-a) + (c-a)(d-b)\big)t + \big(-bc + ad\big)\, dt =$$

$$\int_0^1 \big(-bc + ad\big)\, dt = -bc + ad = \det\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Hence, if $\langle x_0, y_0\rangle$, $\langle x_1, y_1\rangle$, $\ldots$, $\langle x_n, y_n\rangle$, $\langle x_{n+1}, y_{n+1}\rangle = \langle x_0, y_0\rangle$ is a polygon in the plane traversed counterclockwise,

$$2\text{Area} = \sum_{i=1}^{n+1} \det\begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix}$$

$$\frac{1}{2}\int_C \langle 0, x^2\rangle \bullet d\mathbf{r} = \frac{1}{2}\int_0^1 \left\langle 0, \big((a + (c-a)t\big)^2\right\rangle \bullet \langle c - a, d - b\rangle\, dt =$$

$$\frac{1}{2}\int_0^1 \big((a + (c-a)t\big)^2 (d-b)\, dt = \frac{(d-b)}{2}\int_0^1 \big((a + (c-a)t\big)^2\, dt =$$

$$\frac{(d-b)}{2}\int_0^1 a^2 + 2a(c-a)t + (c-a)^2 t^2\, dt = \frac{(d-b)}{2}\left(a^2 + a(c-a) + \frac{(c-a)^2}{3}\right) =$$

$$\frac{(d-b)}{2}\frac{(3ac + (c-a)^2)}{3} = \frac{(d-b)(a^2 + ac + c^2)}{6}$$

Hence

$$6\cdot M_y = \sum_{i=1}^{n+1}(y_i - y_{i-1})(x_i^2 + x_i x_{i-1} + x_{i-1}^2) = \sum_{i=1}^{n+1}(x_i^2 y_i + x_i x_{i-1} y_i + x_{i-1}^2 y_i - x_i^2 y_{i-1} - x_i x_{i-1} y_{i-1} - x_{i-1}^2 y_{i-1})$$

The monomial $x_i^2 y_i$ will be canceled by the same monomial with a minus sign in the next term so

$$6\cdot M_y = \sum_{i=1}^{n+1}(y_i - y_{i-1})(x_i^2 + x_i x_{i-1} + x_{i-1}^2) = \sum_{i=1}^{n+1}(x_i x_{i-1} y_i + x_{i-1}^2 y_i - x_i^2 y_{i-1} - x_i x_{i-1} y_{i-1})$$

Now check

$$6 \cdot M_y = \sum_{i=1}^{n+1} (x_i + x_{i-1}) \det \begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix}$$

You may use any of a number of arguments to see

$$6 \cdot M_x = \sum_{i=1}^{n+1} (y_i + y_{i-1}) \det \begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix}$$

so

$$\bar{x} = \frac{1}{3} \frac{\displaystyle\sum_{i=1}^{n+1} (x_i + x_{i-1}) \det \begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix}}{\displaystyle\sum_{i=1}^{n+1} \det \begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix}} \qquad \bar{y} = \frac{1}{3} \frac{\displaystyle\sum_{i=1}^{n+1} (y_i + y_{i-1}) \det \begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix}}{\displaystyle\sum_{i=1}^{n+1} \det \begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix}}$$

These formulas can be found in old books on surveying and new books on computer graphics.

There is an instrument called a planimeter which is used to compute the area enclosed by a plane curve. The book and the web site have additional material on planimeters showing that they are just mechanical manifestations of Green's method for computing area. There are also some apps calling themselves "planimeters" which measure area of a region on a google map after you put some pins around the region. I have not reverse engineered the software but I strongly suspect that these apps just compute the area of the polygon using the formula above.

2. Further applications

Going from line integrals around a boundary to double integrals is straightforward, just compute $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$. Going from a double integral to a line integral is more interesting since you need to find a field such that $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$ is your function and there are always lots of choices.

There are some other sorts of applications which are more subtle. If $\langle M_0, N_0 \rangle$ and $\langle M_1, N_1 \rangle$ have continuous partials on D and if $\dfrac{\partial N_1 - N_0}{\partial x} - \dfrac{\partial M_1 - M_0}{\partial y} = 0$, then $\displaystyle\oint_{\partial D} \langle M_0, N_0 \rangle \bullet d\mathbf{r} = \oint_{\partial D} \langle M_1, N_1 \rangle \bullet d\mathbf{r}$ since they are both equal to $\displaystyle\iint_D \left(\frac{\partial N_0}{\partial x} - \frac{\partial M_0}{\partial y} \right) dA$.

Example. Integrate $\displaystyle\oint_C (y + e^{\sin(x^2)})\, dx + (xy + e^{\cos(y^2)})\, dy$ counterclockwise around the unit circle. Note $\dfrac{\partial\, y + e^{\sin(x^2)}}{\partial y} = \dfrac{\partial\, y}{\partial y}$ and $\dfrac{\partial\, xy + e^{\cos(y^2)}}{\partial x} = \dfrac{\partial\, xy}{\partial x}$ so

$$\oint_C (y+e^{\sin(x^2)})\, dx+(xy+e^{\cos(y^2)})\, dy = \oint_C y\, dx+xy\, dy = \int_0^{2\pi} \sin(t)\big(-\sin(t)\big)+\cos(t)\sin(t)\cos(t)\big)dt =$$

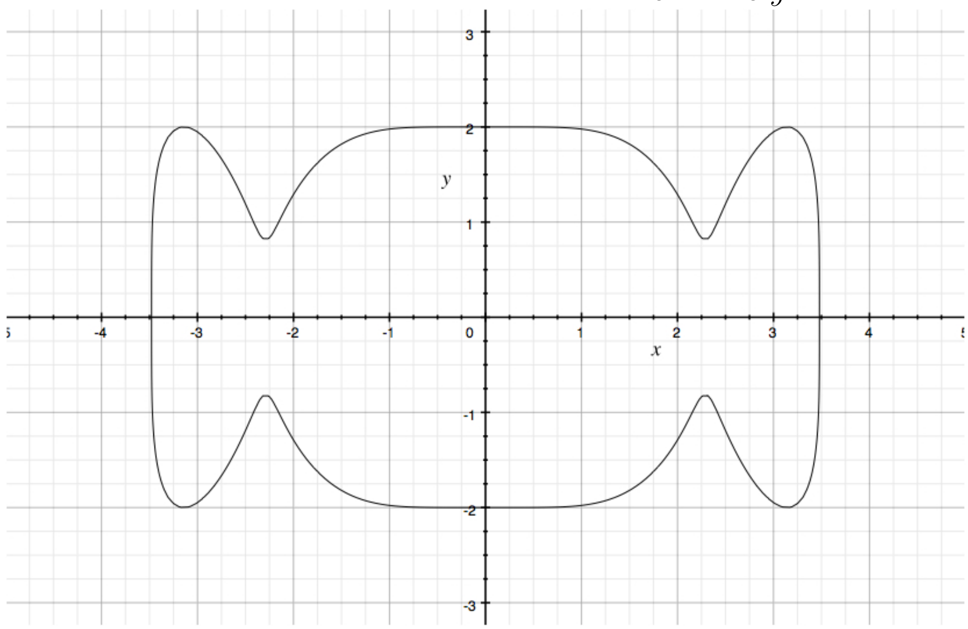
$$\int_0^{2\pi} \cos^2(t)\sin(t) - \sin^2(t) dt = \frac{-\cos^3(t)}{3} - \frac{t - \frac{\sin(2t)}{2}}{2}\Bigg|_0^{2\pi} = \left(\frac{-1}{3} - \frac{2\pi}{2}\right) - \left(\frac{-1}{3}\right) = -\pi$$

If you try to do the line integral directly you will get nowhere. In this case you could also do the double integral $\displaystyle\iint_C (y-1)\, dA =$ (Moment about x-axis) $-$ Area.

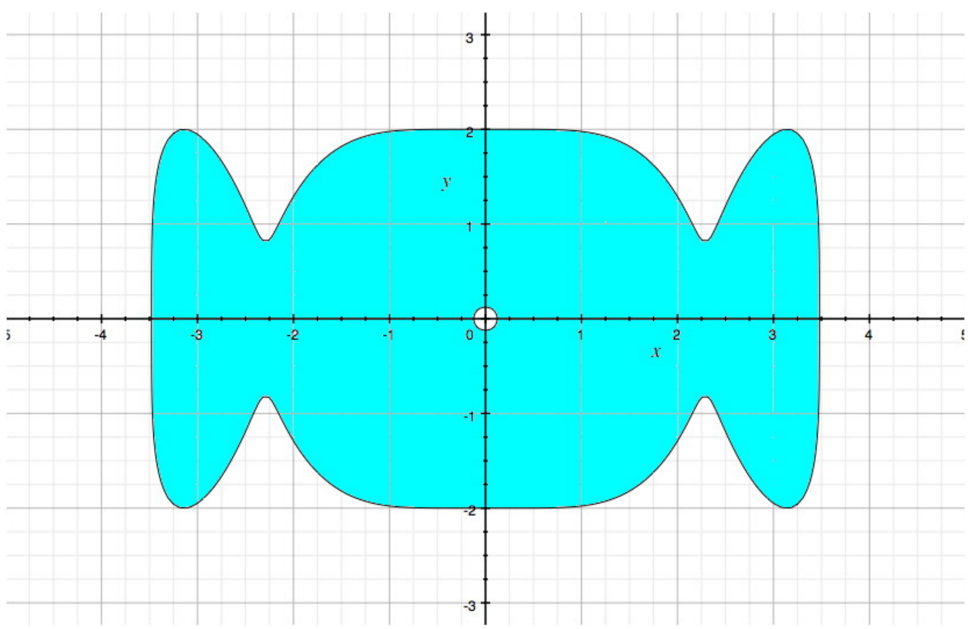
3. Change of curve

Given a region D with a conservative field $\langle P, Q\rangle$ defined on D minus some points, line integrals along complicated curves can be replaced by line integrals along simpler curves.

Example. Below is a graph of $\sin^2(x)x^4 + y^4 = 16$. Let $\mathbf{F} = \langle P, Q\rangle = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ and compute $\int_C \mathbf{F} \bullet d\mathbf{r}$ around this curve. We already checked that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$.



We can not applied Green's Theorem directly because $\mathbf{F}$ is defined only on $\mathbb{R} - \big\{(0,0)\big\}$ which is not the entire interior of the curve. However is we let D be the light blue region, then $\mathbf{F}$ is defined on D.



In this case ∂D is the outer curve $\sin^2(x)x^4 + y^4 = 16$ and the circle $x^2 + y^2 = \frac{1}{8^2}$. The outer curve is oriented counterclockwise and the inner circle is oriented clockwise. If C_1 is $\sin^2(x)x^4 + y^4 = 16$ oriented counterclockwise and if C_2 is the circle $x^2 + y^2 = \frac{1}{8^2}$ oriented counterclockwise, then

$$\int_{\partial D} \mathbf{F} \bullet d\mathbf{r} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} - \int_{C_2} \mathbf{F} \bullet d\mathbf{r}$$

We checked $\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = 2\pi$ so $\int_{C_1} \mathbf{F} \bullet d\mathbf{r} = 2\pi$ even though we have no hope of doing the integral directly.

Did you see the "fast one" in that last bit. We actually checked $\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = 2\pi$ where C_3 is the unit circle, not the one of radius $\frac{1}{8}$. But another application of the above principal shows that $\int_{C_3} \mathbf{F} \bullet d\mathbf{r} = 2\pi$ for every circle centered at the origin.

4. The normal form of Green's Theorem

Another way to write Green's Theorem is

$$\oint_{\partial D} \langle P, Q \rangle \bullet \mathbf{T}\, ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

When discussing line integrals, we found that $\int_{\partial D} \langle P, Q \rangle \bullet \mathbf{N}\, ds$ has important applications to fluid flow. Fluid flow includes such unlikely items as Maxwell's Equations in addition to more conventional possibilities. Hence it is important to realize that there is a Green's Theorem for this integral too:

$$\oint_{\partial D} \langle P, Q \rangle \bullet \mathbf{N}\, ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

The proof is easy. By formula (3) in the line integral handout $\oint_{\partial D} \langle P, Q \rangle \bullet \mathbf{N}\, ds = \oint_{\partial D} -Q dx + P dy = \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial (-Q)}{\partial y} \right) dA = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA.$

If $\langle P, Q \rangle$ is incompressible then $\oint_{\partial D} \langle P, Q \rangle \bullet \mathbf{N}\, ds = 0$ for any closed curve. It follows that

$$\langle P, Q \rangle \textit{ is incompressible if (and only if) } \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0.$$

This follows since if $\frac{\partial P}{\partial x}(a,b) + \frac{\partial Q}{\partial y}(a,b) > 0$ at some point (a,b), then it is positive in some disk D centered at the point and hence $\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA > 0$ which contradicts $\oint_{\partial D} \langle P, Q \rangle \bullet \mathbf{N}\, ds = 0$ plus Green's Theorem. It follows that $\frac{\partial P}{\partial x}(a,b) + \frac{\partial Q}{\partial y}(a,b) \leqslant 0$. Similarly it follows that $\frac{\partial P}{\partial x}(a,b) + \frac{\partial Q}{\partial y}(a,b) \geqslant 0$, so $\frac{\partial P}{\partial x}(a,b) + \frac{\partial Q}{\partial y}(a,b) = 0$.

Suppose $\langle P, Q \rangle$ is both conservative and incompressible. Let p be a potential function for it in some region D. Then p is an *harmonic function* on D, which just means that p satisfies Laplace's equation

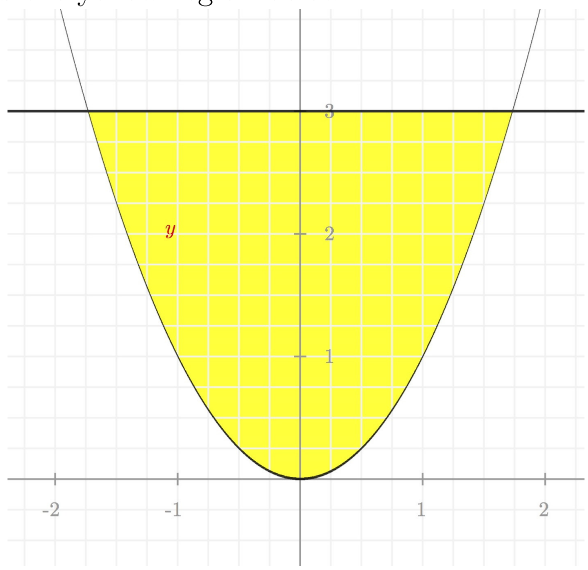
$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

Harmonic functions are extremely interesting, useful and well-studied. The book on page 932 shows $e^y \sin(x)$ is harmonic. So are $e^y \cos(x)$, $e^x \sin(y)$ and $e^x \cos(y)$. Linear functions are harmonic.

Any linear combination of harmonic functions is harmonic. $x^2 - y^2$ and xy are harmonic. So is $x^3 - 3xy^2$.

5. Why was our double integral zero?

Why is $\displaystyle\iint_D x\,dA = 0$ if D is the yellow region below?



Here is the answer and, more importantly, what you need to write if you are doing this for partial credit. (You don't need to write your explanation in italics.)

The integral $\displaystyle\iint_D x\,dA$ *is the moment about the* y *axis of the region* D*. In this case,* D *is symmetric about the* y *axis so the moment about the* y*-axis is* 0.

In general, if L is any line cutting through a region D such that D is symmetric about L then the centroid of D lies on that line. If you know the y coordinate of the centroid, $\bar{y}$, you know $\displaystyle\iint_D x\,dA = \text{Area}\cdot\bar{y}$.