MATH 20550 Triple Integrals in cylindrical and spherical coordinates Fall 2016

1. Coordinates

1.1. Cylindrical coordinates. $(r, \theta, z) \mapsto (x, y, z)$

$$\begin{aligned} x =& r\cos\theta \\ y =& r\sin\theta \\ z =& z \end{aligned}$$

Cylindrical coordinates are just polar coordinates in the plane and z. Useful formulas

$$\begin{aligned} r =& \sqrt{x^2+y^2} \\ \tan\theta =& \frac{y}{x}, x \neq 0; \qquad x = 0 \implies \theta = \frac{\pi}{2} \end{aligned}$$

These are just the polar coordinate useful formulas.

Cylindrical coordinates are useful for describing cylinders.

$$r = f(\theta) \quad z \geqslant 0$$

is the cylinder above the plane polar curve $r = f(\theta)$.

$$r^2 + z^2 = a^2$$

is the sphere of radius a centered at the origin.

$$r = mz \quad m > 0 \text{ and } z \geqslant 0$$

is the cone of slope m with cone point at the origin.

1.2. Spherical coordinates. $(\rho, \theta, \phi) \mapsto (x, y, z)$

$$\begin{aligned} x =& \rho\cos\theta\sin\phi \\ y =& \rho\sin\theta\sin\phi \\ z =& \rho\cos\phi \end{aligned}$$

Useful formulas

$$\begin{aligned} \rho =& \sqrt{x^2+y^2+z^2} \\ \rho\sin\phi =& \sqrt{x^2+y^2} \\ \tan\theta =& \frac{y}{x}, x \neq 0; \qquad x = 0 \implies \theta = \frac{\pi}{2} \\ \tan\phi =& \frac{\sqrt{x^2+y^2}}{z}, z \neq 0; \qquad z = 0 \implies \phi = \frac{\pi}{2} \end{aligned}$$

You can also change spherical coordinates into cylindrical coordinates.

$$\begin{aligned} z =& \rho\cos\phi \\ r =& \rho\sin\phi \\ \theta =& \theta \end{aligned}$$

Standard graphs in spherical coordinates:

$$\rho = a$$

is the sphere of radius a centered at the origin.

$$\phi = c$$

is the cone of slope $\tan(c)$ with cone point at the origin.

$$\theta = c$$

is the vertical plane over the line $y = \tan(c)x$.

$$\rho \cos\phi = c$$

is the plane $z = c$.

$$\rho = 2d\cos\phi$$

is the sphere of radius $|d|$ centered at $(0, 0, d)$.

$$\rho = 2d\cos\theta\sin\phi$$

is the sphere of radius $|d|$ centered at $(d, 0, 0)$.

$$\rho = 2d\sin\theta\sin\phi$$

is the sphere of radius $|d|$ centered at $(0, d, 0)$.

Indeed,

$$\rho = 2a\cos\theta\sin\phi + 2b\sin\theta\sin\phi + 2c\cos\phi$$

is the sphere of radius $\sqrt{a^2 + b^2 + c^2}$ centered at (a, b, c).

2. Triple integrals

As usual, the goal is to evaluate some triple integral over some solid in space. If the solid is S, then

$$\iiint_S f\, dV$$

does not depend on any particular coordinate system (which is why I have not written $f(x, y, z)$). All you have to be able to do is to evaluate f at points in S (no matter how the points are described to you) and to compute the volume of the pieces into which you have partitioned S. Maybe f is given to you in Cartesian coordinates, $f(x, y, z)$, or maybe in terms of cylindrical coordinates, $f(r, \theta, z)$, or maybe in terms of spherical coordinates, $f(\rho, \theta, \phi)$. Given a formula in one coordinate system you can work out formulas for f in other coordinate systems but behind the scenes you are just evaluating a function, f, at a point $\mathbf{p} \in S$. If you use a different coordinate system, the formula for f looks different but it is still the same function. If we agree on a point $\mathbf{p} \in S$ and all go off and use our various formulas, we will all get the same number for the value of f at that point.

Aside: Remember that when we first introduced the triple integral we *estimated* the triple integral just given a verbal description of S and a table of values for f. We partitioned S so that in each piece we could choose one of the points from our table and then we wrote down the Riemann sum and that was our approximation.

2.1. **Cylindrical coordinates.** Suppose we have described S in terms of cylindrical coordinates. This means that we have a solid C in (r, θ, z) space and when we map C into space using cylindrical coordinates we get S. If we cut C up into little boxes we get little slices of pie in space so

$$\iiint_C f \cdot |r| \; dV = \iiint_S f \, dV$$

To use this formula usefully we will need to be able to evaluate f at points given to us in cylindrical coordinates.

2.2. **Spherical coordinates.** Suppose we have described S in terms of spherical coordinates. This means that we have a solid Γ in (ρ, θ, ϕ) space and when we map Γ into space using spherical coordinates we get S. If we cut Γ up into little boxes we get little pieces in space as described in the book

$$\iiint_\Gamma f \cdot \rho^2 \, |\sin \phi| \; dV = \iiint_S f \, dV$$

To use this formula usefully we will need to be able to evaluate f at points given to us in spherical coordinates.

2.3. **Example.** Suppose you want to integrate x^2 over a ball of radius a centered at the origin, $\iiint_S x^2\,dV$. In cylindrical coordinates S is $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$, $-\sqrt{a^2-r^2} \leqslant z \leqslant \sqrt{a^2-r^2}$. Hence

$$\iiint_S x^2\,dV = \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3 \cos^2\theta\,dz\,d\theta\,dr$$

In spherical coordinates S is $0 \leqslant \rho \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \phi \leqslant \pi$. Hence

$$\iiint_S x^2\,dV = \int_0^a \int_0^{2\pi} \int_0^{\pi} \rho^4 \cos^2\theta \sin^3\phi\,d\phi\,d\theta\,d\rho$$

By now you should be able to see

$$\iiint_S x^2\,dV = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} x^2\,dz\,dy\,dx$$

in Cartesian coordinates.

I'm not overly excited about doing any of these integrals but the spherical coordinates one is the easiest. Since you are integrating over a box in (ρ, θ, ϕ) space,

$$\int_0^a \int_0^{2\pi} \int_0^{\pi} \rho^4 \cos^2\theta \sin^3\phi\,d\phi\,d\theta\,d\rho = \left(\int_0^a \rho^4\ d\rho\right)\left(\int_0^{2\pi} \cos^2\theta\ d\theta\right)\left(\int_0^{\pi} \sin^3\phi\ d\phi\right) =$$

$$\frac{a^5}{5} \cdot \pi \cdot \frac{2}{3} = \frac{2a^5}{15}\pi$$

2.4. **Example.** Suppose you want the volume of the solid between the cones of slope 1 and slope $\frac{1}{2}$ and inside the cylinder over the circle of radius 3 centered at the origin in the xy plane, $\iiint_S 1\,dV$.

In cylindrical coordinates $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant r \leqslant 3$ and $\frac{r}{2} \leqslant z \leqslant r$ so

$$\iiint_S 1\,dV = \int_0^3 \int_0^{2\pi} \int_{\frac{r}{2}}^{r} r\,dz\,d\theta\,dr$$

In spherical coordinates S is $0 \leqslant \theta \leqslant 2\pi$, $\pi/4 \leqslant \phi \leqslant \arctan(2)$. The ρ coordinate starts at 0 and keeps going until it hits the cylinder. This happens when $r = 3$ so $\rho \sin\phi = 3$ or $\rho = 3\csc\phi$. $0 \leqslant \rho \leqslant 3\csc\phi$, Hence

$$\iiint_S 1\,dV = \int_0^{2\pi} \int_{\pi/4}^{\arctan(2)} \int_0^{3\csc\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

This time cylindrical looks easiest but not necessarily by much.

$$\int_0^3 \int_0^{2\pi} \int_{\frac{r}{2}}^{r} r\,dz\,d\theta\,dr = \int_0^3 \int_0^{2\pi} r\,z\Big|_{z=r/2}^{z=r} d\theta\,dr = \int_0^3 \int_0^{2\pi} \frac{r^2}{2} d\theta\,dr =$$

$$2\pi \int_0^3 \frac{r^2}{2}\,dr = 2\pi \frac{r^3}{6}\Big|_0^3 = 9\pi$$

In spherical coordinates

$$\int_0^{2\pi} \int_{\pi/4}^{\arctan(2)} \int_0^{3\csc\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} \int_{\pi/4}^{\arctan(2)} \frac{\rho^3}{3} \sin\phi \Big|_{\rho=0}^{\rho=3\csc\phi} d\phi\,d\theta =$$

$$\int_0^{2\pi} \int_{\pi/4}^{\arctan(2)} 9\csc^3\phi \sin\phi\,d\phi\,d\theta = 9 \int_0^{2\pi} \int_{\pi/4}^{\arctan(2)} \csc^2\phi\,d\phi\,d\theta =$$

$$9 \int_0^{2\pi} -\cot\phi\Big|_{\pi/4}^{\arctan(2)} d\theta = 9 \int_0^{2\pi} \big(-1/2 + 1\big)\big)\,d\theta = 9\pi$$

2.5. **Example.** Convert

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dx\, dy$$

into a spherical coordinate iterated integral
(from here, example 2.)

Let us start by describing the solid. Note $\int_0^3 \int_0^{\sqrt{9-y^2}} \cdots dx\, dy$ describes the quarter of the disk of radius 3 in the xy plane in the first quadrant.

Above this quarter-disk the z coordinate starts at $z = \sqrt{x^2+y^2} = r$ which is the cone of slope 1 and ends at $z = \sqrt{18-x^2-y^2}$ which is the upper hemisphere of radius $\sqrt{18} = 3\sqrt{2}$. Hence the solid is inside the cylinder parallel to the z axis of radius 3, above the 45° cone and below the sphere centered at the origin of radius $2\sqrt{3}$.

Hence in spherical coordinates one has $0 \leqslant \phi \leqslant \pi/4$, $0 \leqslant \theta \leqslant \pi/2$ and $0 \leqslant \rho \leqslant 3\sqrt{2}$. Hence

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dx\, dy = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{3\sqrt{2}} \rho^4 \sin(\phi)\, d\rho\, d\theta\, d\phi =$$

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \left.\frac{\rho^5}{5}\right|_0^{3\sqrt{2}} d\theta\, d\phi = \frac{\pi}{4}\frac{\pi}{2}\frac{243}{5} \cdot 16\sqrt{2} = \frac{486\sqrt{2}}{5}\pi^2$$