Department of Mathematics
University of Notre Dame
MATH 20550 - Calculus III
Summer 2015

Name Key

Final Exam

July 24, 2015

This exam has 13 problems worth a total of **145 points**. You will have 105 minutes to work on it. All answers should be given as exact, closed form numbers as opposed to decimal approximations. **For full credit, you must show all work.** Calculators and all other electronic devices are strictly forbidden.

Please read and sign the Honor Pledge:

Honor Pledge: As a member of the Notre Dame community, I will not participate in or tolerate academic dishonesty.

Signature: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	15	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	15	
12	10	
13	15	
Total:	145	

1. (10 points) Find the Jacobian of the transformation

$$x = \frac{u}{v} \qquad y = \frac{v}{w} \quad Z = \frac{w}{u}$$

$$\frac{\partial(\cancel{x},y,\cancel{z})}{\partial(u,v,w)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} & 0 \\ 0 & \frac{1}{w} & -\frac{v}{w^2} \\ \frac{-w}{u^2} & 0 & \frac{1}{u} \end{vmatrix} = \frac{1}{uvw} - \left(-\frac{u}{v^2}\right)\left(0 - \left(-\frac{v}{w^2}\right)\left(\frac{-w}{u^2}\right)\right)$$

$$= 0$$

2. (10 points) Is the vector field

$$\vec{F} = (3x^2 - 2y^2)\hat{\imath} + (4xy + 3)\hat{\jmath}$$

conservative? If so, find a potential function.

If $\vec{F}$ is conservative then $\exists f$ s.t. $\nabla f = \vec{F}$ then

$f = \int (3x^2 - 2y^2)\,dx = x^3 - 2xy^2 + g(y)$

$f_y = -4xy + g'(y) \neq 4xy + 3$ for any $g(y)$

So $\vec{F}$ is not conservative

3. (10 points) Compute

$$\iint \text{curl}\vec{F} \cdot d\vec{S}$$

where S is the sphere of radius 16 oriented outward, and $\vec{F} = \langle xyz, x^2y^2z^2, x^3y^3z^3 \rangle$.

S is a closed surface so we can apply the divergence theorem.

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iiint_E \text{div}(\text{curl}\,\vec{F})\, dV$$

$\text{div}(\text{curl}\,\vec{F}) = 0$ so

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = 0$$

4. (15 points) Compute the flux of $\vec{F} = \langle xye^z, xy^2z^3, -ye^z \rangle$ through the box bound by the coordinate planes and the planes $x = 3$, $y = 2$, $z = 1$, where the box has outward orientation.

$$\cancel{\iint_S xye^z\,dx}$$

$$\cancel{\iint_S F}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \mathrm{div}(\vec{F})\,dV$$

$$\mathrm{div}(\vec{F}) = ye^z + 2xyz^3 - ye^z = 2xyz^3$$

$$\iiint_E \mathrm{div}(\vec{F})\,dV = \int_0^3 \int_0^2 \int_0^1 2xyz^3\,dz\,dy\,dx = 2\left(\frac{9}{2}\right)\left(\frac{4}{2}\right)\left(\frac{1}{4}\right)$$

$$= \frac{9}{2}$$

5. (10 points) Find and classify all the critical points of

$$f(x,y) = xy - 2x - 2y - x^2 - y^2$$

We need to find x, y s.t

$\nabla f(x,y) = \langle 0,0 \rangle = \langle y-2-2x, x-2-2y \rangle$ so we have

$$\begin{cases} y-2-2x=0 & ① \\ x-2-2y=0 & ② \end{cases}$$

① implies $y=2x+2$ plugging this into ② we have $x-2-2(2x+2)=0$

$=-3x+(-6)=0$ So $x=-2$ then $y=-2$.

To classify we must determin the Hessian

$f_{xx}=-2$

$H(x,y) = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3$

So $(-2,-2)$ is a local max.

6. (10 points) Give a vector function which represents the curve of intersection between the hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.

If $x = \cos\theta$, $y = \sin\theta$ then $z = \cos^2\theta - \sin^2\theta$

So

$$\vec{r}(\theta) = \langle \cos\theta, \sin\theta, \cos^2\theta - \sin^2\theta \rangle$$

7. (10 points) Is there a vector field $\vec{G}$ on $\mathbb{R}^3$ such that

$$\text{curl}\vec{G} = \langle xyz, -y^2z, yz^2 \rangle = \vec{F}$$

? Explain.

If $\vec{F}$ is the curl of something then $\text{div}(\vec{F}) = 0$, so let's see if this nessisary condition holds.

$\text{div}(\vec{F}) = yz - 2yz + 2yz = yz \neq 0$!

So $\vec{F}$ cannot be the curl of something.

8. (10 points) The plane $2x+y+2z=9$ intersects the sphere $x^2+y^2+z^2=9$ tangentially at exactly one point. Find the point of intersection.

$f(x,y,z)=2x+y+2z$ then $2x+y+2z=9$ is a level surface of f and if $g(x,y,z)=x^2+y^2+z^2$ then $x^2+y^2+z^2=9$ is a level surface of g. So we need to find (x,y,z) s.t.

$\nabla f = c\nabla g$, for some c.

$\nabla f = \langle 2,1,2\rangle = c\nabla g = \langle 2cx, 2cy, 2cz\rangle$

subject to $2x+y+2z=9$ and $x^2+y^2+z^2=9$. Doing a little algebra or by inspection we see $c=\frac{1}{2}, x=2, y=1, z=2$

So $(2,1,2)$.

9. (10 points) Find an equation for the sphere which has $(2, 1, 4)$ and $(4, 3, 10)$ as antipodal points (the are connected by a line through the center of the sphere).

$$2r = d((2,1,4),(4,3,10)) = \sqrt{2^2+2^2+6^2} = \sqrt{44} \Rightarrow r = \frac{\sqrt{44}}{2}$$

The center is the halfway point between the two points, so

$$\frac{(2,1,4)+(4,3,10)}{2} = (3,2,7)$$

So our sphere is

$$(x-3)^2+(y-2)^2+(z-7)^2 = \frac{44}{4} = 11$$

10. Compute the limits or show they do not exist

(a) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{xy}$$

(b) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{xy+\cos(x)}{\frac{x+e^y}{x^2y+\cos(x)e^y}}$$

a) Consider $(x,0)\to(0,0)$ then

$$\lim_{(x,0)\to(0,0)} \frac{xy}{x^2+2y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

Consider $(x,x)\to(0,0)$ then

$$\lim_{(x,x)\to(0,0)} \frac{xy}{x^2+2y^2} = \lim_{x\to 0} \frac{x^2}{3x^2} = \frac{1}{3} \neq 0 \quad \text{DNE}$$

b. $\cos x$ and e^x are continuous so $xy+\cos(x)$ is continuous, $x+e^y$ is continuous, and $x^2y+\cos(x)e^y$ is continuous. Furthermore all denominators are non-zero at $(0,0)$ so our limit is continuous.

$$\lim_{(x,y)\to(0,0)} \frac{xy+\cos x}{\frac{x+e^y}{x^2y+\cos(x)e^y}} = \frac{1}{\frac{1}{1}} = 1$$

11. (15 points) Compute the line integral

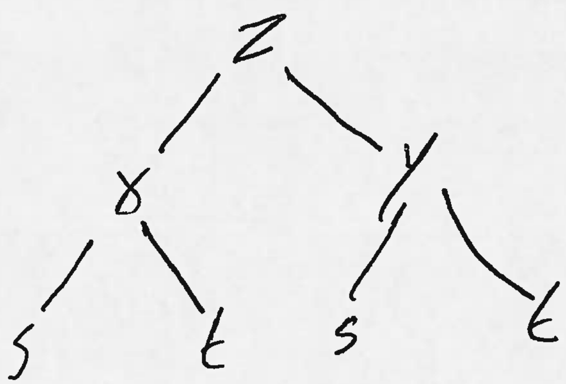
$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = \langle y - \cos y, x \sin y \rangle$ and C is the circle $(x-3)^2 + (y+4)^2 = 4$, oriented clockwise.

By Green's Theorem (keeping in mind orientation)

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_D \left(\sin y \cancel{dA} - (1 - \sin y)\right) dA = -\iint_D -1 \, dA = 4\pi$$

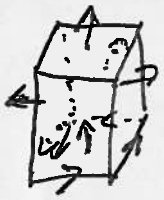
12. (10 points) Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ where $z = x^2 - 3y^2$, $x = st$, $y = s + t^2$.



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x)(t) - 6y(1) = 2st^2 - 6(s+t^2)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 2xs - 6y(2t) = 2s^2t - 12(st + t^3)$$

13. (15 points) Compute



$$\iint_S \text{curl}\vec{F} \cdot d\vec{S}$$

where S is the top and four sides of the faces of the box with verticies $(\pm 1, \pm 1 \pm 1)$ (the box without the bottom), given outward orientation, and $\vec{F}(x, y, z) = \langle xyz, xy, x^2yz \rangle$.

5 sides is too many so lets try surface swapping.

For consistant orientation the orientation at the bottom must be uppward. So let S_2 be the bottom of the box then.

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl}(\vec{F}) \cdot d\vec{S}.$$

The surface is parametrized by $\vec{r}(x,y) = \langle x, y, 0 \rangle$ so $\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle$

Now $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xy & x^2yz \end{vmatrix} = \langle x^2z - x, xy - 2xyz, y - xz \rangle$

$\text{curl}(\vec{F})(\vec{r}(x,y)) = \langle -x, xy, y \rangle$

Then

$\text{curl}(\vec{F})(\vec{r}(x,y)) \cdot d\vec{S} = y$ So

$$\iint_{S_2} \text{curl}(\vec{F}) \cdot d\vec{S} = \int_{-1}^{1}\int_{-1}^{1} y \, dx\,dy = 0$$