1.(5pts) Two particles travel along the space curves

$$\mathbf{r}_1(t) = \left\langle t, t^2 + 1, -t \right\rangle \text{ and } \mathbf{r}_2(t) = \langle 1 + 2t, 3 + t, 4 + 3t \rangle .$$

Do the two particles collide and do their paths intersect?

(a) Yes, they collide at $(2, 5, -2)$.

(b) No, they do not collide, but their paths intersect at $(-1, 2, 1)$.

(c) Yes, they collide at $(-1, 2, 1)$.

(d) No, they do not collide, and their paths do not intersect.

(e) No, they do not collide, but their paths intersect at $(2, 5, -2)$.

2.(5pts) Find the vector projection of **b** onto **a**.

$$\mathbf{a} = \langle 2, -2, 1 \rangle , \ \mathbf{b} = \langle 6, -1, 4 \rangle .$$

(a) $\langle 11, 8, -6 \rangle$ (b) $\langle -4, 4, 2 \rangle$ (c) 2 (d) $\langle -11, -8, 6 \rangle$ (e) $\langle 4, -4, 2 \rangle$

3.(5pts) Find the position vector at time t (seconds) of a particle with mass 5 (kilograms) experiencing a force $\mathbf{F}(t) = \langle -5\cos t, 0, -5\sin t\rangle$ (in Newtons) with initial velocity $\mathbf{v}(0) = \langle 0, 3, 2\rangle$ (in meters per second) and initial position vector $\mathbf{r}(0) = \langle 1, 3, 0\rangle$ (in meters).

(a) $\mathbf{r}(t) = \langle 5\cos t, 15, 5\sin t - 5\rangle$

(b) $\mathbf{r}(t) = \langle -\sin t, 3, \cos t + 1\rangle$

(c) $\mathbf{r}(t) = \langle \cos t, 3, \sin t\rangle$

(d) $\mathbf{r}(t) = \langle \cos t, 3t + 3, t + \sin t\rangle$

(e) $\mathbf{r}(t) = \langle \cos t, 0, \sin t\rangle$

4.(5pts) Find the area of the parallelogram with vertices $A(0, 1, 1)$, $B(1, 2, 4)$, $C(-1, 1, 0)$, and $D(0, 2, 3)$.

(a) $-\sqrt{6}$ (b) $\sqrt{6}$ (c) -4 (d) $\sqrt{41}$ (e) 4

5.(5pts) Not on this exam. Use the chain rule to find $\dfrac{\partial z}{\partial s}$ when $z = x^3y^2$, $x = s\cos(2t)$, $y = s\sin t$.

(a) $\dfrac{\partial z}{\partial s} = 3s^2t^2$ (b) $\dfrac{\partial z}{\partial s} = s^5\cos^3(2t)\sin^2 t$ (c) $\dfrac{\partial z}{\partial s} = 2s^3t$

(d) $\dfrac{\partial z}{\partial s} = 5s^4\cos^3(2t)\sin^2 t$ (e) $\dfrac{\partial z}{\partial s} = 3s^4\cos^3(2t)\sin^2 t$

6.(5pts) Not on this exam. Find the second partial derivative f_{xy} of the function $f(x,y) = x^3y^2 + \sin x$.

(a) $f_{xy} = 6xy^2 - \sin x$ (b) $f_{xy} = 3x^2y^2 + \cos x$ (c) $f_{xy} = 2x^3y$

(d) $f_{xy} = 2x^3$ (e) $f_{xy} = 6x^2y$

7.(5pts) Which of the following represents the line through the points $P(-8, 2, -1)$ and $Q(0, 2, -3)$?

(a) $4x - z = 3$

(b) $x = -8,\ y = 2 + 2t,\ z = -1 - 3t$

(c) $x = -8 + 8t,\ y = 2,\ z = -1 - 2t$

(d) $x + 6y - 4z = 0$

(e) $x = -8t,\ y = 2 + 2t,\ z = -3 - t$

8.(5pts) Find the derivative of the vector function $\mathbf{r}(t) = t\,\mathbf{i} + \sqrt{t}\,\mathbf{j} + \dfrac{t}{1+t}\mathbf{k}$.

(a) $\mathbf{r}'(t) = \mathbf{i} + \dfrac{1}{2\sqrt{t}}\mathbf{j} + \dfrac{1}{(1+t)^2}\mathbf{k}$

(b) $\mathbf{r}'(t) = \sqrt{t^2 + t + \dfrac{t^2}{(1+t)^2}}$

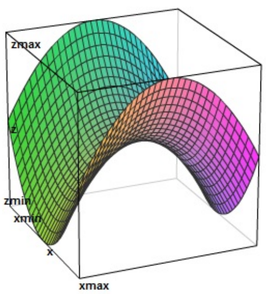
(c) $\mathbf{r}'(t) = \mathbf{i} + \sqrt{t}\,\mathbf{j} + \dfrac{t}{1+t}\mathbf{k}$

(d) $\mathbf{r}'(t) = 0$

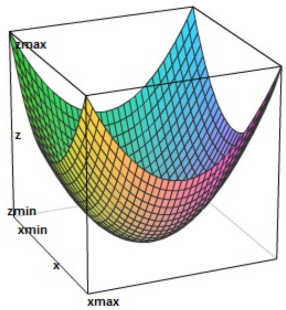
(e) $\mathbf{r}'(t) = 1 + \dfrac{1}{2\sqrt{t}} + \dfrac{1}{(1+t)^2}$

9.(5pts) Not on this exam. Select the correct graph and the correct contour plot of level curves for the function

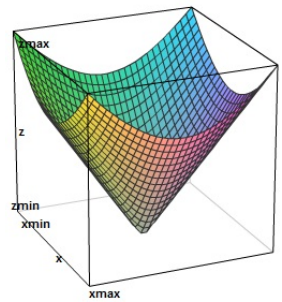
$$f(x,y) = x^2 - y^2.$$



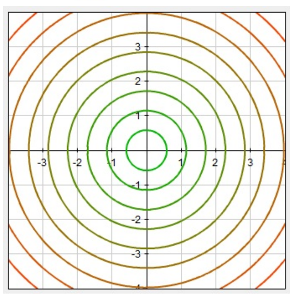
I.



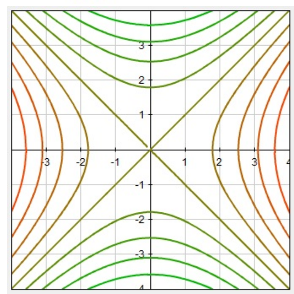
II.



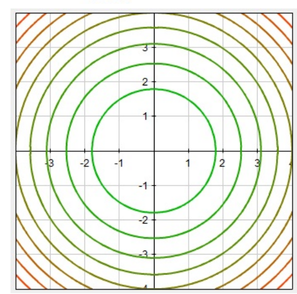
III.



A.



B.



C.

(a) I and B (b) I and A (c) II and A

(d) II and C (e) III and C

10.(5pts) Find the length of the curve given by $\mathbf{r}(t) = \left\langle \frac{1}{3}t^3, 8t, -2t^2 \right\rangle$ from the point $(0,0,0)$ to the point $(9,24,-18)$.

(a) 33 (b) $\dfrac{1923}{5}$ (c) $\dfrac{9}{4}\sqrt{329}$ (d) $3\sqrt{109}$ (e) 15.6

11.(12pts) Find an equation of the plane through the points $P(0, 1, -1)$, $Q(1, 2, 0)$, and $R(2, -2, 1)$.

12.(12pts) Find the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ for the path given by $\mathbf{r}(t) = \langle -\cos t, t, \sin t\rangle$ at the point $(-1, 0, 0)$.

13.(12pts) Find a vector equation for the tangent line to the curve given by $\mathbf{r}(t) = \left\langle 1 + \sqrt{t}, t^3 - 2, \ln t \right\rangle$ at the point $(2, -1, 0)$.

14.(12pts) The curves $\mathbf{r}_1(t) = \langle t^2 - 1, 0, 3t - 3 \rangle$ and $\mathbf{r}_2(s) = \langle s, -1 + \cos s, \sin s \rangle$ intersect at the origin. Find their angle of intersection.

1. **Solution.** To check if they intersect, we equate components of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ and solve for t. Comparing the first components, we must have that $t = 1 + 2s$, and comparing the third components, we must have that $-t = 4 + 3s$. Hence $0 = 5 + 5s$ and $s = -1$. Then $t = -1$ as well and $\mathbf{r}_1(-1) = \mathbf{r}_2(-1) = (-1, 2, 1)$ so the two particles collide.

---

2. **Solution.** The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by $\text{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$. In this case, $\mathbf{a}\cdot\mathbf{b} = 12+2+4 = 18$ and $\mathbf{a}\cdot\mathbf{a} = 2^2+(-2)^2+1^2 = 9$, so $\text{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{18}{9}\langle 2, -2, 1\rangle = \langle 4, -4, 2\rangle$.

---

3. **Solution.** From Netwton's second law, $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a}$ is acceleration, and thus the particle's acceleration at time t is $\mathbf{a}(t) = \langle -\cos t, 0, -\sin t\rangle$. To get the position function, we must integrate twice, each time choosing appropriate constants of integration to agree with the given $\mathbf{v}(0)$ and $\mathbf{r}(0)$. Thus $\mathbf{v}(t) = \langle -\sin t, 3, \cos t + 1\rangle$ and $\mathbf{r}(t) = \langle \cos t, 3t+3, t+\sin t\rangle$.

---

4. Solution. The area of the parallelogram is given by the magnitude of the cross products of the vectors forming two adjacent edges of the parallelogram: $\left|\vec{AB} \times \vec{AC}\right|$. A quick calculation:

$$\begin{aligned}\left|\vec{AB} \times \vec{AC}\right| &= \left|\langle 1,1,3\rangle \times \langle -1,0,-1\rangle\right| \\ &= \left|\langle -1,-2,1\rangle\right| \\ &= \sqrt{6}.\end{aligned}$$

---

5. Solution. From the chain rule, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$. Computing each of these:

$$\begin{aligned}\frac{\partial z}{\partial x} &= 3x^2y^2 = 3(s^2\cos^2(2t))(s^2\sin^2 t) = 3s^4\cos^2(2t)\sin^2 t, \\ \frac{\partial z}{\partial y} &= 2x^3y = 2(s^3\cos^3(2t))(s\sin t) = 2s^4\cos^3(2t)\sin t, \\ \frac{\partial x}{\partial s} &= \cos(2t), \\ \frac{\partial y}{\partial s} &= \sin t.\end{aligned}$$

Thus we have that

$$\frac{\partial z}{\partial s} = 3s^4\cos^3(2t)\sin^2 t + 2s^4\cos^3(2t)\sin^2 t = 5s^4\cos^3(2t)\sin^2 t.$$

---

6. Solution. According to Clairot's Theorem, we can take the derivatives in whichever order we wish. $f_y = 2x^3y$, and $f_{yx} = 6x^2y$.

---

7. Solution. In order to get a line, we need a point on the line and a vector parallel to that line. In this case, we can use the point $P(-8, 2, -1)$ and the vector $\vec{PQ} = \langle 8, 0, -2 \rangle$. This leads us to the parametric equations $x = -8 + 8t$, $y = 2$, $z = -1 - 2t$.

Failing that, only three of the provided answers are lines (the other two are planes), and it is straightforward to check that of these only the correct answer contains both points provided.

8. Solution. Derivatives of vector functions can be computed component-wise, and so $\mathbf{r}'(t) = \mathbf{i} + \dfrac{1}{2\sqrt{t}}\mathbf{j} + \dfrac{1}{(1+t)^2}\mathbf{k}$.

9. Solution. We can see that the level curves of the function $x^2 - y^2$ are of the form $x^2 - y^2 = k$ for some constant k. These are hyperbolas, and so the contour plot is plot B. The graph matching this contour plot is I.

10. Solution. To find the arclength, one must calculate the definite integral of $|\mathbf{r}'(t)|$ from the initial point to the terminal point. Notice that $\mathbf{r}(0) = \langle 0, 0, 0\rangle$ and $\mathbf{r}(3) = \langle 9, 24, -18\rangle$, so that we must find $\displaystyle\int_0^3 |\mathbf{r}'(t)|\ dt$. We calculate that

$$\mathbf{r}'(t) = \left\langle t^2, 8, -4t \right\rangle,$$

so that

$$|r'(t)| = \sqrt{t^4 + 64 + 16t^2} = \sqrt{(t^2+8)^2} = (t^2+8).$$

Thus our integral becomes

$$\int_0^3 |\mathbf{r}'(t)|\ dt = \int_0^3 t^2 + 8\,dt = \frac{1}{3}t^3 + 8t|_0^3 = 33.$$

---

11. Solution. The normal for this plane must be perpendicular to the vectors $\vec{PQ}$ and $\vec{PR}$. Hence it must be perpendicular to $\vec{PQ} \times \vec{PR}$. We calculate that

$$\vec{PQ} \times \vec{PR} = \langle 1, 1, 1\rangle \times \langle 2, -3, 2\rangle = \langle 5, 0, -5\rangle.$$

So we can choose as the normal for our plane the vector $\mathbf{n} = \langle 1, 0, -1\rangle$. Taking P as the base point (though any of the points will give the same equation), we get that the equation of the plane is

$$1(x-0) + 0(y-1) + (-1)(z-(-1)) = 0,$$

or

$$x - z = 1.$$

---

12. Solution. Notice that the curve given by $\mathbf{r}(t)$ passes through the point $(-1, 0, 0)$ when $t = 0$. We calculate

$$\begin{aligned}
\mathbf{r}'(t) &= \langle \sin t, 1, \cos t \rangle \\
|\mathbf{r}'(t)| &= \sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2} \\
\mathbf{T}(t) &= \frac{\mathbf{r}(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}} \langle \sin t, 1, \cos t \rangle \\
\mathbf{T}'(t) &= \frac{1}{\sqrt{2}} \langle \cos t, 0 - \sin t \rangle \\
|\mathbf{T}'(t)| &= \frac{1}{\sqrt{2}} \sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}} \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle \cos t, 0, -\sin t \rangle \\
\mathbf{T}(0) &= \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle, \ \mathbf{N}(0) = \langle 1, 0, 0 \rangle \\
\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) &= \frac{1}{\sqrt{2}} (\langle 0, 1, 1 \rangle \times \langle 1, 0, 0 \rangle) = \frac{1}{\sqrt{2}} \langle 0, -1, -1 \rangle
\end{aligned}$$

Better:

$$\begin{aligned}
r'(t) = \langle \sin t, 1, \cos t \rangle &\qquad \text{so at } t = 0 \quad \mathbf{r}'(0) = \langle 0, 1, 1 \rangle \\
\mathbf{r}''(t) = \langle \cos t, 0, -\sin t \rangle &\qquad \text{so at } t = 0 \quad \mathbf{r}''(0) = \langle 1, 0, 0 \rangle
\end{aligned}$$

$\mathbf{T}$ and $\mathbf{r}'$ point in the same direction so $\mathbf{T} = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. $\mathbf{B}$ and $\mathbf{r}' \times \mathbf{r}''$ point in the same direction as $\mathbf{B}$ and $\mathbf{r}' \times \mathbf{r}'' = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, 1, -1 \rangle$ and $\mathbf{B} = \left\langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$. Finally $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and $\det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \langle 2, 0, 0 \rangle$ points in the direction of $\mathbf{N}$ and so $\mathbf{N} = \langle 1, 0, 0 \rangle$.

---

13. Solution. We begin by noticing that the curve passes through this point when $t = 1$. Now, the line in question must be parallel to the vector $\mathbf{r}'(1)$ and pass through the point $(2, -1, 0)$. We calculate that

$$\mathbf{r}'(t) = \left\langle \frac{1}{2\sqrt{t}}, 3t^2, \frac{1}{t} \right\rangle,$$

so that $\mathbf{r}'(1) = \left\langle \frac{1}{2}, 3, 1 \right\rangle$. Thus one equation for the line is given by

$$\langle x, y, z \rangle = \left\langle 2 + \frac{t}{2}, -1 + 3t, t \right\rangle.$$

Other equations are also possible.

---

14. Solution. This intersection occurs when $t = 1$ and $s = 0$. The angle of intersection of the curves is the angle between their tangent vectors at the given point. We calculate that

$$\mathbf{r}_1'(t) = \langle 2t, 0, 3 \rangle, \ \mathbf{r}_1(1) = \langle 2, 0, 3 \rangle$$
$$\mathbf{r}_2'(s) = \langle 1, -\sin s, \cos s \rangle, \ \mathbf{r}_2(0) = \langle 1, 0, 1 \rangle.$$

All that remains is to find the angle between $\langle 2, 0, 3 \rangle$ and $\langle 1, 0, 1 \rangle$, which can be derived from the formula $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$.

$$\langle 2, 0, 3 \rangle \cdot \langle 1, 0, 1 \rangle = 5$$
$$|\langle 2, 0, 3 \rangle| = \sqrt{13}, \ |\langle 1, 0, 1 \rangle| = \sqrt{2}$$
$$\cos\theta = \frac{5}{\sqrt{13}\sqrt{2}},$$

so that

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{26}}\right).$$
