1.(6pts) Find $\displaystyle\int\int_D \frac{2y}{x^2+1} dA$ where $D = \{(x,y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant \sqrt{x}\}$.

(a) $\frac{1}{2}\ln 2$ (b) $-\frac{1}{2}\ln 2$ (c) 0 (d) 1 (e) $\ln 2$

2.(6pts) The point $(1,1)$ is a critical point of $f(x,y) = x^2 + y^2 + 1$. This critical point $(1,1)$ of $f(x,y)$ is

(a) a local minimum point; (b) a saddle point; (c) indeterminant type;

(d) a local maximum point; (e) neither maximum nor minimum.

3.(6pts) Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies within the region $\Omega = \{(x,y,z)\big|x \geqslant 0, y \geqslant 0, z \geqslant 0\}$.

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) 4π (e) 0

4.(6pts) Find the maximum value of $f(x,y,z) = xyz$ subject to $x^2 + 2y^2 + 3z^2 = 6$.

(a) $\frac{2}{\sqrt{3}}$ (b) 1 (c) 0 (d) $-\frac{2}{\sqrt{3}}$ (e) 6

5.(6pts) A lamina occupies the part of the unit disk in the first quadrant. If the density function is $\rho(x,y) = x^2 2 + y^2$, using polar coordinates find the total mass.

(a) $\frac{\pi}{6}$ (b) 1 (c) $\frac{1}{6}$ (d) π (e) 2

6.(6pts) Find the maximum volume of a rectangular box such that the sum of lengths of its 12 edges is 24.

(a) 8 (b) 1 (c) 0 (d) 12 (e) $(12)^3$

7.(6pts) Find the equation of tangent plane at the point $(1,2,3)$ to the surface

$$x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 6$$

(a) $x + y + z - 6 = 0$ (b) $3x + 2y + z - 6 = 0$ (c) $x + \frac{y}{2} + \frac{z}{3} - 3 = 0$

(d) $x + y + z - 1 = 0$ (e) $x + 2y + 3z - 14 = 0$

8.(6pts) Let $f(x,y) = (1+xy)(x+y)$. Find all critical points of $f(x,y)$.

(a) $(1,-1)$, $(-1,1)$ (b) $(1,1)$, $(-1,-1)$ (c) $(1,1)$, $(1,-1)$, $(-1,-1)$, $(-1,1)$

(d) $(1,1)$, $(-1,1)$ (e) $(1,-1)$, $(-1,-1)$

9.(6pts) Find the volume of the solid bounded by the surface $z = 6 - xy$ and the planes $x = 2$, $x = -2$, $y = 0$, $y = 3$ and z $= 0$.

(a) 72 (b) 3 (c) 36 (d) 0 (e) 6

10.(6pts) Let $f(x, y, z) = (\sqrt{2})e^{x^2+y^2-z^2}$ and $P = (3, 4, 5)$. Find the maximum rate of change of f at P.

(a) 20 (b) -20 (c) $20e$ (d) 10 (e) $10e$

11.(10pts) Find maximum and minimum points of the function $f(x, y) = 2y + z$ subject to constraints $x + y + z = 1$ and $x^2 + y^2 = 4$.

12.(10pts) Evaluate the integral $\displaystyle\int_0^8 \int_{y^{\frac{1}{3}}}^2 e^{x^4}\,dx\,dy$

13.(10pts) Find the mass and the x-coordinate for center of mass of the lamina that occupies the region D and has the given density function $\rho(x, y) = x + y$, where D is the triangular region with the vertices $(0, 0)$, $(1, 1)$ and $(4, 0)$.

Please set up as iterated integrals which include the limits of integration, but do not solve.

1. Solution. $\int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x^2+1} dy\ dx = \int_0^1 \frac{y^2}{x^2+1}\Big|_0^{\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1} dx = \frac{1}{2} \ln |x^2+1|\Big|_0^1 = \frac{1}{2} \ln 2.$

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2. Solution. $\det \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \det \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$ and $2 > 0$ so local minima.

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3. Solution. The surface in question is the graph of $z = \sqrt{1-x^2-y^2}$ lying over $A = \{(x,y) | x \geqslant 0, y \geqslant 0\}$ $\int \int_A \sqrt{\frac{\partial z}{\partial x}^2 + \frac{\partial z}{\partial y}^2 + 1}\, dA = \int \int_A \sqrt{\frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} + 1}\, dA = \int \int_A \sqrt{\frac{1}{1-x^2-y^2}}\, dA = \int_0^{\pi/2} \int_0^1 \frac{r}{\sqrt{1-r^2}} dr d\theta = -\int_0^{\pi/2} \sqrt{1-r^2}\Big|_0^1 d\theta = \int_0^{|pi/2} d\theta = \frac{\pi}{2}$

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4. Solution. $\nabla f = \langle yz, xz, xy \rangle$, $\nabla g = \langle 2x, 4y, 6z \rangle$ so $\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$ so $xyz = 2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2$.

If $\lambda \neq 0$, $x^2 = 2y^2 = 3z^2$ or $3x^2 = 6$ or $x = \pm\sqrt{2}$. It follows that $y = \pm 1$ and $z = \pm\frac{\sqrt{2}}{\sqrt{3}}$. The value of f at these eight critical points is $\pm\frac{2}{\sqrt{3}}$.

If $\lambda = 0$ then $xyz = 0$ so $x = 0$ or $y = 0$ or $z = 0$ and all the values are 0. Hence the maximum value is $\frac{2}{\sqrt{3}}$.

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5. Solution. $\int \int_A \rho\ dA = \int_0^{\pi/2} \int_0^1 r^2 dr d\theta = \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^1 d\theta = \int_0^{\pi/2} \frac{1}{3} d\theta = \frac{\pi}{6}.$

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6. Solution. A rectangular box has a length, a width and a height: ℓ, w and h. There are four different copies of each kind so the length is $4\ell + 4w + 4h$ and the constraint is $4\ell + 4w + 4h = 24$ or $\ell + w + h = 6$. We are asked to maximize $V = \ell wh$. $\nabla V = \langle wh, \ell h, w\ell \rangle$ and $\nabla g = \langle 1, 1, 1 \rangle$ or $\langle wh, \ell h, w\ell \rangle = \lambda \langle 1, 1, 1 \rangle$.

Hence $\ell wh = \lambda \ell = \lambda w = \lambda h$.

If $\lambda \neq 0$, $\ell = w = h$ so $\ell = w = h = 2$ and $V = 8$. If $\lambda = 0$, two of ℓ, w or h must be 0 and hence $V = 0$, so 8 is the maximum.

7. Solution. If $F(x,y,z) = x^2 + \dfrac{y^2}{2} + \dfrac{z^2}{3}$ then our surface is the level surface $F(x,y,z) = 6$. Hence the normal vector to the tangent plane is $\nabla F(1,2,3)$ or $\left\langle 2x, y, \dfrac{2}{3}z \right\rangle\Big|_{(1,2,3)} = \langle 2,2,2\rangle$. $\langle 2,2,2\rangle \bullet \langle x-1, y-2, z-3\rangle = 0$ or $2x + 2y + 2z = 12$ which is seen to be the same as $x + y + z - 6 = 0$.

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8. Solution. The critical points are zeros of the gradient of $f(x,y) = x + y + x^2y + xy^2$ is $\nabla f(x,y,z) = \langle 1 + 2xy + y^2, 1 + 2xy + x^2\rangle$ so

$$\begin{aligned} 1 + 2xy + y^2 &= 0 \\ 1 + 2xy + x^2 &= 0 \end{aligned}$$

and hence $x^2 = y^2$ and $x = \pm y$.

If $x = y$, the first equation is $1 + 3y^2 = 0$ which has no solutions.

If $x = -y$, the first equation $1 - 2y^2 + y^2 = 0$ so $y = \pm 1$ and the two critical points are $(1,-1)$ and $(-1,1)$.

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9. Solution. $\displaystyle\int\int\int_V 1dV = \int\int_A \int_0^{6-xy} dzdA = \int\int_A 6 - xy\ dA = \int_{-2}^{2}\int_0^3 (6-xy)dydx = \int_{-2}^{2} 6y - \frac{xy^2}{2}\Big|_0^3 dx = \int_{-2}^{2} 18 - 18xdx = 18x - 9x^2\Big|_{-2}^{2} = 0 - (-36-36) = 72.$

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10. Solution. $\nabla f = \left\langle \sqrt{2}(2x)e^{x^2+y^2-z^2}, \sqrt{2}(2y)e^{x^2+y^2-z^2}, -\sqrt{2}(2z)e^{x^2+y^2-z^2}\right\rangle$. At P, $\nabla f(P) = \left\langle 6\sqrt{2}, 8\sqrt{2}, 10\sqrt{2}\right\rangle$ The maximum rate of change is $\left|\nabla f\right| = \sqrt{2\cdot 36 + 2\cdot 64 + 2\cdot 100} = \sqrt{2(100+100)} = 20$.

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11. Solution. The constraint is an ellipse, hence bounded and closed so there are maxima and minima. By Lagrange, $\langle 0,2,1\rangle = \lambda \langle 1,1,1\rangle + \mu \langle 2x, 2y, 0\rangle$, so

$$\begin{aligned} 0 &= \lambda + 2\mu x \\ 2 &= \lambda + 2\mu y \\ 1 &= \lambda \end{aligned}$$

$$\begin{aligned} -1 &= 2\mu x \\ 1 &= 2\mu y \end{aligned}$$

Hence $x = -\dfrac{1}{2\mu}$ and $y = \dfrac{1}{2\mu}$ and from $x^2 + y^2 = 1$ we see $\dfrac{2}{4\mu^2} = 4$ or $\mu^2 = \dfrac{1}{8}$ and $\mu = \pm\dfrac{1}{2\sqrt{2}}$. It follows that $x = \pm\sqrt{2}$ and $y = -x$. Then $x + y = 0$ and hence $z = 1$ so the critical points are $\left(\sqrt{2}, -\sqrt{2}, 1\right)$ and $\left(-\sqrt{2}, \sqrt{2}, 1\right)$.

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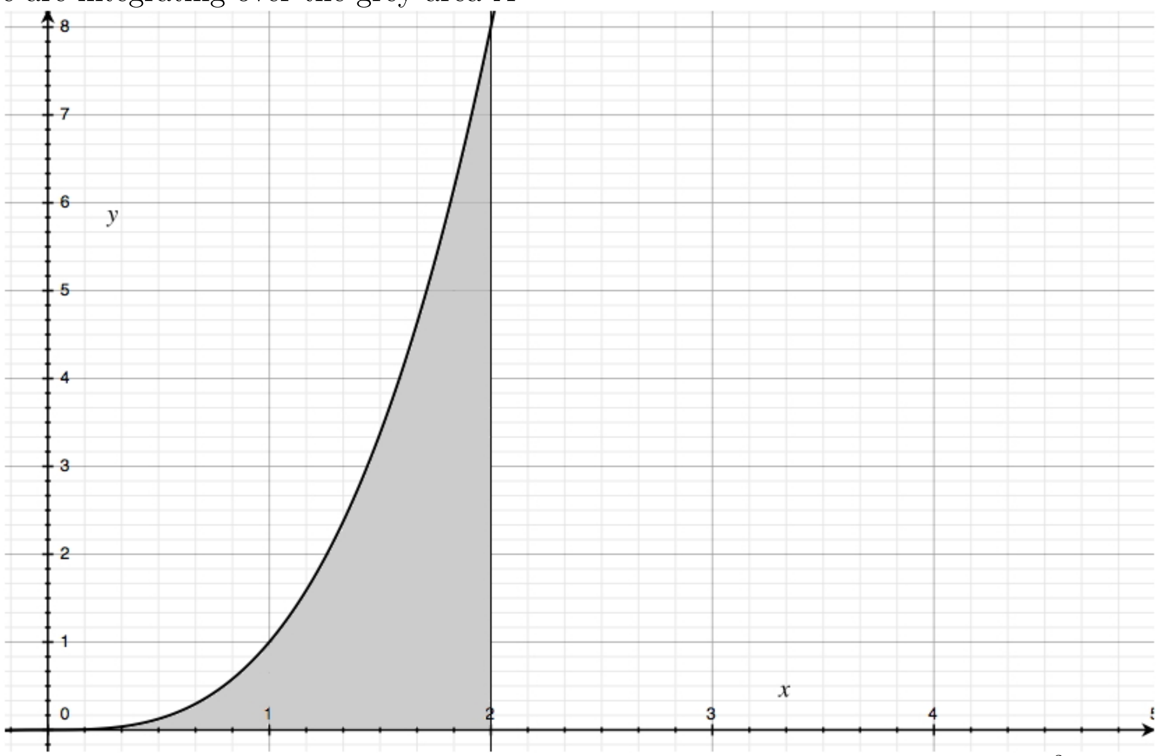
12. Solution. We have no hope of integrating e^{x^4} so we need to switch the order of integration. We are integrating over the grey area A



where the curve is $y = x^3$. Hence $\int_0^8 \int_{y^{\frac{1}{3}}}^2 e^{x^4}\,dx\,dy = \int\int_A e^{x^4}\,dA = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx = \int_0^2 x^3 e^{x^4}\,dx = \frac{1}{4}e^{x^4}\Big|_0^2 = \frac{e^{16}}{4} - \frac{1}{4}$.

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13. Solution. The total mass is $M = \int\int_A (x+y)dA$ and the moment about the y-axis is $M_y = \int\int_A y(x+y)dA$ where A is the triangle. The x-coordinate of the center of mass is $\frac{M_y}{M}$.

The set up for both integrals can be the same but there are two ways to do the integral so here we will do one one way and the other the other. In practice this makes the problem as difficult as possible.

$$M = \int\int_A (x+y)dA = \int_0^1 \int_y^{4-3y} (x+y)dx\,dy$$

$$M_y = \int\int_A y(x+y)dA = \int_0^1 \int_0^x y(x+y)dy\,dx + \int_1^4 \int_0^{(4-x)/3} y(x+y)dy\,dx$$
