Name: SOLUTIONS Date: 09/01/2016

M20550 Calculus III Tutorial
Worksheet 1

1. Find the vector given by the projection of $\mathbf{v} = \langle 3, 1, 4\rangle$ onto $\mathbf{a} = \langle 1, 2, -2\rangle$.

Solution: The projection of $\mathbf{v}$ onto $\mathbf{a}$ is given by $\text{Proj}_{\mathbf{a}}\mathbf{v} = \dfrac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{a}|^2}\,\mathbf{a}$.

We have $|\mathbf{a}|^2 = 1^2+2^2+(-2)^2 = 9$ and $\mathbf{a}\cdot\mathbf{v} = 3(1)+(1)(2)+4(-2) = 3+2-8 = -3$.

Therefore, $\text{Proj}_{\mathbf{a}}\mathbf{v} = \dfrac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{a}|^2}\,\mathbf{a} = \dfrac{-3}{9}\langle 1,2,-2\rangle = -\dfrac{1}{3}\langle 1,2,-2\rangle = \left\langle -\dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{2}{3}\right\rangle$.

2. Find **two** vectors that are perpendicular to the plane that passes through the three points $P(1, 4, 5)$, $Q(-2, 5, -2)$, and $R(1, -1, 0)$.

Solution: First, let's make two vectors out of the points P, Q, R on the plane:

$$\overrightarrow{PQ} = \langle -2-1, 5-4, -2-5\rangle = \langle -3, 1, -7\rangle.$$

$$\overrightarrow{PR} = \langle 1-1, -1-4, 0-5\rangle = \langle 0, -5, -5\rangle.$$

A vector that is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\overrightarrow{PQ}\times\overrightarrow{PR}$:

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix}\mathbf{k}$$
$$= \langle -40, -15, 15\rangle.$$

To find another vector perpendicular to the plane, we find a vector that is parallel to $\langle -40, -15, 15\rangle$. One of which is $\langle -8, -3, 3\rangle$ since $5\langle -8, -3, 3\rangle = \langle -40, -15, 15\rangle$.

3. Is

$$x^2 - 2x + y^2 + z^2 + 7 = 1 - 5x + 2z$$

an equation of a sphere? If so, find the center of the sphere.

Solution: After completing the square for each of the variables x, y, and z, one obtains the equation $(x+\frac{3}{2})^2 + y^2 + (z-1)^2 = -\frac{11}{4}$. This is clearly not the equation of a sphere since we require that the radius squared be a positive number.

4. Let L be a straight line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$. At what point does this line intersect the yz-plane?

> ***Solution:*** First, let's write the equation of L. The vector equation of the line passing through the points A and B is given by
>
> $$\langle x, y, z\rangle = \mathbf{r}_0 + t\mathbf{v},$$
>
> where $\mathbf{r}_0 = \langle 2, 4, -3\rangle$ is the position vector of the point A and $\mathbf{v}$ is the parallel vector. In this case, $\mathbf{v}$ can be chosen to be $\overrightarrow{AB} = \langle 3-2, -1-4, 1-(-3)\rangle = \langle 1, -5, 4\rangle$. So, we get the vector equation for L as follows:
>
> $$\langle x, y, z\rangle = \langle 2, 4, -3\rangle + t\langle 1, -5, 4\rangle = \langle 2+t, 4-5t, -3+4t\rangle.$$
>
> Now L intersects the yz-plane when the x-coordinate of L is zero. Hence, we get $2+t = 0$ or $t = -2$. When $t = -2$, $y = 4-5(-2) = 14$ and $z = -3+4(-2) = -11$. Therefore, L intersects the yz-plane at the point $(0, 14, -11)$.

5. A tow truck drags a stalled car along a road. The chain makes an angle of 30 degrees with the horizontal and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

> ***Solution:*** Let $\mathbf{F}$ and $\mathbf{D}$ be the force and displacement vectors, respectively. From the problem statements, we get $|\mathbf{F}| = 1500$ and $|\mathbf{D}| = 1$. So, the work done is given by
>
> $$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos(30) = 1500\cdot 1\cos(30) = 750\sqrt{3}\,\mathrm{KJ}.$$

6. (a) Find an equation of the sphere that passes through the origin and has center $(2, -2, 1)$.

   (b) What is an equation of the intersection of this sphere with the yz-plane?

> ***Solution:*** (a) The sphere must touch the origin. So, the radius is given by the distance between the center and the origin (this is the same as the length of the position vector of the point $(2, -2, 1)$). Thus, we get $r^2 = 2^2 + (-2)^2 + 1^2 = 9$. And the equation of the sphere with the given center is
>
> $$(x-2)^2 + (y+2)^2 + (z-1)^2 = 9.$$

(b) Visually, the intersection of a sphere with the yz-plane should be a circle. On the yz-plane, the x-coordinate is zero. Putting $x = 0$ into the equation of the sphere above, we get

$$(0-2)^2 + (y+2)^2 + (z-1)^2 = 9 \implies (y+2)^2 + (z-1)^2 = 5$$

So, the equations of the intersection of this sphere with the yz-plane is

$$(y+2)^2 + (z-1)^2 = 5, \quad \text{and} \quad x = 0.$$

This is a circle with center $(0, -2, 1)$ and radius $\sqrt{5}$.

7. Find the volume of the parallelepiped spanned by the vectors

$$\mathbf{u} = \langle 1, 3, -5 \rangle, \mathbf{v} = \langle -1, 0, 2 \rangle, \text{ and } \mathbf{w} = \langle 0, -3, 0 \rangle.$$

Solution: The volume of the parallelepiped spanned by the vectors

$$\mathbf{u} = \langle 1, 3, -5 \rangle, \mathbf{v} = \langle -1, 0, 2 \rangle, \text{ and } \mathbf{w} = \langle 0, -3, 0 \rangle,$$

is given by

$$\text{Vol} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

We have

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 3 & -5 \\ -1 & 0 & 2 \\ 0 & -3 & 0 \end{vmatrix} = 1(0-(-6)) - 3(0-0) + (-5)(3-0) = 6 - 15 = -9.$$

So the volume is

$$\text{Vol} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |-9| = 9.$$

8. If the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by $\text{Comp}_{\mathbf{a}}\mathbf{b} = 1$, what is $\text{Comp}_{2\mathbf{a}}3\mathbf{b}$?

Solution:

$$\text{Comp}_{2\mathbf{a}}3\mathbf{b} = \frac{2\mathbf{a} \cdot 3\mathbf{b}}{|2\mathbf{a}|} = \frac{6(\mathbf{a} \cdot \mathbf{b})}{2|\mathbf{a}|} = \frac{6}{2} \cdot \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = 3\,\text{Comp}_{\mathbf{a}}\mathbf{b} = 3(1) = 3$$

9. Find a vector equation of the line through the point $(1, -1, 1)$ and parallel to the line $x + 2 = \frac{1}{2}y = z - 3$.

Solution: To write the equation of a line, we need one point on the line and a parallel vector, which gives the direction of the line.

We have $(1,-1,1)$ is on the line. So, the position vector of this point is $\langle 1,-1,1\rangle$. The parallel vector of our line should be the same as the parallel vector of the line $x+2=\frac{1}{2}y=z-3$ since two lines are parallel.

Now, $x+2=\frac{1}{2}y=z-3$ is the same as $\dfrac{x+2}{1}=\dfrac{y}{2}=\dfrac{z-3}{1}$. From these symmetric equations, we get the parallel vector $\langle 1,2,1\rangle$. So, our parallel vector is $\langle 1,2,1\rangle$.

Finally, the vector equation of our line is given by

$$\langle x,y,z\rangle=\langle 1,-1,1\rangle+t\,\langle 1,2,1\rangle$$

10. Find the area of the triangle with vertices $(1,1,1)$, $(2,3,1)$, and $(4,2,2)$.

Solution: We know the area of the parallelogram spanned by two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by $|\mathbf{u}\times\mathbf{v}|$. We can make two vectors out of the three points $(1,1,1)$, $(2,3,1)$, and $(4,2,2)$. Let $\mathbf{u}$ be the vector from $(1,1,1)$ to $(2,3,1)$ and $\mathbf{v}$ be the vector from $(1,1,1)$ to $(4,2,2)$, then

$$\mathbf{u}=\langle 2-1,3-1,1-1\rangle=\langle 1,2,0\rangle \quad\text{and}\quad \mathbf{v}=\langle 4-1,2-1,2-1\rangle=\langle 3,1,1\rangle\,.$$

And

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&0\\3&1&1\end{vmatrix}=\langle 2,-1,-5\rangle\,.$$

Now, the area of the triangle with vertices $(1,1,1)$, $(2,3,1)$, and $(4,2,2)$ will be half of the area of the parallelogram spanned by the two vectors $\mathbf{u}$ and $\mathbf{v}$ resulted from these three points. Thus, the required area is given by

$$\text{Area}_{\Delta}=\frac{1}{2}\,|\mathbf{u}\times\mathbf{v}|=\frac{1}{2}\,\big|\,\langle 2,-1,-5\rangle\,\big|=\frac{1}{2}\sqrt{2^2+(-1)^2+(-5)^2}=\frac{1}{2}\sqrt{30}.$$