Name: SOLUTIONS Date: 09/08/2016

M20550 Calculus III Tutorial
Practice Problems

1. Find the unit tangent, the (principal) unit normal, and the binormal vectors to the curve $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, 3t^2 \rangle$ at $t = \pi$.

> ***Solution:*** We have $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, 3t^2 \rangle$. So
>
> $$\mathbf{r}'(t) = \langle 2\cos 2t, -2\sin 2t, 6t \rangle \implies \mathbf{r}'(\pi) = \langle 2, 0, 6\pi \rangle .$$
>
> $$\mathbf{r}''(t) = \langle -4\sin 2t, -4\cos 2t, 6 \rangle \implies \mathbf{r}''(\pi) = \langle 0, -4, 6 \rangle .$$
>
> Also,
>
> $$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \langle 2, 0, 6\pi \rangle \times \langle 0, -4, 6 \rangle = \langle 24\pi, -12, -8 \rangle = 4 \langle 6\pi, -3, -2 \rangle .$$
>
> Then,
>
> $$\mathbf{T}(\pi) = \frac{\mathbf{r}'(\pi)}{|\mathbf{r}'(\pi)|} = \frac{\langle 2, 0, 6\pi \rangle}{|\langle 2, 0, 6\pi \rangle|} = \frac{1}{\sqrt{4 + 36\pi^2}} \langle 2, 0, 6\pi \rangle .$$
>
> $$\mathbf{B}(\pi) = \frac{\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)}{|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)|} = \frac{4 \langle 6\pi, -3, -2 \rangle}{4 |\langle 6\pi, -3, -2 \rangle|} = \frac{1}{\sqrt{13 + 36\pi^2}} \langle 6\pi, -3, -2 \rangle .$$
>
> $$\begin{aligned} \mathbf{N}(\pi) = \mathbf{B}(\pi) \times \mathbf{T}(\pi) &= \frac{1}{\sqrt{13 + 36\pi^2}} \langle 6\pi, -3, -2 \rangle \times \frac{1}{\sqrt{4 + 36\pi^2}} \langle 2, 0, 6\pi \rangle \\ &= \frac{1}{\sqrt{13 + 36\pi^2}} \frac{1}{\sqrt{4 + 36\pi^2}} \langle 6\pi, -3, -2 \rangle \times \langle 2, 0, 6\pi \rangle \\ &= \frac{1}{\sqrt{13 + 36\pi^2}} \frac{1}{\sqrt{4 + 36\pi^2}} \left\langle -18\pi, -4 - 36\pi^2, 6 \right\rangle . \end{aligned}$$

2. Find the equation for the normal and osculating planes to the curve $\mathbf{r}(t) = 2\cos(3t)\mathbf{i} + t\mathbf{j} + 2\sin(3t)\mathbf{k}$ at the point $(-2, \pi, 0)$.

> ***Solution:*** First, we note that t corresponds to the point $(-2, \pi, 0)$ is $t = \pi$ since $\mathbf{r}(t) = \langle 2\cos(3t), t, 2\sin(3t) \rangle = \langle -2, \pi, 0 \rangle$ implies $t = \pi$ by looking at the second component.
>
> A normal vector of the normal plane at $t = \pi$ is $\mathbf{r}'(\pi)$. We have
>
> $$\mathbf{r}'(t) = \langle -6\sin(3t), 1, 6\cos(3t) \rangle \implies \mathbf{r}'(\pi) = \langle 0, 1, -6 \rangle .$$
>
> So, the normal plane at the point $(-2, \pi, 0)$ is given by
>
> $$\langle 0, 1, -6 \rangle \cdot \langle x, y, z \rangle = \langle 0, 1, -6 \rangle \cdot \langle -2, \pi, 0 \rangle \implies y - 6z = \pi.$$

A normal vector of the osculating plane at $t = \pi$ is $\mathbf{r}^{'}(\pi) \times \mathbf{r}^{''}(\pi)$. We have, $\mathbf{r}^{''}(t) = \langle -18\cos(3t), 0, -18\sin(3t)\rangle$ and so $\mathbf{r}^{''}(\pi) = \langle 18, 0, 0\rangle$. Then,

$$\mathbf{r}^{'}(\pi) \times \mathbf{r}^{''}(\pi) = \langle 0, 1, -6\rangle \times \langle 18, 0, 0\rangle = 18\,\langle 0, 1, -6\rangle \times \langle 1, 0, 0\rangle = 18\,\langle 0, -6, -1\rangle\,.$$

So, we can take $\langle 0, 6, 1\rangle$ to be a normal vector for this osculating plane. And the equation is

$$\langle 0, 6, 1\rangle \bullet \langle x, y, z\rangle = \langle 0, 6, 1\rangle \bullet \langle -2, \pi, 0\rangle \implies 6y + z = 6\pi.$$

3. A particle moves with position function $\mathbf{r}(t) = \langle \cos t, \sin t, \cos^2 t\rangle$. Find the tangential and normal components of acceleration when $t = \pi/4$.

Solution: We have

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, -2\cos t \sin t\rangle = \langle -\sin t, \cos t, -\sin(2t)\rangle$$

$$\implies \mathbf{r}'(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right\rangle,$$

$$\mathbf{r}^{''}(t) = \langle -\cos t, -\sin t, -2\cos(2t)\rangle \implies \mathbf{r}^{''}(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\rangle.$$

And so

$$a_T = \frac{\mathbf{r}'(\pi/4) \bullet \mathbf{r}^{''}(\pi/4)}{|\,\mathbf{r}'(\pi/4)\,|} = 0.$$

We know $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$. Since $a_T = 0$, we get $\mathbf{a} = a_N\mathbf{N}$. So,

$$|\mathbf{a}| = a_N|\mathbf{N}| = a_N \cdot 1 = a_N.$$

Thus,

$$\begin{aligned} a_N = |\mathbf{a}| &= |\,\mathbf{r}''(\pi/4)\,| \\ &= \left|\left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\rangle\right| \\ &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\ &= 1 \end{aligned}$$

4. Find the arc length of the curve $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3\right\rangle$, $0 \le t \le 1$.

Solution:

First, we need the derivative:

$$\mathbf{r}'(t) = \left\langle 2, 2t, t^2 \right\rangle$$

and its magnitude

$$\left|\mathbf{r}'(t)\right| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2 + t^2)^2} = 2 + t^2 \quad \text{since } 2 + t^2 > 0.$$

The arc length is then

$$L = \int_0^1 \left|\mathbf{r}'(t)\right| dt = \int_0^1 (2 + t^2) dt = 2t + \frac{1}{3}t^3 \Big|_0^1 = \frac{7}{3} - 0 = \frac{7}{3}.$$