Name: SOLUTIONS Date: 10/06/2016

M20550 Calculus III Tutorial
Worksheet 6

1. Write an equation of the tangent line to the curve of intersection between the two surfaces defined by $z = x^2 + y^2$ and $x^2 + 2y^2 + z^2 = 7$ at the point $(-1, 1, 2)$.

   Hint: Think about the geometry of the gradient vectors. You don't have to parametrize the curve to do this problem.

> ***Solution:*** The surface $z = x^2 + y^2$ can be written as the level surface $F(x, y, z) = x^2 + y^2 - z = 0$; and so the gradient of F is
>
> $$\nabla F(x, y, z) = \langle 2x, 2y, -1 \rangle .$$
>
> Also, the gradient of the level surface $G(x, y, z) = x^2 + 2y^2 + z^2 = 7$ is
>
> $$\nabla G(x, y, z) = \langle 2x, 4y, 2z \rangle .$$
>
> The tangent vector at $(-1, 1, 2)$ of the curve of intersection between these two surfaces is perpendicular to both vectors $\nabla F(-1, 1, 2) = \langle -2, 2, -1 \rangle$ and $\nabla G(-1, 1, 2) = \langle -2, 4, 4 \rangle$. And
>
> $$\nabla F(-1, 1, 2) \times \nabla G(-1, 1, 2) = \langle -2, 2, -1 \rangle \times \langle -2, 4, 4 \rangle = \langle 12, 10, -4 \rangle .$$
>
> Thus, $\langle 12, 10, -4 \rangle$ is a parallel vector of the tangent line to the curve of intersection at $(-1, 1, 2)$. Thus, an equation of the required tangent line is
>
> $$\langle x, y, z \rangle = \langle -1, 1, 2 \rangle + t \langle 12, 10, -4 \rangle .$$

2. Find the tangent plane and the normal line to the surface $x^2y + xz^2 = 2y^2z$ at the point $P = (1, 1, 1)$.

> ***Solution:*** The given surface is the zero level surface of the function $F(x, y, z) = x^2y + xz^2 - 2y^2z$. So, the normal vector to the tangent plane at the point $P(1, 1, 1)$ is given by $\nabla F(1, 1, 1)$. We have
>
> $$\nabla F(x, y, z) = \langle 2xy + z^2, x^2 - 4yz, 2xz - 2y^2 \rangle \implies \nabla F(1, 1, 1) = \langle 3, -3, 0 \rangle.$$
>
> Thus, the equation of the tangent plane at $(1, 1, 1)$ is
>
> $$3(x - 1) - 3(y - 1) = 0 \implies x - y = 0,$$
>
> and the equation for the normal line at $(1, 1, 1)$ is
>
> $$\langle x, y, z \rangle = \langle 1, 1, 1 \rangle + t\langle 3, -3, 0 \rangle = \langle 1 + 3t, 1 - 3t, 1 \rangle.$$

3. Find a point on the surface $z = x^2 - y^3$ where the tangent plane is parallel to the plane $x + 3y + z = 0$.

Solution: First, rewrite $z = x^2 - y^3$ into the level surface $F(x, y, z) = z - x^2 + y^3 = 0$ then $\nabla F(x, y, z) = \langle -2x, 3y^2, 1\rangle$. Since we want a point (x, y, z) such that the tangent plane at this point is parallel to the plane $x+3y+z = 0$, we can have x, y, z satisfy:

$$\left\langle -2x, 3y^2, 1\right\rangle = \langle 1, 3, 1\rangle \text{ where } \langle 1, 3, 1\rangle \text{ is a normal vector of } x + 3y + z = 0.$$

Thus, we get $-2x = 1 \implies x = -\frac{1}{2}$, and $3y^2 = 3 \implies y = \pm 1$. We only need one point so pick $y = 1$ and with $x = -\frac{1}{2}$, we get $z = \left(-\frac{1}{2}\right)^2 - (1)^3 = -\frac{3}{4}$. So, $\left(-\frac{1}{2}, 1, -\frac{3}{4}\right)$ is one point we're looking for.

4. Find all the critical points of $f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$.

Solution: We want to find all points such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$. We have

$$\begin{cases} f_x(x, y) = 6xy - 12x = 0 & (1) \\ f_y(x, y) = 3y^2 + 3x^2 - 12y = 0 & (2) \end{cases}$$

Equation (1) implies $6x(y - 2) = 0 \implies x = 0$ or $y = 2$.

- When $x = 0$, equation (2) is equivalent to $3y^2 - 12y = 0 \implies 3y(y - 4) = 0 \implies y = 0$ or $y = 4$. So, we get the points $(0, 0)$ and $(0, 4)$.
- When $y = 2$, equation (2) is equivalent to $12 + 3x^2 - 24 = 0 \implies x^2 = 4 \implies x = -2$ or $x = 2$. So, we get the points $(-2, 2)$ and $(2, 2)$ here.

Thus, all the critical points of f are $(0, 0)$, $(0, 4)$, $(-2, 2)$, $(2, 2)$.

5. Find the local maximum and the local minimum value(s) and saddle point(s) of the function $z = x^3 + y^3 - 3xy + 1$.

Solution: First, let's find all the critical points of $z = x^3 + y^3 - 3xy + 1$:

$$\begin{cases} z_x(x,y) = 3x^2 - 3y = 0 \implies y = x^2 & (1) \\ z_y(x,y) = 3y^2 - 3x = 0 & (2) \end{cases}$$

With $y = x^2$, equation (2) becomes $3x^4 - 3x = 0 \implies 3x(x^3 - 1) = 0 \implies x = 0$ or $x = 1$. Thus, all the critical points are $(0,0)$ and $(1,1)$.

Now, we will use the Second Derivative Test to test each critical point. We want to compute

$$D(x,y) = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = z_{xx}z_{yy} - z_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9.$$

And we have

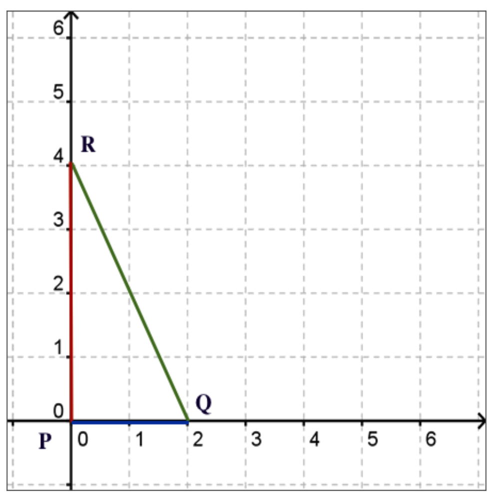
$$D(0,0) = -9 < 0 \implies (0,0) \text{ is a saddle point.}$$

$$D(1,1) = 36 - 9 > 0 \text{ and } z_{xx}(1,1) = 6 > 0 \implies z(1,1) \text{ is a local minimum.}$$

In conclusion, the local minimum value of z is $z(1,1) = 1^3 + 1^3 - 3(1)(1) + 1 = 0$. And $(0,0)$ is the saddle point of z, i.e. $z(0,0)$ is neither a local minimum nor local maximum.

6. Identify the absolute maximum and absolute minimum values attained by $g(x,y) = x^2y - 2x^2$ within the triangle T bounded by the points $P(0,0)$, $Q(2,0)$, and $R(0,4)$.

Solution: The picture for the triangle T:



First, we find all critical points in the interior of the triangle:

$$\begin{cases} g_x(x,y) = 2xy - 4x = 0 & (1) \\ g_y(x,y) = x^2 = 0 & (2) \end{cases}$$

Equation (2) tells us that x must be zero. And when $x = 0$, equation (1) is true automatically giving us the points $(0, y)$ for $0 \leq y \leq 4$ are the solutions of this system of equations. So, all the critical points are exactly the boundary PR of the triangle. So, we get no critical point inside the triangle. We move on to analyze the boundaries.

On the boundary PR, we have $x = 0$ and $0 \leq y \leq 4$. And, $g(0, y) = 0$.

On the boundary PQ, we have $0 \leq x \leq 2$ and $y = 0$. And, $g(x, 0) = -2x^2$. The graph of $-2x^2$ is a parabola concaves downward. So, $g(x, 0) = -2x^2$ with $0 \leq x \leq 2$ attains a maximum value of 0 when $x = 0$ and a minimum value of -8 when $x = 2$.

On the boundary QR, we have $y = -2x + 4$ with $0 \leq x \leq 2$. And, $g(x, -2x + 4) = x^2(-2x + 4) - 2x^2 = -2x^3 + 2x^2$, for $0 \leq x \leq 2$. The critical numbers of $-2x^3 + 2x^2$ for $0 \leq x \leq 2$ are $x = 0$ and $x = \frac{2}{3}$. So, g has a minimum of 0 at $x = 0$ and a maximum of $\frac{8}{27}$ at $x = \frac{2}{3}$, $y = \frac{8}{3}$ on this boundary.

Here is a summary of the results:

(x, y)	$g(x, y)$
$(0, y)$	0
$(2, 0)$	-8
$\left(\frac{2}{3}, \frac{8}{3}\right)$	$\frac{8}{27}$

So, we conclude that on the whole triangle (including boundaries), the function has an absolute maximum of $\frac{8}{27}$ at $\left(\frac{2}{3}, \frac{8}{3}\right)$ and an absolute minimum of -8 at $(2, 0)$.

7. Identify the absolute maximum and absolute minimum values attained by $z = 4x^2 - y^2 + 1$ within the region R bounded by the curve $4x^2 + y^2 = 16$.

Solution: First, we find the critical points in the interior of the region R. We have

$$\begin{cases} z_x(x, y) = 8x = 0 & \Longrightarrow x = 0 \\ z_y(x, y) = -2y = 0 & \Longrightarrow y = 0 \end{cases}$$

So, the only critical point inside R is $(0, 0)$.

Move on to the boundary $4x^2 + y^2 = 16$. Note that this is the ellipse $\frac{x^2}{2^2} + \frac{y^2}{4^2} = 1$. On this boundary, we have $y^2 = 16 - 4x^2$ and $-2 \leq x \leq 2$. So, we get $z =$

$4x^2 - (16 - 4x^2) + 1 = 8x^2 - 15$ for $-2 \leq x \leq 2$. The critical number here satisfies $16x = 0 \implies x = 0$. With $x = 0$, $y^2 = 16 - 4 \cdot 0^2 \implies y = \pm 4$. So, the critical points on the boundary are $(0, -4)$ and $(0, 4)$. And the end points here are $(-2, 0)$ and $(2, 0)$.

Finally, let's compute the values of z at all the points of "interest":

(x, y)	$z = 4x^2 - y^2 + 1$
$(0, 0)$	1
$(0, -4)$	-15
$(0, 4)$	-15
$(-2, 0)$	17
$(2, 0)$	17

In conclusion, the absolute maximum value of z is 17 and it occurs at the points $(-2, 0)$ and $(2, 0)$. The absolute minimum value of z is -15 and it occurs at the points $(0, -4)$ and $(0, 4)$.

Another way of finding extrema on the **boundary** is to use the method of Lagrange Multipliers. In this language, we want to find the extrema of $z = 4x^2 - y^2 + 1$ subject to the constraint $g(x, y) = 4x^2 + y^2 = 16$. We have $\nabla z = \lambda \nabla g$ for some constant λ. So, we get the system of equations:

$$\begin{cases} 8x = \lambda 8x & (1) \\ -2y = \lambda 2y & (2) \\ 4x^2 + y^2 = 16 & (3) \end{cases}$$

Equation (1) $\Leftrightarrow 8x(1 - \lambda) = 0 \implies x = 0$ or $\lambda = 1$.

- If $x = 0$, then from equation (3) we get $y = \pm 4$. And so we get $(0, \pm 4)$ as the points of interest.
- If $\lambda = 1$, then from equation (2) we get $y = 0$. With $y = 0$, equation (3) gives $x = \pm 2$. So, the points of interest are $(\pm 2, 0)$.

Then, we can create the table like we did above to find the absolute max and min of z.

8. Find the point(s) on the surface $y^2 = 9 + xz$ that are closest to the origin.

Solution: The distance between any point (x, y, z) on the given surface to the origin is given by

$$d = \sqrt{x^2 + y^2 + z^2}.$$

Basically, for this problem, we want to find the absolute minimum of d on the surface $y^2 = 9 + xz$. To avoid the square root, we can minimize the function $L = d^2 = x^2 + y^2 + z^2$ instead.

We want to eliminate one variable in L. We have $y^2 = 9 + xz$. So, we get $L(x, z) = x^2 + (9 + xz) + z^2$. Now, let's find the critical point(s) for L:

$$\begin{cases} L_x = 2x + z = 0 & (1) \\ L_z = x + 2z = 0 & (2) \end{cases}$$

Solving the above system of equation we get $x = 0$ and $z = 0$. So, the only critical point is $(0, 0)$. Now, we use the Second Derivative test to classify this critical point:

$$D(x, z) = (L_{xx})(L_{zz}) - (L_{xz})^2 = 2 \cdot 2 - 1^2 = 3 > 0.$$

So, $D(0, 0) = 3 > 0$ and $L_{xx}(0, 0) = 2 > 0$. And since $(0, 0)$ is the only critical point of $L(x, z)$, we get that at $(0, 0)$, L attains an absolute minimum.

To get the points we want, we need to find y when $x = 0 = z$. From the equation $y^2 = 9 + xz$, we get $y = \pm 3$. Finally, the points on the surface $y^2 = 9 + xz$ that are closest to the origin is $(0, -3, 0)$ and $(0, 3, 0)$.