Name: SOLUTIONS Date: 11/10/2016

M20550 Calculus III Tutorial
Worksheet 9

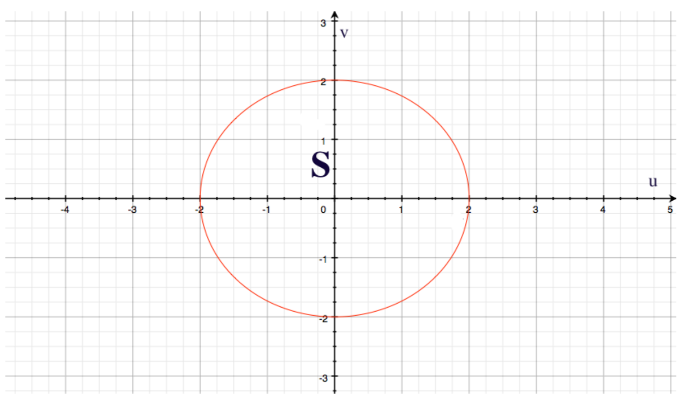
1. Compute $\displaystyle\iint_R \frac{1}{2}\, dA$ where R is the region bounded by $2x^2 + 2xy + y^2 = 8$ using the change of variables given by $x = u + v$ and $y = -2v$.

Solution: We know R is the region bounded by $2x^2 + 2xy + y^2 = 8$. Using the transformation $x = u + v$ and $y = -2v$, the boundary $2x^2 + 2xy + y^2 = 8$ will turn into

$$2(u + v)^2 + 2(u + v)(-2v) + (-2v)^2 = 8.$$

$$\implies u^2 + v^2 = 4.$$

So, the transformation of R, denote S, is the region bounded by the circle $u^2 + v^2 = 4$ in the uv-plane.



Before proceeding to compute the double integral, we need to find the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = (1)(-2) - (1)(0) = -2.$$

Thus,

$$\begin{aligned}
\iint_R \frac{1}{2}\,dA &= \iint_S \frac{1}{2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dA \\
&= \int_0^{2\pi}\int_0^2 \frac{1}{2}|-2|r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{r=2}\,d\theta \\
&= \int_0^{2\pi} 2\,d\theta \\
&= 4\pi.
\end{aligned}$$

2. Let R be the parallelogram enclosed by the lines $x+3y=0$, $x+3y=2$, $x+y=1$, and $x+y=4$. Evaluate the following integral by making appropriate change of variables

$$\iint\limits_R \frac{x+3y}{(x+y)^2}\,dA.$$

Solution: Observe the set of equations:

$$x+3y=0 \qquad\qquad x+3y=2$$
$$x+y=1 \qquad\qquad x+y=4$$

So, if we let

$$u = x+3y \quad \text{and } v = x+y,$$

then the transformation of R, denote S, is given by the region bounded by the lines

$$u=0 \qquad\qquad u=2$$
$$v=1 \qquad\qquad v=4$$

So, S is the region bounded by the rectangle $[0,2]\times[1,4]$ in the uv-plane.

Next, we need to compute the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

In order to compute these partials, we need to write x and y in terms of u and v. We have

$$\begin{aligned} x + 3y &= u \quad (eq\ 1) \\ x + y &= v \quad (eq\ 2) \end{aligned}$$

$(eq\ 1) - (eq\ 2)$ is equivalent to $2y = u - v \implies y = \frac{1}{2}u - \frac{1}{2}v$. And $(eq\ 1) - 3(eq\ 2)$ gives $-2x = u - 3v \implies x = -\frac{1}{2}u + \frac{3}{2}v$. So,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}.$$

And so, we get

$$\begin{aligned} \iint\limits_R \frac{x+3y}{(x+y)^2}\,dA &= \iint\limits_S \frac{u}{v^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA \\ &= \int_1^4 \int_0^2 \frac{u}{v^2}\left|-\frac{1}{2}\right| du\,dv \\ &= \int_1^4 \frac{1}{4}u^2v^{-2}\Big|_{u=0}^{u=2} dv \\ &= \int_1^4 v^{-2}\,dv \\ &= -\frac{1}{v}\Big|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}. \end{aligned}$$

3. Evaluate the line integral $\int_C (z - 2xy)\,ds$ along the curve C given by $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$, $0 \le t \le \frac{\pi}{2}$.

Solution: $\int_C (z - 2xy)\,ds$ is a line integral with respect to arc length (because of the ds at end). Since $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$, we get $x(t) = \sin t$, $y(t) = \cos t$, $z(t) = t$. So, $z - 2xy = t - 2\sin t \cos t$. And $\mathbf{r}'(t) = \langle \cos t, -\sin t, 1\rangle$. So,

$$ds = |\mathbf{r}'(t)|dt = \sqrt{(x')^2 + (y')^2 + (z')^2}\,dt = \sqrt{\cos^2 t + (-\sin t)^2 + 1^2} = \sqrt{2}\,dt.$$

Thus, for $0 \le t \le \frac{\pi}{2}$,

$$\begin{aligned}\int_C (z - 2xy)\, ds &= \int_0^{\pi/2} (t - 2\sin t \cos t)\sqrt{2}\, dt \\ &= \sqrt{2}\left[\frac{1}{2}t^2 - \sin^2 t\right]_0^{\pi/2} \\ &= \sqrt{2}\left[\frac{\pi^2}{8} - 1\right].\end{aligned}$$

4. Find $\int_C 2xy^3\, ds$ where C is the upper half of the circle $x^2 + y^2 = 4$.

Solution: First, let's parametrize the curve C. C is the upper half of the circle $x^2 + y^2 = 4$. So, we can let

$$x(t) = 2\cos t, \qquad y(t) = 2\sin t \qquad \text{for } 0 \le t \le \pi.$$

Then, $x'(t) = -2\sin t$ and $y'(t) = 2\cos t$. Therefore,

$$ds = \sqrt{(x')^2 + (y')^2}\, dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = \sqrt{4\sin^2 t + 4\cos^2 t} = 2\, dt.$$

Thus, for $0 \le t \le \pi$,

$$\begin{aligned}\int_C 2xy^3\, ds &= \int_0^{\pi} 2\,(2\cos t)\,(2\sin t)^3\; 2\, dt \\ &= \int_0^{\pi} 32\left(\sin^3 t\right)(\cos t)\; dt \\ &= 8\left[\sin^4 t\right]_0^{\pi} \\ &= 0.\end{aligned}$$

5. Calculate the line integral $\int_C (y^2 + x)\, dx + 4xy\, dy$ where C is the arc of $x = y^2$ from $(1, 1)$ to $(4, 2)$.

Solution: First, we need to parametrize the curve C. Since C is a part of the curve $x = y^2$, we can let $y = t$; then we have $x = t^2$. Moreover, since the curve C is the part from $(1,1)$ to $(4,2)$, we get $1 \le y \le 2$. So, we have $1 \le t \le 2$. Thus, a parametrization of C is as follows:

$$x(t) = t^2, \qquad y(t) = t \qquad \text{for } 1 \le t \le 2.$$

Now, $\int_C (y^2 + x)\,dx + 4xy\,dy$ is a line integral with respect to x and y because we see the dx and dy. Here,

$$dx = x'(t)\,dt = 2t\,dt \quad \text{and} \quad dy = y'(t)\,dt = 1\,dt.$$

So, for $1 \le t \le 2$,

$$\begin{aligned}\int_C (y^2 + x)\,dx + 4xy\,dy &= \int_1^2 \left[\left(t^2 + t^2\right) 2t + 4(t^2)(t) \right] dt \\ &= \int_1^2 8t^3\,dt \\ &= \left[2t^4\right]_1^2 \\ &= 2^5 - 2 = 30.\end{aligned}$$

6. Evaluate the line integral $\int_C z^2\,dx + x\,dy + y\,dz$ where C is the line segment from $(1,0,0)$ to $(4,1,2)$.

Solution: First, we parametrize C, the line segment **from** $(1,0,0)$ **to** $(4,1,2)$. For $0 \le t \le 1$, C can be written as the vector function

$$\mathbf{r}(t) = \langle 1,0,0 \rangle + t\Big(\langle 4,1,2 \rangle - \langle 1,0,0 \rangle \Big) = \langle 1,0,0 \rangle + t\,\langle 3,1,2 \rangle.$$

So, $x(t) = 1 + 3t$, $y(t) = t$, and $z(t) = 2t$ for $0 \le t \le 1$. Then,

$$dx = x'(t)\,dt = 3\,dt, \quad dy = y'(t)\,dt = 1\,dt, \quad dz = z'(t)\,dt = 2\,dt.$$

Hence, for $0 \leq t \leq 1$,

$$\begin{aligned}\int_C z^2\,dx + x\,dy + y\,dz &= \int_0^1 \Big[(2t)^2(3) + (1+3t)(1) + t(2)\Big]\,dt \\ &= \int_0^1 \big[12t^2 + 5t + 1\big]\,dt \\ &= \left[4t^3 + \frac{5}{2}t^2 + t\right]_0^1 \\ &= \frac{15}{2}.\end{aligned}$$

7. Compute $\displaystyle\int_C x^2\,ds$ where C is the intersection of the surface $x^2+y^2+z^2=4$ and the plane $z=\sqrt{3}$.

Solution: The intersection of the sphere $x^2+y^2+z^2=4$ and the plane $z=\sqrt{3}$ is the circle

$$x^2+y^2+\left(\sqrt{3}\right)^2 = 4, \quad z=\sqrt{3}$$

$$\text{or simply} \quad x^2+y^2=1, \quad z=\sqrt{3}.$$

Thus, a parametrization of C could be

$$\mathbf{r}(t) = \left\langle \cos t, \sin t, \sqrt{3}\right\rangle \quad \text{for } 0 \leq t \leq 2\pi.$$

Then, $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle \implies |\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$.
So $ds = |\mathbf{r}'(t)|\,dt = 1\,dt$. Finally, for $0 \leq t \leq 2\pi$,

$$\begin{aligned}\int_C x^2\,ds &= \int_0^{2\pi} \left(\cos^2 t\right)\,dt \\ &= \int_0^{2\pi} \frac{1}{2}\big(1+\cos 2t\big)\,dt \\ &= \frac{1}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} \\ &= \pi.\end{aligned}$$