Name: SOLUTIONS Date: 12/08/2016

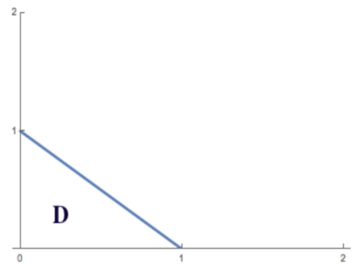
M20550 Calculus III Tutorial
Worksheet 11

1. Compute the surface integral $\iint_S (x+y+z)\,dS$, where S is a surface given by $\mathbf{r}(u,v) = \langle u+v, u-v, 1+2u+v\rangle$ and $0 \le u \le 2,\ 0 \le v \le 1$.

> ***Solution:*** First, we know
>
> $$\iint_S (x+y+z)\,dS = \iint_D \Big[(u+v)+(u-v)+(1+2u+v)\Big]|\mathbf{r}_u \times \mathbf{r}_v|\,dA,$$
>
> where D is the domain of the parameters $u,\ v$ given by $0 \le u \le 2,\ 0 \le v \le 1$.
> We have $\mathbf{r}_u = \langle 1,1,2\rangle$ and $\mathbf{r}_v = \langle 1,-1,1\rangle$. Then,
> $\mathbf{r}_u \times \mathbf{r}_v = \langle 1,1,2\rangle \times \langle 1,-1,1\rangle = \langle 3,1,-2\rangle$. So,
>
> $$|\mathbf{r}_u \times \mathbf{r}_v| = |\langle 3,1,-2\rangle| = \sqrt{3^2+1^2+(-2)^2} = \sqrt{14}.$$
>
> Thus,
>
> $$\begin{aligned}\iint_S (x+y+z)\,dS &= \int_0^1\int_0^2 (4u+v+1)\sqrt{14}\,du\,dv\\ &= 11\sqrt{14}.\end{aligned}$$

2. Let S be the portion of the graph $z = 4-2x^2-3y^2$ that lies over the region in the xy-plane bounded by $x=0,\ y=0$, and $x+y=1$. Write the integral that computes $\iint_S \left(x^2+y^2+z\right)\,dS$.

> ***Solution:*** First, we need a parametrization of the surface S. Since S is a surface given by the equation $z = 4-2x^2-3y^2$, we can choose x and y to be the parameters. So,
>
> $$\mathbf{r}(x,y) = \left\langle x, y, 4-2x^2-3y^2\right\rangle,$$
>
> and the domain D of the parameters $x,\ y$ is given by the region in the xy-plane bounded by $x=0,\ y=0$, and $x+y=1$ (see picture below)



Now, $\mathbf{r}_x = \langle 1, 0, -4x \rangle$ and $\mathbf{r}_y = \langle 0, 1, -6y \rangle$. So, $\mathbf{r}_x \times \mathbf{r}_y = \langle 4x, 6y, 1 \rangle$ and $|\mathbf{r}_x \times \mathbf{r}_y| = |\langle 4x, 6y, 1 \rangle| = \sqrt{16x^2 + 36y^2 + 1}$. Thus,

$$\begin{aligned}\iint_S \left(x^2 + y^2 + z\right) dS &= \iint_D x^2 + y^2 + (4 - 2x^2 - 3y^2)|\mathbf{r}_x \times \mathbf{r}_y|\, dA \\ &= \int_0^1 \int_0^{-x+1} (4 - x^2 - 2y^2)\sqrt{16x^2 + 36y^2 + 1}\; dy\, dx.\end{aligned}$$

3. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ and S is a surface given by

$$x = 2u, \quad y = 2v, \quad z = 5 - u^2 - v^2,$$

where $u^2 + v^2 \leq 1$. S has downward orientation.

Solution: We have $\mathbf{r}(u, v) = \langle 2u, 2v, 5 - u^2 - v^2 \rangle$, so $\mathbf{r}_u = \langle 2, 0, -2u \rangle$ and $\mathbf{r}_v = \langle 0, 2, -2v \rangle$ and so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2, 0, -2u \rangle \times \langle 0, 2, -2v \rangle = \langle 4u, 4v, 4 \rangle .$$

Note that $\mathbf{r}_u \times \mathbf{r}_v = \langle 4u, 4v, 4 \rangle$ gives unit normal vectors pointing upward (z-component is positive). But, S has downward orientation so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -\iint\limits_{u^2+v^2 \leq 1} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\; dA.$$

Now, $\mathbf{F}\big(\mathbf{r}(u, v)\big) = \langle 2v, -2u, 5 - u^2 - v^2 \rangle$. So

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \left\langle 2v, -2u, 5 - u^2 - v^2 \right\rangle \cdot \langle 4u, 4v, 4 \rangle = 20 - 4u^2 - 4v^2.$$

Thus,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -\iint\limits_{u^2+v^2\leq 1} (20 - 4u^2 - 4v^2)\, dA.$$

$$\overset{\text{polar}}{=} -\int_0^{2\pi} \int_0^1 (20 - 4r^2) r\, dr\, d\theta$$

$$= -18\pi.$$

4. Compute the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 0$ and $z = 2$ with normal pointing away from the origin.

Solution: We want to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 0$ and $z = 2$ with normal pointing away from the origin.

Note that this is not a closed surface (it has no top nor bottom), otherwise, we would use Divergence Theorem. This flux integral doesn't seem to be difficult to compute directly. First, we parametrize S: let $x = 2\cos u$, $y = 2\sin u$, $z = v$. Then

$$\mathbf{r}(u, v) = \langle 2\cos u, 2\sin u, v\rangle, \quad \text{domain } D \text{ is } \ 0 \leq u \leq 2\pi,\ 0 \leq v \leq 2.$$

Then, $\mathbf{r}_u = \langle -2\sin u, 2\cos u, 0\rangle$ and $\mathbf{r}_v = \langle 0, 0, 1\rangle$. So,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2\sin u, 2\cos u, 0\rangle \times \langle 0, 0, 1\rangle = \langle 2\cos u, 2\sin u, 0\rangle.$$

Now, let's check our orientation. Let's take the point where $u = \pi/2$ and $v = 1$, ie $(x, y, z) = (0, 2, 1)$. At the point $(0, 2, 1)$, the unit normal vector points in the direction of the vector $(\mathbf{r}_u \times \mathbf{r}_v)(\pi/2, 1) = \langle 0, 2, 0\rangle$. This means the unit normal vector is pointing away from the origin. So, our parametrization of S gives the correct orientation for S. Moving on!

Now, $\mathbf{F}\big(\mathbf{r}(u, v)\big) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \langle 2\cos u, 2\sin u, v\rangle \cdot \langle 2\cos u, 2\sin u, 0\rangle = 4$.

Thus,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA$$

$$= \iint_D 4\, dA.$$

$$= \int_0^{2\pi} \int_0^2 4\, dv\, du$$

$$= 16\pi.$$

5. Let S be the surface defined as $z = 4 - 4x^2 - y^2$ with $z \geq 0$ and oriented upward. Let $\mathbf{F} = \langle x - y, x + y, ze^{xy} \rangle$. Compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$. (*Hint*: use one of the theorems you learned in class.)

Solution: This question uses Stokes' theorem: S is a surface with boundary, and we are taking the flux integral of $\nabla \times \mathbf{F}$, the curl of $\mathbf{F}$.

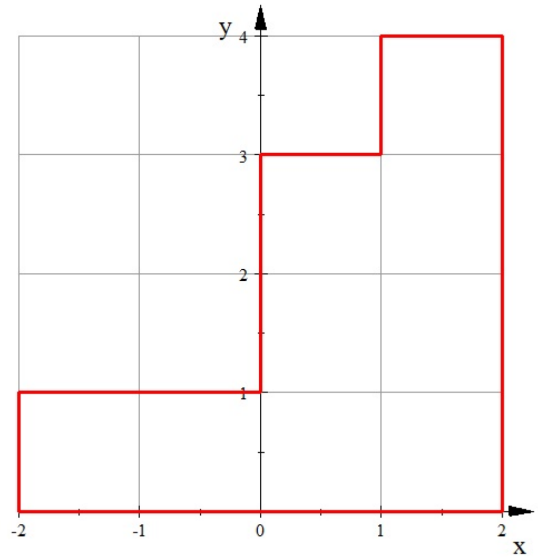
The boundary of S is given by $z = 0, 4x^2 + y^2 = 4$, and since S is oriented with upward orientation, the boundary of S has counterclockwise orientation when viewed from above. Thus, a parametrization of the boundary is given by

$$\mathbf{r}(t) = \langle \cos t, 2\sin t, 0 \rangle, 0 \leq t \leq 2\pi.$$

Thus, by Stokes' Theorem, we have

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\
&= \int_0^{2\pi} \langle \cos t - 2\sin t, \cos t + 2\sin t, 0 \rangle \cdot \langle -\sin t, 2\cos t, 0 \rangle \, dt \\
&= \int_0^{2\pi} \left(-\sin t \cos t + 2\sin^2 t + 2\cos^2 t + 4\sin t \cos t \right) dt \\
&= \int_0^{2\pi} (2 + 3\sin t \cos t) \, dt = \left(2t + \frac{3}{2}\sin^2 t \right) \Big|_0^{2\pi} \\
&= 4\pi.
\end{aligned}$$

6. Evaluate $\int_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy$ where C is the curve below and C is oriented in clockwise direction.



Solution: This problem uses Green's theorem. One main clue is the shape of the curve C (it has 8 pieces!). Let D be the region enclosed by the curve C. And since the orientation of C is clockwise, instead of counterclockwise, we have

$$\begin{aligned}\int_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy &= -\iint_D \left[(3 + 5x^4y^4) - (5x^4y^4 - 2)\right] dA \\ &= -\iint_D 5\, dA \\ &= -5\iint_D 1\, dA \\ &= -5 \cdot \text{Area}(D) \\ &= -5 \cdot 9 \\ &= -45.\end{aligned}$$

7. Let S be the boundary surface of the region bounded by $z = \sqrt{36 - x^2 - y^2}$ and $z = 0$, with outward orientation. Find $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} - 2yz\mathbf{k}$.

Solution: This is a closed surface, so the divergence theorem works nicely here.

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(-2yz) = 1 + 2y - 2y = 1$$

Call the solid H (since it's half of a ball). So, the divergence theorem gives

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_H 1\, dV = \text{volume}(H)$$

The solid H is half of the ball of radius 6, and so its volume is

$$\text{volume}(H) = \frac{1}{2}\left(\frac{4}{3}\pi(6)^3\right) = \frac{2}{3}(216\pi) = 144\pi.$$

8. *(A Challenging Problem)* Evaluate

$$\int_C (y^3 + \cos x)dx + (\sin y + z^2)dy + x\, dz$$

where C is the closed curve parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 2t\rangle$ with counterclockwise direction when viewed from above. (*Hint*: the curve C lies on the surface $z = 2xy$.)

Solution: If you rewrite this integral as $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ and note that the curve C lies in $\mathbb{R}^3$ and not in the plane (otherwise we'd use Green's theorem), we see that Stokes' theorem applies to it. The hint provides the surface to fill in the curve with.

First, we need to parametrize the surface $z = 2xy$:

$$\mathbf{p}(x, y) = \langle x, y, 2xy\rangle, \quad (x, y) \in D = \left\{(x, y) | x^2 + y^2 \leq 1\right\}$$

as the parametrization. C has counterclockwise orientation when viewed from above, so this means that the surface, call it S, we fill it in with must have upward orientation.

$$\begin{aligned} \mathbf{p}_x &= \langle 1, 0, 2y\rangle \\ \mathbf{p}_y &= \langle 0, 1, 2x\rangle \\ \mathbf{p}_x \times \mathbf{p}_y &= \langle -2y, -2x, 1\rangle \end{aligned}$$

Notice that $\mathbf{p}_x \times \mathbf{p}_y$ points upward, since the $\hat{k}$-component is positive, so this is the correct choice for the orientation. Now, we need the curl of $\mathbf{F}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 + \cos x & \sin y + z^2 & x \end{vmatrix} = \left\langle -2z, -1, -3y^2\right\rangle$$

Finally, we apply Stokes' Theorem

$$\begin{aligned}
\int_C (y^3+\cos x)dx+(\sin y+z^2)dy+x\,dz &= \iint_S (\text{curl }\mathbf{F})\cdot d\mathbf{S} \\
&= \iint_D (\text{curl }\mathbf{F})\cdot(\mathbf{p}_x\times\mathbf{p}_y)dA \\
&= \iint_D \left\langle -4xy,-1,-3y^2\right\rangle\cdot\langle -2y,-2x,1\rangle\,dA \\
&= \iint_D \left(8xy^2+2x-3y^2\right)dA \\
&= \int_0^{2\pi}\int_0^1 \left(8r^3\cos\theta\sin^2\theta+2r\cos\theta-3r^2\sin^2\theta\right)r\ drd\theta \\
&= \int_0^{2\pi}\left(\frac{8}{5}r^5\cos\theta\sin^2\theta+\frac{2}{3}r^3\cos\theta-\frac{3}{4}r^4\sin^2\theta\right)\Bigg|_0^1 d\theta \\
&= \int_0^{2\pi}\left(\frac{8}{5}\cos\theta\sin^2\theta+\frac{2}{3}\cos\theta-\frac{3}{4}\sin^2\theta\right)d\theta \\
&= \int_0^{2\pi}\left(\frac{8}{5}\cos\theta\sin^2\theta+\frac{2}{3}\cos\theta-\frac{3}{4}\left(\frac{1-\cos 2\theta}{2}\right)\right)d\theta \\
&= \left(\frac{8}{15}\sin^3\theta+\frac{2}{3}\sin\theta-\frac{3\theta}{8}+\frac{3}{16}\sin 2\theta\right)\Bigg|_0^{2\pi} \\
&= -\frac{3}{4}\pi
\end{aligned}$$