Thickenings of $\mathbf{Z}/n\mathbf{Z}$–manifolds and remarks on codimension two submanifolds

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Thickenings of CW complexes were formalized by Wall, [**12**]: a thickening of a finite CW complex, K, is a manifold W and a simple homotopy equivalence $f: K \to W$. The thickening is *orientable*, *Spin*, or whatever provided W is. Morgan and Sullivan introduced $\mathbf{Z}/n\mathbf{Z}$–manifolds in [**5**]. The main point of this note is the simple observation that $\mathbf{Z}/n\mathbf{Z}$–manifolds have explicit, orientable thickenings of the smallest dimension possible.

Recall that a $\mathbf{Z}/n\mathbf{Z}$–manifold is an oriented manifold with boundary M with ∂M divided into n pieces, $\partial M = \bigcup\limits_{j=1}^{n} (\partial M)_j$, together with another manifold δM and n orientation preserving diffeomorphisms $d_j: (\partial M)_j \to \delta M$. We leave to the reader the straightforward generalization to PL or TOP manifolds. The $\mathbf{Z}/n\mathbf{Z}$–manifold itself is the CW complex obtained from this data by using the disjoint union of the d_j to obtain a map $d: \partial M \to \delta M$ and then setting $M/\delta M$ to be the union of M and the mapping cylinder of d along ∂M. We denote it by $M/\delta M$, suppressing the diffeomorphisms d_j from the notation. The *dimension* of the $\mathbf{Z}/n\mathbf{Z}$–manifold is the dimension of M. Morgan and Sullivan define a tangent bundle $\tau_{M/\delta M}$ for a $\mathbf{Z}/n\mathbf{Z}$–manifold which is a bundle over $M/\delta M$. We recall the definition below.

The observation alluded to above can be summed up with

Theorem 1. *A $\mathbf{Z}/n\mathbf{Z}$–manifold of dimension m has an orientable thickening of dimension $m+1$,*

$$\iota: M/\delta M \longrightarrow W \ .$$

The simple homotopy equivalence ι is an embedding and $\iota^*(\tau_W) = \tau_{M/\delta M} \oplus \epsilon$ where ϵ is the trivial line bundle. Moreover, there is a map $r: \partial W \to M/\delta M$ so that W is the mapping cylinder of r and ι is the standard inclusion of the range into the mapping cylinder.

Before beginning the proof, let us agree that the mapping cylinder of any map $\alpha: A \to B$ is the quotient space of $A \times [0,1]$ disjoint union B identified by $(a, 0) \sim \alpha(a)$. With this bit of notation, the proof of the Theorem goes as follows. Let $W = \big(M \times [-1,1]\big) \cup \big(\delta M \times D^2\big)$ glued together as follows. Consider D^2 as the unit disk in the complex plane. Define n disjoint intervals in $\partial D^2 = S^1$, $e_j: [-1,1] \to S^1$, $j = 1, \ldots, n$ by $e_j(t) = \frac{2\pi(j+\frac{t}{4})i}{n}$. This puts the $e_j(0)$ at the nth roots of unity. Use $d_j \times e_j$ to embed $(\partial M)_j \times [-1,1]$ in $\delta M \times S^1$. This defines the gluing needed to construct W. It is straightforward to check that W is an oriented smooth manifold with boundary.

To precisely describe the remaining structures, fix radial arcs, $T_j: [0,1] \to D^2$ with $T_j(t) = \frac{2\pi jti}{n} = t \cdot e_j(0)$. Define additional arcs $E_j: [-1,1] \to S^1$ $j = 1, \ldots, n$ by $E_j(t) = \frac{2\pi(j+\frac{1}{2}+\frac{t}{4})i}{n}$. To simplify the formulae to follow, let us agree that the subscript j in d_j, ∂M_j, e_j and so on, is defined mod n. So for example $\partial M_{n+1} = \partial M_1$ and $d_{n+1} = d_1$.

Define $\iota: M/\delta M \to W$ by sending $m \in M$ to $m \times 0$ and the point $m \times t$ in the mapping cylinder, $m \in \partial M_j$, to $\partial M_j \times t \to \delta M \times t \cdot e_j(0)$ using d_j. It is an embedding.

Figure 2 shows the case $n = 3$ and shows $\delta M \times D^2$ with a bit of $M \times [-1,1]$ glued to it along the $(\partial M)_j \times [-1,1]$. The rest of $M \times [-1,1]$ is left to the reader's imagination. The bockstein is the dot at the center and the image of ι is the union of the dotted radial lines. The images of the E_j are labelled.

The result should be clear from this figure, but here are some further details.

The boundary of W may be described roughly as taking two copies of M and gluing ∂M_j to ∂M_{j+1} via the evident diffeomorphism, a "shifted double" if you will. More precisely, construct a differentiable manifold by taking two copies of M, $M \times \{-1, 1\}$ together with n copies of $\delta M \times [-1,1]$ and gluing them together as follows. The jth copy will be denoted δM_j. Glue $\partial M_j \times 1$ to $\delta M_j \times -1$ by d_j and glue $\partial M_{j+1} \times -1$

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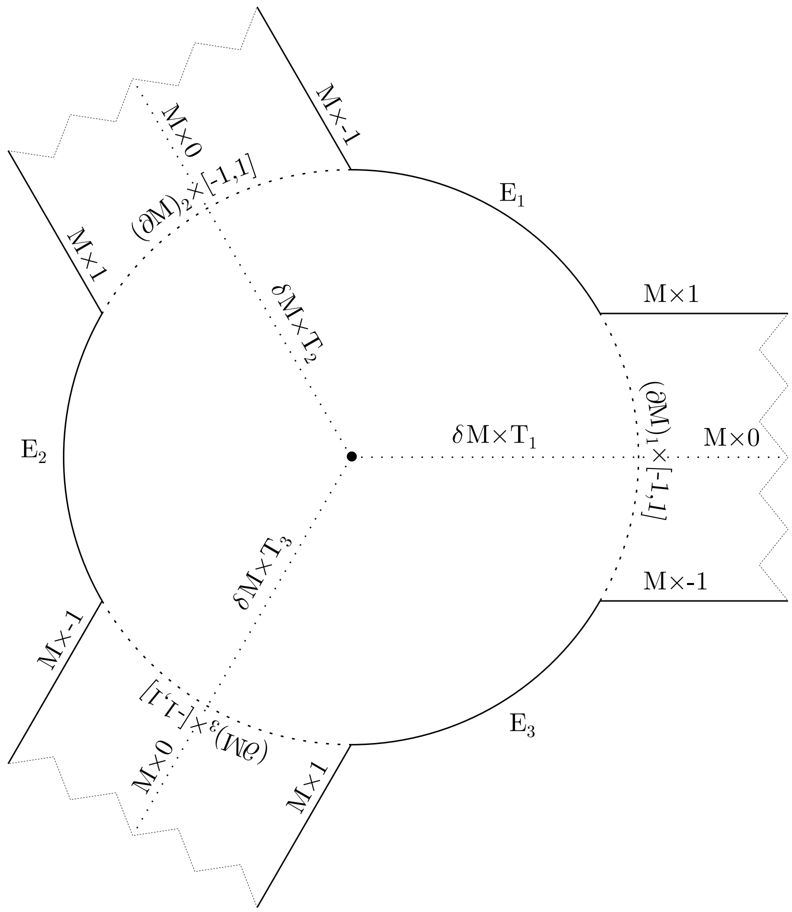

Figure 2

to $\delta M_j \times 1$ by d_{j+1}. Let V denote the result and construct an embedding $E\colon V \to \partial W \subset W$ as follows. Map $M \times \pm 1 \subset V$ to $M \times \pm 1 \subset W$ by the identity. Embed $\delta M_j \times [-1,1]$ in $\delta M \times D^2$ by sending $m \times t$ to $m \times E_j(t)$. Check that E defines a diffeomorphism between V and ∂W.

Define a map $\hat{r}\colon V \to M/\delta M$ as follows: $\hat{r}(m \times \pm 1) = m$ for all $m \in M$; for all $m \in \delta M_j$ $\hat{r}(m \times t) = d_j^{-1}(m) \times -t$ in the mapping cylinder, $-1 \le t \le 0$; $\hat{r}(m \times t) = d_{j+1}^{-1}(m) \times t$ for $0 \le t \le 1$. Let $r\colon \partial W \to M/\delta M$ be $\hat{r} \circ E^{-1}$. Check that $\hat{r}$ is well–defined and continuous.

It will suffice to identify the mapping cylinder of $\hat{r}$ rather than that of r. To identify the mapping cylinder of $\hat{r}$ with W, let $X_n \subset D^2$ be the union of the T_j, $j = 1, \ldots, n$. It is an n–spoked wheel and divides the disk into n slices. Define maps $R_j\colon [-1,1] \to X_n$, $j = 1, \ldots, n$ by $R_j(t) = \begin{cases} -t \cdot e_j(0) & -1 \le t \le 0 \\ t \cdot e_{j+1}(0) & 0 \le t \le 1 \end{cases}$ and let Y_n denote the mapping cylinder of n disjoint copies of $[-1,1]$ using the disjoint union of the R_j. Choose a homeomorphism $\iota_n\colon Y_n \to D^2$ so that ι_n restricted to X_n is the inclusion with which we started. We further insist that ι_n restricted to the jth copy of $[-1,1]$ is just E_j. Finally, we require that ι_n restricted to the jth copy of $-1 \times [0,1]$ is just $\iota_n(-1 \times t) = e_j(1-t)$, $j = 1, \ldots, n$; restricted to the jth copy of $1 \times [0,1]$ is just $\iota_n(1 \times t) = e_{j+1}(t-1)$ $j = 1, \ldots, n-1$; restricted to the nth copy of $1 \times [0,1]$ is just $\iota_n(1 \times t) = e_1(t-1)$.

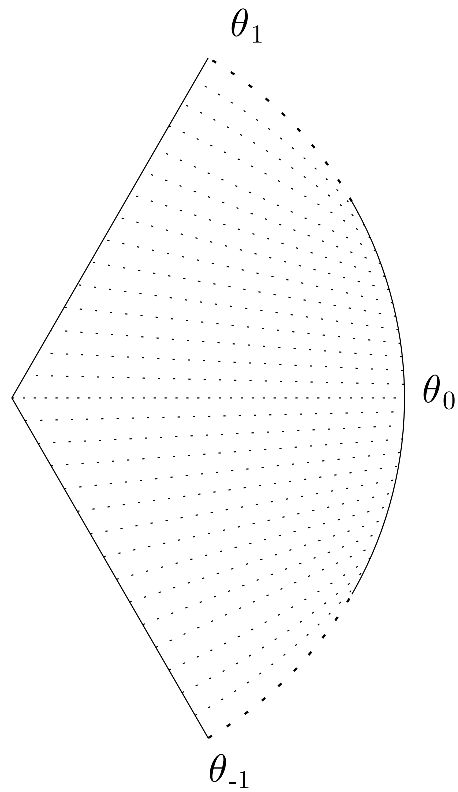

Figure 3

Figure 3 shows one slice of the disk. The solid rays $\theta_{\pm 1}$ denote two adjacent spokes of the wheel. The solid arc of the circle is $[-1, 1]$ mapped onto the circle by the appropriate E_j. The dotted lines represent the mapping cylinder $[0, 1]$ coordinate. As the picture illustrates, the identification of the mapping cylinder with our slice can be chosen symmetric about the bisector θ_0. We can then stack n of these slices together to get the full disk.

Since a mapping cylinder has the simple homotopy type of its range, we have a thickening in the sense of Wall.

We now turn to the tangent bundle result. Pick a cuspidal embedding of X_n in D^2 instead of the standard one and use this to get the cuspidal embedding of $M/\delta M$ in W. (Figure 4 below illustrates the two embeddings for X_3. Compare with the illustration on p.474 of **[5]**.) The subbundle of τ_W consisting of vectors tangent to the image is the Morgan–Sullivan tangent bundle to $M/\delta M$. There is an evident one dimensional normal bundle which is orientable. Orient it so that on the S^1 it give the usual orientation. With this convention, orientations of W are in bijection with orientations of $M/\delta M$.

Remark: Recall that $\mathbf{Z}/2\mathbf{Z}$–manifolds are actually manifolds (possibly non–orientable). One can see that the thickening constructed here is the total space of the orientation line bundle over the manifold $M/\delta M$. Formula (1.4) on p.474 of Morgan–Sullivan **[5]** follows.

Remarks: The above construction is fairly functorial since all but one of the choices just involves the disk and so can be fixed for all $\mathbf{Z}/n\mathbf{Z}$–manifolds. The one choice that is not of this sort is the choice of ordering for the boundary components. This choice does not affect the $\mathbf{Z}/n\mathbf{Z}$–manifold but it does affect the thickening. There are $n!$ such choices so we get $n!$ potentially different thickenings. One can see that if two identifications differ by a cyclic permutation, then the resulting thickenings are diffeomorphic. Our set of thickenings up to diffeomorphism contains at most $(n-1)!$ elements and one can construct examples for which there are $(n-1)!$ distinct diffeomorphism types. (For such an example let M be the disjoint union of once punctured closed manifolds M_i where M_i is the connected sum of i copies of CP^n. One can recover the ordering up to

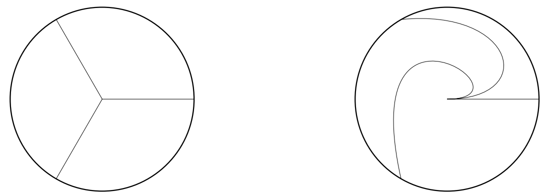

Figure 4

cyclic permutation from the homotopy type of the boundary of W.)

Remark: Morgan and Sullivan want $\mathbf{Z}/n_1\mathbf{Z}$ manifolds in order to have a geometric theory of bordism with $\mathbf{Z}/n_1\mathbf{Z}$ coefficients. They observe that in order to get natural maps between theories with different coefficients one must order the boundary components. These maps have nice descriptions on the thickenings.

1. Start with a thickened $\mathbf{Z}/n_1\mathbf{Z}$ manifold and let $n_2 = n_1 \cdot n_3$. Take the evident n_3–fold cyclic branched cover, branched along the bockstein. This is a thickening of a $\mathbf{Z}/n_2\mathbf{Z}$ manifold. The projection map from the total space of the branched cover to the base covers the Morgan–Sullivan map on bordism with coefficients induced by the injection $\mathbf{Z}/n_1\mathbf{Z} \to \mathbf{Z}/n_2\mathbf{Z}$.
2. Start with a thickened $\mathbf{Z}/n_2\mathbf{Z}$ manifold and suppose $n_2 = n_1 \cdot n_3$. Take a new bockstein consisting of n_3 copies of δM, labeled δM_i. Glue the ∂M_j to the δM_i satisfying $j \equiv i \bmod n_3$ to get a thickening of a $\mathbf{Z}/n_1\mathbf{Z}$ manifold. There is an embedding from the thickening of the $\mathbf{Z}/n_1\mathbf{Z}$ manifold just constructed to the thickening of original $\mathbf{Z}/n_2\mathbf{Z}$ manifold. (Just map each of the n_3 2–disks for the δM_i to the 2–disk for δM by rotating the ith disk through an angle of $\frac{2\pi(i-1)}{n_2}$ and extend.) This map covers the Morgan–Sullivan map on bordism with coefficients induced by the surjection $\mathbf{Z}/n_2\mathbf{Z} \to \mathbf{Z}/n_1\mathbf{Z}$.

Remarks: The thickening construction is functorial on the category for which the maps are smooth maps which preserve the ordering on the boundary. The thickenings are examples of homotopically stratified spaces with three strata in the sense of Quinn. Finally, the double of W gives a thickening of $M/\delta M$ in one dimension higher. It can also be constructed by preforming a construction similar to the one above but using D^3 instead of D^2. The diffeomorphism type of these thickenings is independent of the chosen ordering.

Remark: We leave it to the reader to pursue the k–ad (in the sense of Wall [**13**]) version of this construction.

Some remarks on representation

Corollary 5. *Let X be a CW complex with finite skeleta. Let $x \in H^k(X;\mathbf{Z}_{(2)})$ be a cohomology class. Then there exists a closed, compact orientable manifold, N^n, and a map $f\colon N \to X$ such that $f^*(x) \neq 0$. If x has infinite order, we may take $n = k$; otherwise we may take $n = k+1$.*

The proof of the Corollary is immediate from Steenrod representability as proved by Thom [**9**], Wall [**11**] and Connor & Floyd [**2**]. If x has infinite order there is a class $c \in H_k(Z;\mathbf{Z}_{(2)})$ with $\langle x, c\rangle \neq 0$. Represent c and the result follows. If x has finite order 2^ℓ there is a class $c \in H_{k-1}(X;\mathbf{Z}_{(2)})$ of finite order 2^ℓ with the following property. The homomorphism $c\colon \mathbf{Z}/2^\ell\mathbf{Z} \to H_{k-1}(X;\mathbf{Z}_{(2)})$ induced by c gives rise via Universal Coefficients to a homomorphism $c^*\colon H^k(X;\mathbf{Z}_{(2)}) \to \mathbf{Z}/2^\ell\mathbf{Z}$ with $c^*(x) \neq 0$. By the Thom, Wall and Connor & Floyd results above, we can find a $\mathbf{Z}/2^\ell\mathbf{Z}$ manifold and a map $g\colon M/\delta M \to X$ so that $\delta g\colon \delta M \to X$ represents c, from which we deduce $g^*(x) \neq 0$. Use Theorem 1 to thicken $M/\delta M$ and let $G\colon D(W) \to X$ denote the evident map from the double of the thickening to X induced by g. Then $G^*(x) \neq 0$ and we let $f = G$. ∎

To paraphrase the above proof, detecting cohomology classes involves being able to represent homology classes of finite order by mapping in elements of the same order, hence $\mathbf{Z}/n\mathbf{Z}$ manifolds. A more complicated but still useful representation theorem proceeds as follows. Let X have finite skeleta and consider $x \in$

$H_r(X \times BSO; \mathbf{Z})$. Suppose x has order 2^ℓ and that x is in the image of $MSO_r(X)$. Then there exist manifolds M^m for all $m \geq r+2$, a map $f\colon M \to X$, and a class $\chi \in H_r(M; \mathbf{Z})$ of order 2^ℓ such that

1. $(f \times \tau_M)_*(\chi) = x$
2. $(f \times \tau_M)_*\big(H_r(M; \mathbf{Z})\big)$ is the cyclic subgroup generated by x
3. $(f \times \tau_M)_* = 0$ for $* > r$.

Applications to codimension 2 submanifolds

By a result of Thom's [**9**], a mod 2 cohomology class $x \in H^2(M; \mathbf{Z}/2\mathbf{Z})$ has a dual submanifold if and only if the map $x\colon X \to K(\mathbf{Z}/2\mathbf{Z}, 2)$ factors through the Thom class $U\colon TO(2) \to K(\mathbf{Z}/2\mathbf{Z}, 2)$. (The Bockstein associated to the exact sequence $0 \to \mathbf{Z} \xrightarrow{\times n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$ is denoted δ_n.) Suzuki [**8**] writes down a partial Postnikov decomposition for $TO(2)$. Let $\mathcal{P}_2$ denote the Pontrjagin square and let E be the fibre of the map

$$K(\mathbf{Z}/2\mathbf{Z}, 2) \times K(\mathbf{Z}/2\mathbf{Z}, 4) \times K(\mathbf{Z}/2\mathbf{Z}, 8) \xrightarrow{\left(\delta_4\mathcal{P}_2(\iota_2)+\delta_2\iota_4\right) \times \left((Sq^1\iota_2)^2+\iota_2\iota_4\right)} K(\mathbf{Z}, 5) \times K(\mathbf{Z}/2\mathbf{Z}, 6)$$

There is a map $TO(2) \to E$ which induces an isomorphism in mod 2 and integral cohomology through dimension 8. More precisely, the next k invariant is in dimension 9. The proof is straightforward. There is a twisted Thom class $U_\omega \in H^2(TO(2); \mathbf{Z}^\omega)$ and by Massey's formula [**4**], $\mathcal{P}_2(U) = U_\omega \cup U_\omega + \theta_2(w_1^2 U)$ where $\theta_2\colon \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z}$ is the usual injection. It follows that $\delta_4\mathcal{P}_2(U) + \delta_2(w_1^2 U) = 0$. Define

$$I\colon TO(2) \to K(\mathbf{Z}/2\mathbf{Z}, 2) \times K(\mathbf{Z}/2\mathbf{Z}, 4) \times K(\mathbf{Z}/2\mathbf{Z}, 8)$$

by $I^*(\iota_2) = U$, $I^*(\iota_4) = w_1^2 U$ and $I^*(\iota_8) = w_1^6 U$. Massey's formula and Thom's formula, $Sq^1(U) = w_1 U$, together verify that this map lifts to E. There is a class $x \in H^4(E; \mathbf{Z})$ which transgresses to a generator of $H^5\big(K(\mathbf{Z}, 5); \mathbf{Z}\big)$ and we require that under the map $\iota\colon TO(2) \to E$, $\iota^*(x) = U_\omega \cup U_\omega$. This makes ι^* an isomorphism in cohomology with $\mathbf{Z}[\frac{1}{2}]$ coefficients.

Check that I^* is onto in mod 2 cohomology through dimension 15. Serre's results [**7**] on the Poincaré series of Eilenberg–MacLane spaces can be used to check that ι^* is an isomorphism in mod 2 cohomology through dimension 8. It follows that the next k invariant is in dimension 9.

Theorem 6. *If $x \in H^2(M; \mathbf{Z}/2\mathbf{Z})$ has a submanifold dual then there exists a class $y \in H^4(M; \mathbf{Z}/2\mathbf{Z})$ such that $\delta_4\mathcal{P}_2(x) + \delta_2(y) = 0$ and $(Sq^1x)^2 + x \cup y = 0$. If the dimension of M is less than 9, then the condition is also sufficient.*

Remark: Given x a y exists so that $\delta_4\mathcal{P}_2(x) + \delta_2(y) = 0$ if and only if $2 \cdot \delta_4\mathcal{P}_2(x) = 0$, which is just $\delta_2 Sq^2(x) = 0$, a remark of Thom's [**9**]. If there is such a y, then both equations have a solution if and only if $(Sq^1x)^2 + x \cup y \in x \cup H^4(M; \mathbf{Z})$. The equation $\delta_4\mathcal{P}_2(x) + \delta_2(y) = 0$ reduces mod 2 to the equation

$$x \cup Sq^1(x) + Sq^2Sq^1(x) + Sq^1(y) = 0 \ .$$

Remark: There are two cases with easy solutions. Suppose $\delta_2 x = 0$. Note that $\mathcal{P}_2(x)$ is the mod 4 reduction of the square of an integral class lifting x, so $(x, 0)$ is a solution to both the equations. Fix a 1 dimensional cohomology class $\omega \in H^1(M; \mathbf{Z}/2\mathbf{Z})$ and suppose $\delta_2^\omega x = 0$, where δ_2^ω is the twisted integral Bockstein. By Massey's formula [**4**], $\mathcal{P}_2(x) = \bar{x} \cup \bar{x} + \theta_2(\omega^2 \cup x)$, so $(x, \omega^2 \cup x)$ is a solution to the first equation. By a formula of Samelson [**6**], $Sq^1(x) = \omega \cup x$ so the second equation holds as well.

Remark: Consider the more specialized problem of whether w_2 of some bundle over M has a submanifold dual. Now we can calculate in BO and then restrict. Since all integral torsion has exponent 2 we only have to do our calculations mod 2. Our first equation becomes $w_2 Sq^1 w_2 + Sq^2 Sq^1 w_2 + Sq^1 y = 0$. This equation has a solution for $y = w_4 + w_1^2 w_2$. Hence, a necessary condition for w_2 of a bundle to have a submanifold dual is that

$$w_3^2 + w_2 w_4 \in w_2 \cup H^4(M; \mathbf{Z}) \ . \tag{7}$$

This condition is sufficient if the dimension of M is less than 9.

Suzuki [**8**], Thm.7.1 p.110 has much the same formula except he is missing the $w_2 w_4$ term. The difference probably comes from his formula for $\mathcal{P}_2(w_2)$ on p.106 instead of Thomas's [**10**, Thm. C p.71]. Suzuki's formula is inconsistent with the formula for the mod 2 reduction of $\delta_4\mathcal{P}_2(w_2)$.

Since $H^*(BTOP;\mathbf{Z})$ also only has torsion of exponent 2 through dimension 6, the same necessary condition holds for TOP bundles and it remains sufficient for the dimension of M less than 9. For spherical fibrations, the 2 torsion result fails and indeed $\delta_2 Sq^2 w_2 \neq 0 \in H^5(BSG;\mathbf{Z})$.

Theorems and Examples in Codimension 2

We begin with a result of Thom's [**9**, II.2.6 p.55].

Theorem 8. *Any $x \in H^2(M;\mathbf{Z}/2\mathbf{Z})$ has a submanifold dual if the dimension of M is less than 6.*

The proof is immediate since either δ_2 is onto or δ_4 is 0 on a connected manifold.

Example: Thanks to Corollary 5 and the failure of $\delta_2 Sq^2 w_2$ to vanish for spherical fibrations, there is an orientable, smooth, closed, compact 6 manifold M and an orientable spherical fibration ζ over it so that $w_2(\zeta) \in H^2(M;\mathbf{Z}/2\mathbf{Z})$ has no submanifold dual. We can construct simply–connected manifolds and spherical fibrations with the same property in all dimensions greater than 6.

For bundles, the situation is more complicated.

Theorem 9. *Let ζ be a TOP bundle over a closed, compact 6 manifold. Then $w_2(\zeta)$ has a submanifold dual.*

Proof: We may assume that M is connected. Equation (7) is easily satisfied if the map $w_2(\zeta) \cup H^4(M;\mathbf{Z}) \subset H^6(M;\mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ is not trivial, so let us hereafter assume it is. This condition is equivalent to the condition that

$$\cap w_2(\zeta)\colon H_2(M;\mathbf{Z}^{w_1(M)}) \to \mathbf{Z}/2\mathbf{Z}$$

is trivial. But in this case, $w_2(\zeta)$ is the mod 2 reduction of a twisted integral class and hence has a submanifold dual by our previous discussion. ∎

Example: Here is an example of a seven dimensional manifold with a bundle whose w_2 has no submanifold dual. Start with $RP^3 \times RP^2$ and construct the projective bundle on the vector bundle which is the sum of three line bundles whose w_1's are $\alpha+\beta$, α and 0 respectively. Here $\alpha \in H^1(RP^3;\mathbf{Z}/2\mathbf{Z})$ and $\beta \in H^1(RP^2;\mathbf{Z}/2\mathbf{Z})$. Let P^7 denote the total space of this bundle and recall $H^*(P;\mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}[\alpha,\beta,\gamma]$ modulo the relations $\alpha^4=0$, $\beta^3=0$ and $\gamma^3 = \omega_1\gamma^2+\omega_2\gamma+\omega_3$. Then w_2 of the sum of the three line line bundles whose w_1's are α, β and γ respectively has no submanifold dual.

There is still the natural question of when w_2 of the tangent or normal bundle of a manifold has a submanifold dual. The sixth Wu class of a bundle, v_6, can be expressed as a polynomial in the Stiefel–Whitney classes or in the Stiefel–Whitney classes of the inverse bundle, $\bar{w}_i$: explicitly

$$\begin{aligned} v_6 =& \bar{w}_3^2 + \bar{w}_2\bar{w}_4 + \bar{w}_2(\bar{p}_1 + \delta_2\bar{w}_3) \\ =& w_3^2 + w_2w_4 + w_1v_5 + w_2 \cup \delta_2(w_1^3 + w_3) \end{aligned} \tag{10}$$

So equation (7) becomes

$$\begin{gathered} v_6 \in \bar{w}_2 \cup H^4(M;\mathbf{Z}) \\ v_6 + w_1v_5 \in w_2 \cup H^4(M;\mathbf{Z}) \end{gathered} \tag{11}$$

For a manifold of dimension less than 10, v_5 and v_6 of the tangent bundle are trivial, so we have shown

Theorem 12. *For a manifold of dimension ≤ 8, w_2 and $\bar{w}_2$ of the tangent bundle have submanifold duals.*

Examples: In dimension 12 using surgery, we can construct an oriented manifold with $(\tau_M)^*\colon H^*(BSO) \to H^*(M)$ an isomorphism for $* < 6$ and an injection for $* = 6$. For this 12 manifold, the dual to w_2 of the tangent bundle is not realizable by a submanifold. For orientable bundles, w_2 of the normal and tangent bundles are the same.

In dimension 11, Suzuki [**8**] shows that w_2 of the tangent bundle for the 11 manifold $RP^2 \times RP^4 \times RP^5$ is not realized by a dual submanifold. One can check that w_2 of the tangent bundle of the 10 manifold $RP^2 \times RP^2 \times RP^6$ is not the dual to a submanifold either. The proof is straightforward. Compute w_1, w_2 and v_5 for $RP^2 \times RP^2 \times RP^6$: $v_6 = 0$. The group $H^4(RP^2 \times RP^2 \times RP^6; \mathbf{Z})$ is a $\mathbf{Z}/2\mathbf{Z}$ vector space of dimension 5, so the equation $w_1 v_5 \in w_2 \cup H^4(RP^2 \times RP^2 \times RP^6; \mathbf{Z})$ from (11) degenerates into a set of equations in 5 unknowns. These equations are inconsistent.

The remaining question is to find the least dimension of a manifold whose w_2 (tangent or normal) does not have a submanifold dual. Is there a tangential example in dimension 9: is there a normal example in dimension 9, 10 or 11? The necessary condition from equation (11) is always satisfied so the third k–invariant must be analyzed.

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