Fields Institute Communications
Volume **00**, 0000

Complex Spin structures on 3–manifolds

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Abstract. Two closely related descriptions of the set of Spin^c structures on an oriented 3–manifold are given in completely algebraic terms. A third such description was given by Deloup and Massuyeau [DM]. It is hoped that the descriptions given here will prove convenient in applications. The descriptions are natural in that the map induced by a diffeomorphism between the sets of Spin^c structures can be calculated once the linking forms on the manifold and the induced map in homology are known.

1 Basic definitions and standard results

Both the descriptions given here and the one given by Deloup and Massuyeau [DM] depend on the notion of a quadratic function. A function $\psi\colon T \to \mathbb{Q}/\mathbb{Z}$ out of an abelian group T is called *quadratic* provided the function

$$B_\psi(t_1, t_2) = \psi(t_1 + t_2) - \psi(t_1) - \psi(t_2)\colon T \times T \to \mathbb{Q}/\mathbb{Z}$$

is bilinear. Equivalently

$$\psi(t_1 + t_2 + t_3) = \psi(t_1 + t_2) + \psi(t_1 + t_3) + \psi(t_2 + t_3) - \psi(t_1) - \psi(t_2) - \psi(t_3)\ .$$

A quadratic function is called *homogeneous* provided $\psi(-t) = \psi(t)$ for all $t \in T$: the *homogeneity defect* is the homomorphism $d_\psi\colon T \to \mathbb{Q}/\mathbb{Z}$ defined by $d_\psi(t) = \psi(t) - \psi(-t)$ and vanishes if and only if ψ is homogeneous.

If a symmetric bilinear form $\mathcal{L}\colon T \times T \to \mathbb{Q}/\mathbb{Z}$ is fixed, call any quadratic function ψ such that $B_\psi = \mathcal{L}$ a *quadratic enhancement* of $\mathcal{L}$. Let $\mathcal{Q}\mathrm{uad}(\mathcal{L})$ denote the set of quadratic enhancements of $\mathcal{L}$.

Given any group G and G set X, X will be called a *G–torsor* proved the action is simply–transitive: any choice of element $x \in X$ induces a bijection between X and G,

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1991 *Mathematics Subject Classification.* Primary 57R15; Secondary 57M50.

Partially supported by the N.S.F.

There is a group action

$$\mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z}) \times \mathcal{Q}\mathrm{uad}(\mathcal{L}) \to \mathcal{Q}\mathrm{uad}(\mathcal{L})$$

given by sending $f \in \mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$ and $\psi \in \mathcal{Q}\mathrm{uad}(\mathcal{L})$ to $f \bullet \psi(t) = \psi(t) + f(t) \colon T \to \mathbb{Q}/\mathbb{Z}$ which makes $\mathcal{Q}\mathrm{uad}(\mathcal{L})$ into a $\mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$–torsor. The subset of homogeneous, quadratic enhancements, $\mathcal{Q}\mathrm{uad}(\mathcal{L})_h$, is a ${}_2\mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$–torsor, where for an abelian group A, ${}_2A$ denotes the subgroup generated by the elements of order 2.

Given bilinear forms $\mathcal{L}_T$ on T and $\mathcal{L}_S$ on S, an *isometry* between them is a homomorphism $\ell \colon S \to T$ such that $\mathcal{L}_T\big(\ell(s_1), \ell(s_2)\big) = \mathcal{L}_S(s_1, s_2)$. If ψ is a quadratic enhancement of $\mathcal{L}_T$, $\psi \circ \ell \colon S \to \mathbb{Q}/\mathbb{Z}$ is a quadratic enhancement of $\mathcal{L}_S$. If ψ is homogeneous, so is $\psi \circ \ell$. Let $\mathcal{I}\mathrm{som}\,(\mathcal{L})$ denote the group of invertible isometries of $\mathcal{L}$: $\mathcal{I}\mathrm{som}\,(\mathcal{L})$ acts on $\mathcal{Q}\mathrm{uad}(\mathcal{L})$: the subset of homogeneous enhancements is an invariant subset.

Every oriented 3–manifold has Spin^c reductions of its stable tangent bundle. Let $\mathrm{Spin}^c(M)$ denote the set of such reductions. The set $\mathrm{Spin}^c(M)$ is a $H^2(M;\mathbb{Z})$–torsor via an action

$$H^2(M;\mathbb{Z}) \times \mathrm{Spin}^c(M) \to \mathrm{Spin}^c(M)\ .$$

If $x \in H^2(M;\mathbb{Z})$ and $\sigma \in \mathrm{Spin}^c(M)$, let $x \bullet \sigma$ denote the result of the action. Every Spin^c structure $\sigma \in \mathrm{Spin}^c(M)$ has a first Chern class $c_1(\sigma) \in H^2(M;\mathbb{Z})$ and $c_1(x \bullet \sigma) = c_1(\sigma) + 2x$.

Every closed, compact, oriented 3–manifold has a linking form

$$\mathcal{L}_M \colon TH_1(M) \times TH_1(M) \to \mathbb{Q}/\mathbb{Z}$$

where $TH_1(M) \subset H_1(M;\mathbb{Z})$ denotes the torsion subgroup.

2 The work of Deloup and Massuyeau

Deloup and Massuyeau extend the linking form on a closed, compact, oriented 3–manifold to a bilinear function $\mathfrak{b}_M \colon H_2(M;\mathbb{Q}/\mathbb{Z}) \times H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. Then to each Spin^c structure σ they associate a quadratic enhancement of $\mathfrak{b}_M$, $\phi_M(\sigma) \in \mathcal{Q}\mathrm{uad}(\mathfrak{b}_M)$. Evaluation defines a homomorphism

$$\mu_M \colon H^2(M;\mathbb{Z}) \to \mathrm{Hom}(H_2(M;\mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

so $\mathcal{Q}\mathrm{uad}(\mathfrak{b}_M)$ becomes an $H^2(M;\mathbb{Z})$ module. Then Deloup and Massuyeau prove

Theorem 2.1 ([DM, Theorem 2.3]) *The function*

$$\phi_M \colon \mathrm{Spin}^c(M) \to \mathcal{Q}\mathrm{uad}(\mathfrak{b}_M)$$

is injective and $H^2(M;\mathbb{Z})$–equivariant. Deloup and Massuyeau give an algebraic description of the image of ϕ_M.

Remark 2.2 The map ϕ_M is natural in the following sense. If $f \colon M \to N$ is an orientation preserving diffeomorphism then the differential induces a bijection $df^* \colon \mathrm{Spin}^c(N) \to \mathrm{Spin}^c M$. The map f induces an isometry between $\mathfrak{b}_M$ and $\mathfrak{b}_N$ and hence a bijection $f^* \colon \mathcal{Q}\mathrm{uad}(\mathfrak{b}_N) \to \mathcal{Q}\mathrm{uad}(\mathfrak{b}_M)$. The diagram

$$\begin{array}{ccc} \mathrm{Spin}^c(N) & \xrightarrow{df^*} & \mathrm{Spin}^c(M) \\ \big\downarrow{\scriptstyle \phi_N} & & {\scriptstyle \phi_M}\big\downarrow \\ \mathcal{Q}\mathrm{uad}(\mathfrak{b}_N) & \xrightarrow{f^*} & \mathcal{Q}\mathrm{uad}(\mathfrak{b}_M) \end{array}$$

commutes.

Remark 2.3 If M is a rational homology sphere, then ϕ_M is an isomorphism but if $H_1(M;\mathbb{Q}) \neq 0$ then the domain of ϕ_M is countable and the range has the cardinality of the reals.

Remark 2.4 There is a relation between the first Chern class of the Spin^c structure and the homogeneity defect: succinctly the following diagram commutes, [DM, Lemma 2.8].

$$\begin{array}{ccc} \mathrm{Spin}^c(M) & \xrightarrow{\phi_M} & \mathcal{Q}\mathrm{uad}(\mathfrak{b}_M) \\ \downarrow{\scriptstyle c_1} & & {\scriptstyle d}\downarrow \\ H^2(M;\mathbb{Z}) & \xrightarrow{\mu_M} & \mathrm{Hom}(H_2(M;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \end{array}$$

3 The main results

The goal of this note is to find natural, algebraic objects which map into Spin$^c(M)$. There are lots of 3–manifold invariants which are functions out of Spin$^c(M)$ and they will induce functions out of algebraic objects.

The main idea goes back at least as far as [Ta], [KT], or [LW], all of which predate the current interest in Spin^c structures. To begin, consider the set Spin(M) of Spin structures on M. It is an $H^1(M;\mathbb{Z}/2\mathbb{Z})$–torsor.

Any Spin structure is an example of a Spin^c structure so there is a map

$$\Psi\colon \mathrm{Spin}(M) \to \mathrm{Spin}^c(M)\ .$$

The integral Bockstein $\delta\colon H^1(M;\mathbb{Z}/2\mathbb{Z}) \to H^2(M;\mathbb{Z})$ is a homomorphism and Ψ is $H^1(M;\mathbb{Z}/2\mathbb{Z})$–equivariant using δ to get the action on Spin$^c(M)$. It follows that the image of Ψ is precisely the set of Spin^c structures with first Chen class 0, denoted Spin$^c(M)_0$.

Given a Spin structure on M, the papers [Ta] and [KT] exhibit a homogeneous, quadratic enhancement of the linking form $\mathcal{L}_M$ on $TH_1(M)$, the torsion subgroup of $H_1(M;\mathbb{Z})$. Since the set of such homogeneous, quadratic enhancements is a ${}_2H^2(M;\mathbb{Z})$–torsor and since ${}_2H^2(M;\mathbb{Z})$ is the image of δ, it follows that

Theorem 3.1 *There is a bijection $\kappa_M\colon \mathcal{Q}\mathrm{uad}(\mathcal{L}_M)_h \to \mathrm{Spin}^c(M)_0$. Both sets are ${}_2H^2(M;\mathbb{Z})$–torsors and κ_M is equivariant. The bijection κ_M is also natural: given a diffeomorphism $f\colon M \to N$,*

$$\begin{array}{ccc} \mathcal{Q}\mathrm{uad}(\mathcal{L}_N)_h & \xrightarrow{f^*} & \mathcal{Q}\mathrm{uad}(\mathcal{L}_M)_h \\ \downarrow{\scriptstyle \kappa_N} & & {\scriptstyle \kappa_M}\downarrow \\ \mathrm{Spin}^c(N)_0 & \xrightarrow{df^*} & \mathrm{Spin}^c(M)_0 \end{array}$$

commutes.

Corollary 3.2 *There is a natural bijection*

$$H^2(M;\mathbb{Z}) \times_{{}_2H^2(M;\mathbb{Z})} \mathcal{Q}\mathrm{uad}(\mathcal{L}_M)_h \to \mathrm{Spin}^c(M)$$

given by $(x,\psi) \mapsto x \bullet \kappa_M(\psi)$.

The proof of (3.1) is straightforward except perhaps for the naturality. But recall, [KT, p. 209], the enhancement is given as follows. To define $\psi(t)$ take an embedded circle representing t. Use the Lie group framing on the tangent bundle of the circle and the Spin structure on the manifold to put a Spin structure on the stable normal bundle of the embedding. This divides the framings of the normal bundle into even and odd ones. The self–linking number is computed by taking a section of the normal bundle, finding an embedded surface with boundary a multiple of this push off, counting intersections of the original circle with this surface and finally dividing this number by the multiple used. To get the enhancement, use an even push off and divide by twice the multiple used. With this description the naturality is clear.

Remark 3.3 The naturality has the consequence that two orientation preserving diffeomorphisms $f,\, g\colon M \to N$ which induce the same map on $H_1(\_\,;\mathbb{Z})$ induce the same map between the sets of Spin^c structures. More generally, it implies that maps induced by differentials on sets of Spin^c structures can be computed once the induced maps in homology are known.

The information in 3.1 and 3.2 can be repackaged as follows. The torsion subgroup of $H^2(M;\mathbb{Z})$ is isomorphic to $TH_1(M)$, the torsion subgroup of $H_1(M;\mathbb{Z})$, and $\mathcal{Q}\mathrm{uad}(\mathcal{L}_M)$ is a $TH_1(M)$–torsor. The set $\mathrm{Spin}^c(M)_0$ is a subset of $\mathrm{Spin}^c(M)_t$, the set of Spin^c structures with torsion first Chern class. The set $\mathrm{Spin}^c(M)_t$ is a $TH_1(M)$–torsor and so there is an equivariant bijection induced from κ_M

$$\kappa_M\colon \mathcal{Q}\mathrm{uad}(\mathcal{L}_M) \to \mathrm{Spin}^c(M)_t\ .$$

In [LW, p. 271], Looijenga and Wahl give a procedure for passing from a Spin^c structure with torsion first Chern class, to a quadratic enhancement of the linking form. Corollary 3.2 can be repackaged as

Corollary 3.4 *The function $\kappa_M\colon \mathcal{Q}\mathrm{uad}(\mathcal{L}_M) \to \mathrm{Spin}^c(M)_t$ is a natural bijection. It extends to a natural bijection*

$$H^2(M;\mathbb{Z}) \times_{TH_1(M)} \mathcal{Q}\mathrm{uad}(\mathcal{L}_M) \to \mathrm{Spin}^c(M)\ .$$

Remark 3.5 Any function out of $\mathrm{Spin}^c(M)$ extends to a function out of $H^2(M;\mathbb{Z}) \times \mathcal{Q}\mathrm{uad}(\mathcal{L}_M)_h$ or out of $H^2(M;\mathbb{Z}) \times \mathcal{Q}\mathrm{uad}(\mathcal{L}_M)$. Usually several pairs go to the same Spin^c structure, but this seldom presents a difficulty.

Remark 3.6 By the universal coefficients theorem, there is an exact sequence $0 \to \mathbb{Q}/\mathbb{Z} \otimes H_2(M;\mathbb{Z}) \to H_2(M;\mathbb{Q}/\mathbb{Z}) \to TH_1(M) \to 0$. Any quadratic enhancement of the linking form on $TH_1(M)$ extends to a quadratic function on $H_2(M;\mathbb{Q}/\mathbb{Z})$ which can be checked to be a quadratic enhancement of $\mathfrak{b}_M$, giving a function $r\colon \mathcal{Q}\mathrm{uad}(\mathcal{L}_M) \to \mathcal{Q}\mathrm{uad}(\mathfrak{b}_M)$. The same exact sequence defines an injective homomorphism $\mathrm{Hom}(TH_1(M),\mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(H_2(M;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$ and the function r is equivariant. The function r factors as the following composition

$$\mathcal{Q}\mathrm{uad}(\mathcal{L}_M) \xrightarrow{\kappa_M} \mathrm{Spin}^c(M)_t \subset \mathrm{Spin}^c(M) \xrightarrow{\phi_M} \mathcal{Q}\mathrm{uad}(\mathfrak{b}_M)\ .$$

The function $R\colon H^2(M;\mathbb{Z}) \times \mathcal{Q}\mathrm{uad}(\mathcal{L}_M) \to \mathcal{Q}\mathrm{uad}(\mathfrak{b}_M)$ is given by the formula $R(x,\psi) = \mu(x) \bullet r(\psi)$.

Remark 3.7 The function $\mu_M\colon H^2(M;\mathbb{Z}) \to \mathrm{Hom}(H_2(M;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$ when restricted to $TH^2(M)$, the torsion subgroup of $H^2(M;\mathbb{Z})$ lands in $\mathrm{Hom}(TH_1(M),\mathbb{Q}/\mathbb{Z}) \subset \mathrm{Hom}(H_2(M;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$: use $\mu_M\colon TH^2(M) \to \mathrm{Hom}(TH_1(M),\mathbb{Q}/\mathbb{Z})$ to denote the restricted function. The composition

$$\mathcal{Q}\mathrm{uad}(\mathcal{L}_M) \xrightarrow{\kappa_M} \mathrm{Spin}^c(M)_t \xrightarrow{c_1} TH^2(M) \xrightarrow{\mu_M} \mathrm{Hom}(TH_1(M),\mathbb{Q}/\mathbb{Z})$$

is the homogeneity defect.

4 A recall of one way to calculate

Every oriented 3–manifold bounds a simply–connected 4–manifold and every 4–manifold supports Spin^c structures. One way to describe a Spin^c structure on M is to pick a simply–connected W^4 with $\partial W = M$ and a class $c_1 \in H^2(W;\mathbb{Z})$ whose mod 2 reduction is the second Stiefel–Whitney class of W. Since $H^2(W;\mathbb{Z})$ is torsion free, c_1 determines a unique Spin^c structure on W whose first Chern class is c_1. This Spin^c structure can then be restricted to one on M, denoted σ_{W,c_1}.

Let V denote the image of $H^2(W,\partial W;\mathbb{Q})$ in $H^2(W;\mathbb{Q})$. Cup product induces a non–singular, symmetric, bilinear form

$$B_W \colon V \times V \to \mathbb{Q} \ .$$

Let $L \subset V$ denote the image of $H^2(W,\partial W;\mathbb{Z})$: L is a lattice. Let $L^\# = \big\{w \in V \mid B_W(w,x) \in \mathbb{Z}\ \forall x \in L\big\}$. The group $L^\#/L$ is isomorphic to $TH_1(M)$. The pairing B_W induces a bilinear pairing on $L^\#/L$ which under the isomorphism with $TH_1(M)$ becomes minus the linking pairing on M. If the restriction of c_1 to $H^2(M;\mathbb{Z})$ is torsion, it lifts to a class $c_1^\# \in L^\#$. Define

$$\psi_{W,c_1}(t) = \frac{B_W(\bar t,\bar t)}{2} - \frac{B_W(c_1^\#,\bar t)}{2} \colon TH_1(M) \to \mathbb{Q}/\mathbb{Z}$$

by lifting $t \in TH_1(M)$ to $\bar t \in H^2(W,\partial W;\mathbb{Q})$, computing the indicated rational numbers and then reducing them mod $\mathbb{Z}$. Check that ψ_{W,c_1} is well–defined. This uses that the c_1 reduces to the second Stiefel–Whitney class so for $x \in L$, $B_W(x,x)$ and $B_W(c_1,x)$ are the same mod 2. Check that $-\psi_{W,c_1}$ is a quadratic enhancement of the linking form on M.

Since $\sigma_{W,c_1} \in \mathrm{Spin}^c(M)$ denotes the restriction of the Spin^c structure on W determined by c_1, then $c_1(\sigma_{W,c_1})$ is the restriction of c_1 to $H^2(M;\mathbb{Z})$. Check that there are always choices for $c_1 \in H^2(W;\mathbb{Z})$ such that c_1 restricts to 0 in $H^2(M;\mathbb{Z})$.

Theorem 4.1 *Let W be a simply–connected 4–manifold with $\partial W = M$. Fix a $c_1 \in H^2(W;\mathbb{Z}))$ restricting to the second Stiefel–Whitney class of W and restricting to 0 in $H^2(M;\mathbb{Z})$. With notation as above, $\sigma_{W,c_1} \in \mathrm{Spin}^c(M)_0$ and $\kappa_M\big(-\psi_{W,c_1}\big) = \sigma_{W,c_1}$. If c_1 restricts to a torsion class in $H^2(M;\mathbb{Z})$, then $\sigma_{W,c_1} \in \mathrm{Spin}^c(M)_t$ and $\kappa_M\big(-\psi_{W,c_1}\big) = \sigma_{W,c_1}$.*

Proof The proof can be extracted from the discussion in [DM, §2] or from [LW, p. 271]. □

Remark 4.2 In general, σ_{W,c_1} can be identified. Pick some $x \in H^2(W;\mathbb{Z})$ such that $c_1 + 2x$ is torsion in $H^2(M;\mathbb{Z})$. Let x_M denote x restricted to $H^2(M;\mathbb{Z})$. Then the pair $x_M \times \big(-\psi_{W,c_1-2x}\big)$ hits σ_{W,c_1}. The entire calculation is algebraic.

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