SURGERY WITH FINITE FUNDAMENTAL GROUP

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The relationship between the topology of manifolds and homotopy theory has been analysed by surgery [1, 18, 28, 31]. However, the effectiveness of this analysis depends on being able to compute surgery obstructions. In this paper we introduce a new technique for evaluating the (weakly-simple) surgery obstructions for degree-1 normal maps of closed manifolds and apply it to oriented manifolds with finite fundamental group. Our main result is Theorem A below.

Let $(X, \partial X)$ be a finite Poincaré pair of dimension n, and ξ a *TOP* bundle over X which is stably fibre homotopy equivalent to the Spivak normal fibre space ν_X. Let $(M, \partial M)$ be a compact n-dimensional manifold and let

$$(0.1) \qquad h\colon (M, \partial M) \to (X, \partial X); \quad \hat{h}\colon \nu_M \to \xi$$

be a degree-1 normal map with $h|\partial M$ a homotopy equivalence. Then surgery theory gives an element

$$\lambda(h, \hat{h}) \in L_n^h(\mathbb{Z}[\pi_1 X], w_1 X)$$

whose vanishing is necessary and sufficient (when $n \geq 5$ [31], or $n = 4$ if $\pi_1 X$ is virtually polycyclic [4]) for $(h, \hat{h})$ to be normally bordant (rel ∂) to a homotopy equivalence. The special case of (0.1) when M and X are closed manifolds will be called a *closed manifold surgery problem*. We will show that these problems have very restricted obstructions when π_1 is finite, related to the low-dimensional homology of $\pi_1 X$, while $L_n^h(\mathbb{Z}[\pi_1(X)])$ is usually large.

More generally, let $U \subseteq Wh(\pi_1 X)$ be an involution invariant subgroup. Then there is a similar theory if $(X, \partial X)$ is a U-simple Poincaré pair with $h|\partial M$ a U-simple homotopy equivalence, and the resulting obstruction $\lambda^U(h, \hat{h})$ lies in $L_n^U(\mathbb{Z}[\pi_1 X], w_1 X)$. The most important examples are $U = \{0\}$ (simple), $U = SK_1(\mathbb{Z}\pi_1)$ (weakly-simple), and $U = Wh(\mathbb{Z}\pi_1)$, yielding the surgery obstruction groups L^s, L', and L^h respectively. From now on we will only discuss the oriented case ($w_1 X = 1$), and in most cases the surgery obstructions will be evaluated in L'.

Here are two consequences of our main results. First consider the product of a simply-connected surgery problem with a closed manifold P in domain and range.

THEOREM 0.2. *Let P^l be a closed topological manifold with $\pi_1 P$ finite, and $(h\colon M^n \to N^n, \hat{h})$ a simply-connected closed manifold surgery problem. Then the product degree-1 normal map*

$$(h \times \mathrm{id}\colon M^n \times P^l \to N^n \times P^l, \ \hat{h} \times \mathrm{id}_{\nu_P})$$

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is normally cobordant to a weakly simple homotopy equivalence either

   (i) *for $n \equiv 2$ (mod 4) and $l \equiv 0$ (mod 4), if the Euler characteristic of P is even,*
      or

   (ii) *for $n \equiv 0$ (mod 4), if index(P) = 0.*

REMARK. This generalizes the Sullivan, Rourke–Sullivan product formulas [31; § 13B; 27] for the simply-connected index and Arf invariant (take $\pi_1 P = 1$). Part (ii) was first proved in [30]. The analogous result to Part (i) for the L^h obstruction was given in [13, 14]. There a program for solving the closed manifold surgery problem (with finite π_1) was developed, based on Clauwen's factorization [2] of the Ranicki product formula. The present work, which began as a sequel to these papers, uses instead an analysis of the Quinn–Ranicki assembly map (see § 1) together with ideas from [5, 6] and [29, 30].

Our most complete result is for odd-dimensional surgery problems. Recall that if $\bar{\pi}$ is a subquotient of π (that is, $\bar{\pi} = \rho/\rho_0$ where $\rho_0 \lhd \rho \subseteq \pi$) there is a homomorphism $L_n'(\mathbb{Z}\pi) \to L_n'(\mathbb{Z}\bar{\pi})$ induced geometrically by surgery on a covering normal map. Let $C(2)$ denote the cyclic group of order 2 and $Q(2^k)$ the generalized quaternion group of order 2^k.

THEOREM 0.3. *Let N^n be a closed topological manifold with $\pi_1 N$ finite and n odd. Then a closed manifold surgery problem*

$$(h\colon M^n \to N^n, \ \hat{h}\colon \nu_M \to \xi)$$

is normally cobordant to a weakly simple homotopy equivalence if and only if either

   (i) *$n \equiv 1$ (mod 4) and $\lambda'(h, \hat{h})$ maps to zero under transfer-projection to all quaternionic subquotients $Q(2^k)$ of $\pi_1 N$, or*

   (ii) *$n \equiv 3$ (mod 4) and $\lambda'(h, \hat{h})$ maps to zero under projection to all $C(2)$ quotients of $\pi_1 N$.*

To state our main results we need some notation. There exists an H-space G/TOP (see [10, 11]) such that the abelian group $[X/\partial X, G/TOP]$ acts simply transitively on the set of degree-1 normal maps (0.1) with range X rel ∂X. Fix a base-point $(h_0, \hat{h}_0)$ in this set, determined up to normal bordism by a topological bundle structure on ν_X, and let $(h_0, \hat{h}_0) * f$ denote the degree-1 normal map induced by the action of an element f in $[X/\partial X, G/TOP]$ on the base point. Define

(0.4) $\sigma_0^U\colon [X/\partial X, G/TOP] \to L_n^U(\mathbb{Z}[\pi_1 X])$

by the formula

$$\sigma_0^U(f) = \lambda^U((h_0, \hat{h}_0) * f) - \lambda^U(h_0, \hat{h}_0).$$

In Theorem A we will evaluate (0.4), assuming that $\pi_1 X$ is a finite group and $SK_1(\mathbb{Z}[\pi_1 X]) \subseteq U$, by factoring σ_0^U through the low-dimensional homology of $\pi_1 X$. The standard applications are the following.

EXAMPLE 1. In general, a U-simple Poincaré pair $(X, \partial X)$ with reduction $(h_0, \hat{h}_0)$ and $n \geq 5$ admits a topological manifold structure extending one on ∂X if and only if the coset of im(σ_0^U) in $L_n^U(\mathbb{Z}[\pi_1 X])$ contains zero.

EXAMPLE 2. When X is homotopy equivalent to a manifold then (0.4) is just the surgery obstruction of the surgery problem induced by f. This computes the maps σ in the surgery exact sequence [31]:

$$(0.5) \quad [\Sigma(X/\partial X), G/TOP] \xrightarrow{\sigma} L_{n+1}^U(\mathbb{Z}\pi, w) \longrightarrow \mathcal{S}(X, \partial X)$$
$$\longrightarrow [X/\partial X, G/TOP] \xrightarrow{\sigma} L_n^U(\mathbb{Z}[\pi_1 X], w).$$

EXAMPLE 3. By varying X over closed n-dimensional manifolds with $\pi_1 X \cong \pi$, we see that this evaluates the Sullivan–Wall map

$$\Omega_n(B\pi \times G/TOP) \to L_n^U(\mathbb{Z}\pi).$$

In particular, this determines the obstructions for closed manifold surgery problems.

Let V_X denote the total Wu class of ν_X, and

$$k = \{k_{4*+2}\} \in H^{4*+2}(G/TOP; \mathbb{Z}/2)$$

the universal class from [27]. Then given a map $f: X/\partial X \to G/TOP$ we define

$$(0.6) \qquad \mathrm{ARF}_j(f) = \{(V_X^2 \cup f^*(k)) \cap [X, \partial X]\}_{(j)} \in H_j(X; \mathbb{Z}/2)$$

to be the j-dimensional component of the indicated homology class. We let $\mathrm{ARF}(f)$ and $\mathrm{Index}(f)$ denote the change in the ordinary Arf invariant and index of the surgery problem given by f (considered as elements of $L_*(\mathbb{Z})$). Note that $\mathrm{ARF}_0(f)$ is just the Rourke–Sullivan formula for the Arf invariant. Finally let

$$(0.7) \qquad s_r: H_{2^{r+2}}(X; \mathbb{Z}/2) \to H_4(X; \mathbb{Z}/2)$$

(for $r \geqslant 0$) be the Hom-dual of the iterated squaring maps in cohomology.

For any finite group π with Sylow 2-subgroup ρ we define

$$\bar{Y} = \mathrm{Im}(SK_1(\mathbb{Z}\rho) \to SK_1(\mathbb{Z}\pi)) \subseteq Wh(\mathbb{Z}\pi) = K_1(\mathbb{Z}\pi)/\{\pm\pi^{ab}\}.$$

Note that since $\bar{Y}$ is a subgroup of $SK_1(\mathbb{Z}\pi)$, the surgery obstruction groups $L_n^{\bar{Y}}(\mathbb{Z}\pi)$ map into the groups $L_n'(\mathbb{Z}\pi)$ calculated in [34], and so into $L_n^h(\mathbb{Z}\pi)$.

In §1 we construct universal homomorphisms (for all $j \geqslant 0$):

$$(0.8) \qquad \kappa_j^U: H_j(\pi; \mathbb{Z}/2) \to L_{j+2}^U(\mathbb{Z}\pi)_{(2)},$$

for any involution-invariant subgroup $U \subseteq Wh(\mathbb{Z}\pi)$. The following result shows that if $\bar{Y} \subseteq U$, the surgery obstruction map $\sigma_0^U(f)$ can be computed in terms of $\mathrm{Index}(f)$, $\mathrm{ARF}(f)$ and the maps κ_j^U for $1 \leqslant j \leqslant 4$. In the statement of Theorem A the map $c: X \to B\pi_1 X$ classifies the universal cover of X.

THEOREM A. *Let $(X, \partial X)$ be an n-dimensional U-simple Poincaré pair and $(h_0, \hat{h}_0)$ a degree-1 normal map with the torsion of $h|\partial$ in $U \subseteq Wh(\mathbb{Z}[\pi_1 X])$. If $\bar{Y} \subseteq U$ and $\pi_1 X$ is finite, then for any $f: X/\partial X \to G/TOP$,*

$$\sigma_0^U(f) = \lambda^U((h_0, \hat{h}_0)*f) - \lambda^U(h_0, \hat{h}_0) \quad \text{in } L_n^U(\mathbb{Z}[\pi_1 X])$$

is equal to

(a) $\mathrm{Index}(f) + \kappa_2^U\{c_*(\mathrm{ARF}_2(f))\}$ *for $n \equiv 0 \pmod 4$,*
(b) $\kappa_3^U\{c_*(\mathrm{ARF}_3(f))\}$ *for $n \equiv 1 \pmod 4$,*
(c) $\mathrm{ARF}(f) + \kappa_4^U\{c_*(\sum_{r \geqslant 0} s_r(\mathrm{ARF}_{2^{r+2}}(f)))\}$ *for $n \equiv 2 \pmod 4$,*
(d) $\kappa_1^U\{c_*(\mathrm{ARF}_1(f))\}$ *for $n \equiv 3 \pmod 4$.*

We remark that κ_1 is always (split) monic and that κ_3 is detected by quaternionic subquotients of π. More precisely, if $Q(2^k)$ denotes the generalized quaternion group of order 2^k then

$$H_3(Q(2^k)\,;\mathbb{Z}/2)\cong\mathbb{Z}/2$$

and $\kappa_3\{c_*(\mathrm{ARF}_3(f))\}=0$ if and only if the image of $c_*(\mathrm{ARF}_3(f))$ is zero under transfer-projection to all quaternionic sub-quotients. This explains Theorem 0.3 above. Note that $\kappa_1\neq 0$ for $\pi=C(2)$ from [31], and $\kappa_3\neq 0$ for $\pi=Q(2^k)$ from [3]. The new information is that this set of $\mathbb{Z}/2$-valued invariants (often referred to as codimension 1 or 3 Arf invariants) suffices to detect the odd-dimensional surgery obstructions.

In the even-dimensional cases the answer is not so precisely under control (is κ_4 perhaps identically zero ?), and in all dimensions one would like an explicit description of the images of the κ_i. From [5, 30] we know that for $i\geqslant 2$, the image of κ_i^h is contained in the image of

$$H^{i+1}(\mathbb{Z}/2\,;\tilde{K}_0(\mathbb{Z}\pi))\to L_{i+2}^h(\mathbb{Z}\pi).$$

We can also say something about κ_2. Recall that Oliver [20] found examples of finite 2-groups π for which $H^1(\mathbb{Z}/2)\,;Wh'\,(\hat{\mathbb{Z}}_2\pi))$ is non-zero.

THEOREM B. *The composite*

$$H_2(\pi\,;\mathbb{Z}/2)\xrightarrow{\ \kappa_2\ }L_0^{\bar{Y}}(\mathbb{Z}\pi)\longrightarrow L_0^{\bar{Y}}(\hat{\mathbb{Z}}_2\pi)\cong H^1(\mathbb{Z}/2\,;Wh'\,(\hat{\mathbb{Z}}_2\pi))$$

is surjective for π a finite 2-group.

REMARK. Morgan and Pardon (unpublished, proved also in [30, 15]) showed that κ_2 is non-trivial on the image of $H_2(\pi\,;\mathbb{Z})$ in $H_2(\pi\,;\mathbb{Z}/2)$ by considering the surgery obstruction of the product of the Kervaire problem with $S^1\times S^1$ in $L_0^h(\mathbb{Z}[\mathbb{Z}/4\times\mathbb{Z}/2])$. The composite in Theorem B vanishes on the image of integral homology so it gives new examples of closed manifold obstructions in dimensions congruent to 0 (mod 4).

Recently Oliver has shown that the integral homology contributes further to L^s. In fact the image of the 2-adic κ_2^s contains $H^0(\mathbb{Z}/2\,;SK_1(\hat{\mathbb{Z}}_2\pi))$ for π a finite 2-group.

The formulas in Theorem A give the answer in $L_n^h(\mathbb{Z}\pi)$, which is useful for the existence problem of manifold structures, but do not in general give the answer in $L_n^s(\mathbb{Z}\pi)$. In some cases of interest $SK_1(\mathbb{Z}\pi)$ has odd order (e.g. when 4 does not divide the order of π [34]), and then $L_*^s=L_*'$. For another sort of example note, from [20], that, for most groups with periodic cohomology, $SK_1(\mathbb{Z}\pi)$ contains 2-torsion and so $L^s\neq L'$. For all such groups however, the 2-Sylow subgroups have trivial SK_1, and hence the above formulas do hold in $L_*^s(\mathbb{Z}\pi)$. Furthermore, κ_2 and κ_4 are zero in this case, by (7.4).

We now describe the main steps in the proof of Theorem A. It is sufficient to analyse the κ_i with values in $L_*'(\mathbb{Z}\pi)$ for π *a finite 2-group*, and this will be assumed from now on. In fact by naturality, $\kappa_i^{\bar{Y}}$ is the composite

$$H_i(\pi\,;\mathbb{Z}/2)\xrightarrow{\ \mathrm{trf}\ }H_i(\pi_2\,;\mathbb{Z}/2)\xrightarrow{\ \kappa_i\ }L_{i+2}'(\mathbb{Z}\pi_2)\longrightarrow L_{i+2}^{\bar{Y}}(\mathbb{Z}\pi),$$

where $\pi_2\subseteq\pi$ is a 2-Sylow subgroup (compare [32]).

In §1 we describe the surgery assembly map and recall in (1.10) the cohomological formula [**32, 29**], which expresses σ_0^U in terms of the κ_* and certain other universal homomorphisms. In §2 these are shown to reduce to just the ordinary index, so in the rest of the paper we concentrate on the κ_*.

We introduce Wall groups $L_*^s(R\pi)$, where R is the ring of integers in $\mathbb{Q}(\sqrt{5})$ with the Galois involution and prove

THEOREM 1.16. *There exist natural homomorphisms*

$$\bar{\kappa}_i\colon H_i(\pi\,;\mathbb{Z}/2)\to L_i^s(R\pi) \quad and \quad \mathrm{trf}_i\colon L_i^s(R\pi)\to L_{i+2}^s(\mathbb{Z}\pi)$$

such that

$$\kappa_i = \mathrm{trf}_i \circ \bar{\kappa}_i.$$

This immediately gives a similar factorization for the $\kappa_i^{\bar{Y}}$. The reason for our Whitehead torsion assumption $\bar{Y}\subseteq U$ appears at the next step. In the statement $\mathscr{S}_i(\pi)$ is the set of *dihedral* ($i\equiv 1, 2\;(\mathrm{mod}\;4)$) or *quaternion* ($i\equiv 0, 3\;(\mathrm{mod}\;4)$) subquotients of π. Let π^{ab} denote the abelianization of π.

THEOREM 5.4. *If π is a finite 2-group, the map*

$$L_i^{\bar{Y}}(R\pi)\to L_i^{\bar{Y}}(R[\pi^{ab}]) \oplus \sum \{L_i^{\bar{Y}}(R[\rho/\rho_0])\colon \rho_0\lhd\rho \in \mathscr{S}_i(\pi)\},$$

induced by the (sub)quotient maps, is an injection.

The proof of this result depends on the calculation of the groups $L_*^{\bar{Y}}(R\pi)$ carried out in §§3 and 4, using Oliver's logarithmic description of $K_1(\hat{R}_2\pi)/SK_1(\hat{R}_2\pi)$ from [**19**] and the methods of [**34**]. After Theorem 5.4 the $\bar{\kappa}_i$ are completely determined in terms of subquotient maps in homology, once we know the answer for abelian, quaternion, and dihedral groups. This is given in Theorem 6.8. Now Theorem 0.2(i) is a consequence of the fact that $\bar{\kappa}_i$ (and hence $\kappa_i^{\bar{Y}}$) vanishes for $i>3$ on the image of $H_i(\pi\,;\mathbb{Z})$ in $H_i(\pi\,;\mathbb{Z}/2)$.

We remark that the map (0.8)

$$s_r\colon H_{2^r+2}(X\,;\mathbb{Z}/2)\to H_4(X\,;\mathbb{Z}/2)$$

used in Theorem A is natural for push-forward and subquotient maps in homology (since it is dual to an iterated Steenrod square). Therefore to complete the proof of Theorem A, we combine (1.16) and (5.4) with an explicit check in (6.8) for the detecting groups that

$$\bar{\kappa}_{2^r+2} = \bar{\kappa}_4 \circ s_r$$

for all $r>0$. In §7 we determine the image of κ_3 and prove Theorem B.

1. Characteristic class formulae and factorizations

A. *The surgery assembly map*

We begin by giving an alternate description (due to Quinn and Ranicki) of the map from (0.4):

$$\sigma_0^U\colon [X/\partial X, G/TOP]\to L_n^U(\mathbb{Z}\pi_1[X], w)$$

defined by $\sigma_0^U(f) = \lambda^U((h_0, \hat{h}_0) * f) - \lambda^U(h_0, \hat{h}_0)$, where

$$(h_0 \colon (M^n, \partial M) \to (X, \partial X), \quad \hat{h}_0 \colon \nu_M \to \xi)$$

is a fixed normal map. Further details and references are given in an Appendix.

For any pair (π, w) consisting of a group with an orientation character, $\mathbb{L}_0^U(\mathbb{Z}\pi, w)$ is the quadratic L-spectrum and

$$\pi_n(\mathbb{L}_0^U(\mathbb{Z}\pi, w)) = L_n^U(\mathbb{Z}\pi, w).$$

Similarly $\mathbb{L}_U^0(\mathbb{Z}\pi, w)$ denotes the symmetric L-spectrum. For the trivial group we shorten the notation to $\mathbb{L}_0$ and $\mathbb{L}^0$ respectively. Then $\mathbb{L}_0$ is an Ω-spectrum with the space $\mathbb{Z} \times G/TOP$ in dimension zero.

For a space Y and a line bundle with first Stiefel–Whitney class w, let Y^w denote the Thom spectrum of the line bundle where the bottom cell is an S^0.

The *pre-assembly map* is a spectrum map:

$$a_{\pi,w} \colon Y^w \to \mathbb{L}_s^\circ(\mathbb{Z}\pi, w).$$

For w trivial, the definition is on [24, p. 288] and the generalization to our case is straightforward.

The spectrum $\mathbb{L}^0$ is a ring spectrum and $\mathbb{L}_0^U(\mathbb{Z}\pi, w)$ is a module spectrum over it. Any compact topological manifold $(M^n, \partial M)$ with $w = w_1(M)$ has a fundamental class

$$[M, \partial M]_{\mathbb{L}^\circ} \in H_n(M^w, \partial M^w ; \mathbb{L}^0).$$

Let

$$\gamma_0 \colon [X/\partial X, G/TOP] \subseteq H^0(X ; \mathbb{L}_0) \to H_n(X^w ; \mathbb{L}_0)$$

be the map given by capping with $(h_0)_*([M, \partial M]_{\mathbb{L}^\circ})$.

The *surgery assembly map* is the composite

$$A_{\pi,w} \colon B\pi^w \wedge \mathbb{L}_0 \xrightarrow{a_{\pi,w} \wedge 1} \mathbb{L}_s^0(\mathbb{Z}\pi, w) \wedge \mathbb{L}_0 \longrightarrow \mathbb{L}_0^s(\mathbb{Z}\pi, w).$$

Similarly we define the surgery assembly map into $\mathbb{L}_0^U(\mathbb{Z}\pi, w)$ for any involution-invariant subgroup $U \subseteq Wh(\mathbb{Z}\pi)$.

The Quinn–Ranicki factorization theorem (Appendix, Theorem 2). The geometric surgery obstruction map σ_0^U is given by the following composite:

$$[X/\partial X, G/TOP] \xrightarrow{\gamma_0} H_n(X^w ; \mathbb{L}_0) \xrightarrow{c_*} H_n(B\pi^w ; \mathbb{L}_0) \xrightarrow{A_{\pi,w}} L_n^U(\mathbb{Z}\pi, w),$$

where $c \colon (X, w) \to (B\pi, w)$ classifies the universal covering.

B. *Computation of* $c_* \circ \gamma_0 \colon [X/\partial X, G/TOP] \to H_n(B\pi^w ; \mathbb{L}_0)$

First we localize at 2. Recall that, since the L-theory of a finite group contains torsion only at 2, this does not lose information for our applications.

By the results of Morgan and Sullivan [16], Rourke and Sullivan [27], and Milgram [12], we can write

$$H_n(B\pi^w ; \mathbb{L}_0)_{(2)} \xrightarrow{\approx} \bigoplus_{i \geq 0} H_{n-4i}(\pi ; \mathbb{Z}_{(2)}^w) \oplus H_{n-4i-2}(\pi ; \mathbb{Z}/2).$$

Then [32] and [29] give the following 'characteristic variety' formula:

(1.1) $\gamma_0(f) = [u^*(\mathscr{L}) \cup f^*(l) + u^*(\mathscr{L}) \cup f^*(k) + \delta^*(u^*(VSq^1V) \cup f^*(k))] \cap [X]$.

Here u^* denotes the map in cohomology induced by the classifying map $u\colon X \to BSTOP$ of a bundle ξ_+, such that ξ_+ plus the line bundle corresponding to w is the bundle ξ associated to $(h_0, \hat{h}_0)$. The classes l and k are classes in the cohomology of G/TOP. Such classes are defined by Morgan and Sullivan in [16] and by Milgram in [12]. The particular ones we are using are the Morgan–Sullivan ones as in [29], but the reader should have no trouble converting to others if this is desired. Recall that

$$l \in \bigoplus H^{4i}(G/TOP\,;\mathbb{Z}_{(2)}) \quad \text{and} \quad k \in \bigoplus H^{4i+2}(G/TOP\,;\mathbb{Z}/2).$$

The class $\mathscr{L}$ is the 2-local class in $H^{4*}(BSTOP\,;\mathbb{Z}_{(2)})$ as defined in [16]. Note that $\mathscr{L}$ reduces mod 2 to V^2, where V is the total Wu class in $H^{2*}(BSTOP\,;\mathbb{Z}/2)$ and δ^* denotes the integral Bockstein.

C. *The computation of $A_{\pi,w}\colon H_n(B\pi^w\,;\mathbb{L}_0) \to L_n^U(\mathbb{Z}\pi, w)$*
   The pre-assembly map

$$a_{\pi,w}\colon B\pi^w \to \mathbb{L}_s^\circ(\mathbb{Z}\pi, w)$$

is natural with respect to w-preserving group homomorphisms, transfers to subgroups, and the following diagram commutes:

(1.2)

$$
\begin{array}{ccc}
B\pi_1^+ \wedge B\pi_2^+ & \xrightarrow{\;a_{\pi_1} \wedge a_{\pi_2}\;} & \mathbb{L}_s^\circ(\mathbb{Z}\pi_1) \wedge \mathbb{L}_s^\circ(\mathbb{Z}\pi_2) \\
\downarrow & & \downarrow \\
B(\pi_1 \times \pi_2)^+ & \xrightarrow{\;a_{\pi_1 \times \pi_2}\;} & \mathbb{L}_s^\circ(\mathbb{Z}[\pi_1 \times \pi_2])
\end{array}
$$

where $B\pi^+$ is just $B\pi^w$ for w trivial and a_π is $a_{\pi,w}$ for w trivial.

Note that the spectrum $\mathbb{L}_s^\circ(\mathbb{Z})$ is a ring spectrum, with unit $u\colon S^\circ \to \mathbb{L}_s^\circ(\mathbb{Z})$ and the 2-local splitting of $\mathbb{L}_s^\circ(\mathbb{Z})$ in [29] yields a map of ring spectra

$$I\colon K(\mathbb{Z}_{(2)}, 0) \to \mathbb{L}_s^\circ(\mathbb{Z})_{(2)}.$$

Because of the associativity of Cartesian products, the 2-local pre-assembly map $a_{\pi,w}$ is a module map over $K(\mathbb{Z}_{(2)}, 0)$ and we can use this module structure to give a more explicit description of the assembly map in terms of universal homomorphisms.

The homomorphisms we need all arise from the following construction. We fix a family of spectra, which we denote by $\mathbb{E}(\pi, w)$.

In the homotopy category of spectra, we want this assignment to be functorial with respect to w-preserving group homorphisms and transfers to subgroups.

If p_2 denotes the projection to the second factor, we require pairings of spectra

(1.3) $\mu\colon \mathbb{L}_s^\circ(\mathbb{Z}\pi_1) \wedge \mathbb{E}(\pi_2, w) \to \mathbb{E}(\pi_1 \times \pi_2, wp_2)$

which satisfy the following three conditions:

(1.4)   (i) the two ways of computing

$$\mathbb{L}_s^\circ(\mathbb{Z}\pi_1) \wedge \mathbb{L}_s^\circ(\mathbb{Z}\pi_2) \wedge \mathbb{E}(\pi_3, w) \to \mathbb{E}(\pi_1 \times \pi_2 \times \pi_3, wp_3)$$

are homotopic,

(ii) the composite

$$S^\circ \wedge \mathbb{E}(\pi, w) \xrightarrow{u \wedge 1} \mathbb{L}^\circ_s(\mathbb{Z}) \wedge \mathbb{E}(\pi, w) \xrightarrow{\mu} \mathbb{E}(\pi, w)$$

is the identity,

(iii) the pairing is natural for w-preserving group homomorphisms in both the $\mathbb{L}^\circ$ and the $\mathbb{E}$ factors; the pairing is natural with respect to transfers in the $\mathbb{E}$ factor and the identity in the $\mathbb{L}^\circ$ factor.

Given any functor $\mathbb{E}(\pi, w)$ as above, (1.4) says that $\mathbb{E}(\pi, w)$ is an $\mathbb{L}^\circ_s(\mathbb{Z})$-module spectrum. We outline a procedure for constructing homomorphisms after localizing at 2. Let

$$h \colon M(C, i) \to \mathbb{E}(e)_{(2)}$$

be a map where C is a 2-local group and $M(C, i)$ denotes the Moore spectrum with ith integral homology equal to C. We produce a map of Eilenberg–MacLane spectra

$$[h] \colon K(C, i) \cong K(\mathbb{Z}_{(2)}, 0) \wedge M(C, i) \xrightarrow{I \wedge h} \mathbb{L}^\circ_s(\mathbb{Z}) \wedge \mathbb{E}(e)_{(2)} \longrightarrow \mathbb{E}(e)_{(2)}$$

and then construct homomorphisms

(1.5) $$h_{n-i} \colon H_{n-i}(\pi \, ; C^w) \to E_n(\pi, w)_{(2)}$$

by applying π_n to the following composite

$$B\pi^w \wedge K(C, i) \xrightarrow{a_{\pi, w} \wedge [h]} \mathbb{L}^\circ_s(\mathbb{Z}\pi, w) \wedge \mathbb{E}(e)_{(2)} \xrightarrow{\mu} \mathbb{E}(\pi, w)_{(2)}.$$

Because of the conditions in (1.3), the $\{h_j\}$ that we have constructed are natural with respect to w-preserving group homomorphisms and transfers to subgroups, and hence for subquotient maps.

(1.6) EXAMPLE. For the map h take the unit $u \colon S^\circ_{(2)} \to \mathbb{L}^\circ_s(\mathbb{Z})_{(2)}$. This gives rise to homomorphisms

$$h_j \colon H_j(\pi \, ; \mathbb{Z})^w_{(2)} \to L^j_s(\mathbb{Z}\pi, w).$$

We will refer to these particular homomorphisms as $\{I_j\}$.

We can now state another property of the $\{h_j\}$ as constructed in (1.5): the following diagram commutes:

(1.7)
$$
\begin{array}{ccc}
H_n(\pi_1 \, ; \mathbb{Z}_{(2)}) \otimes H_{m-i}(\pi_2 \, ; \mathbb{Z}_{(2)}) & \longrightarrow & H_{n+m-i}(\pi_1 \times \pi_2 \, ; \mathbb{Z}_{(2)}) \\
\downarrow{\scriptstyle I_n \otimes h_{m-i}} & & \downarrow{\scriptstyle h_{n+m-i}} \\
L^n_s(\mathbb{Z}\pi, w) \otimes E_m(\pi_2)_{(2)} & \longrightarrow & E_{n+m}(\pi_1 \times \pi_2)_{(2)}
\end{array}
$$

The proof of (1.7) is an easy diagram chase.

(1.8) EXAMPLE. Let $\kappa \colon M(\mathbb{Z}/2, 2) \to \mathbb{L}^s_\circ(\mathbb{Z})$ denote the unique non-zero homotopy class. This gives rise to homomorphisms

$$\kappa_j \colon H_j(\pi \, ; \mathbb{Z}/2) \to L^s_{j+2}(\mathbb{Z}\pi, w)_{(2)}.$$

(1.9) EXAMPLE. Let $\mathscr{I}: S^\circ \to \mathbb{L}_\circ^s(\mathbb{Z})$ denote the unique homotopy class so that the composite $S^\circ \to \mathbb{L}_\circ^s(\mathbb{Z}) \to \mathbb{L}_s^\circ(\mathbb{Z})$ is 8 times the unit u. This gives homomorphisms

$$\mathscr{I}_j: H_j(\pi; \mathbb{Z}_{(2)}^2) \to L_j^s(\mathbb{Z}\pi, w)_{(2)}.$$

We can now combine formula (1.1) with the κ and $\mathscr{I}$ homomorphisms to give the cohomological formula for the surgery obstruction:

$$(1.10) \quad \sigma_0(f) = \mathscr{I}_* c_* \{(L_+ \cup f^*(l) + \delta^*(V_+ Sq^1 V_+ \cup f^*(k))) \cap [X, \partial X]\}$$
$$+ \kappa_* c_* \{V_+^2 \cup f^*(k) \cap [X, \partial X]\},$$

where $L_+ = u^*(\mathscr{L})$ and $V_+ = u^*(V)$ are the universal classes pulled back to ξ_+. In the oriented case, $\xi_+ = \xi$ and $V_+ = V_X$, the total Wu class of the Spivak normal fibre space ν_X. The term $\mathscr{I}_0 c_* \{L_+ \cup f^*(l) \cap [X, \partial X]\} = \mathrm{Index}(f)$ by the Sullivan product formula [**31**, 13B]. We will show in §2 that $\mathscr{I}_j = 0$ (for $j > 0$), under the assumptions of Theorem A.

D. *Factoring the surgery assembly map*

We prove Theorem A by factoring the $\mathscr{I}$ and κ homomorphisms through a more computable L-theory. For the $\{\mathscr{I}_j\}$ this was done in [**30**] and for the $\{\kappa_j\}$ in [**6**].

(1.11) EXAMPLE. Since $L_0^s(\hat{\mathbb{Z}}_2) = 0$, we can find a map $\tilde{\mathscr{I}}: S^1 \to \mathbb{L}_\circ^s(\mathbb{Z} \to \hat{\mathbb{Z}}_2)$ lifting the $\mathscr{I}$ from (1.9). This lift is not unique (since $L_1^s(\mathbb{Z} \to \hat{\mathbb{Z}}_2) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ there are two such) but either choice produces a family of homomorphisms

$$\tilde{\mathscr{I}}_j: H_j(\pi; \mathbb{Z}_{(2)}^w) \to L_{j+1}^s(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi)_{(2)}$$

which are natural for subquotient maps and satisfy (1.7). This gives a commutative diagram:

$$
\begin{array}{ccc}
H_j(\pi; \mathbb{Z}_{(2)}^w) & \xrightarrow{\;\mathscr{I}_j\;} & L_j^s(\mathbb{Z}\pi, w)_{(2)} \\
& \searrow{\scriptstyle \tilde{\mathscr{I}}_j} & \nearrow \\
& L_{j+1}^s(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi, w)_{(2)} &
\end{array}
$$

We can also smash the map $\tilde{\mathscr{I}}$ above with $M(\mathbb{Z}/2, 0)$ and get a mod 2 version of these results. We still have products and the mod 2 analogue of (1.7) continues to commute using the mod 2 versions of the I_n.

(1.12) EXAMPLE. Let $R = \mathbb{Z}[\varepsilon]$ denote the ring of integers in the quadratic extension $\mathbb{Q}(\sqrt{5})$, where $\varepsilon = \frac{1}{2}(1 + \sqrt{5})$. The Galois involution on R will be denoted $\bar{}$, and the involution on $R\pi$ induced by w is

$$\beta(\textstyle\sum r_g g) \to \sum \bar{r}_g w(g) g^{-1}.$$

By the 's'-decoration on L-groups we will mean torsions in $\tilde{K}_1(R\pi)$ which lie in the subgroup generated by $R^\times \oplus \pi^{ab}$.

Note that $\varepsilon + \bar{\varepsilon} = 1$ so that $\mathbb{L}_\circ^s(R\pi, w) \cong \mathbb{L}_s^\circ(R\pi, w)$. Furthermore $\varepsilon\bar{\varepsilon} = -1$ so

that all the *L*-groups are $\mathbb{Z}/2$-vector spaces [**34**, 3.4.4]. In Proposition 4.11, we show that

$$L_n^s(R, \beta, 1) = \mathbb{Z}/2,\ \mathbb{Z}/2,\ 0,\ 0 \quad \text{(for } n \equiv 0,\ 1,\ 2,\ 3 \text{ (mod 4) respectively).}$$

Let

$$\bar{\kappa}\colon M(\mathbb{Z}/2, 0) \to \mathbb{L}_o^s(R)$$

denote any homotopy class which induces an isomorphism on π_0. We get

$$\bar{\kappa}_j\colon H_j(\pi\,;\mathbb{Z}/2) \to L_j^s(R\pi, w).$$

There is no need to localize since $\mathbb{L}_o^s(R\pi, w)$ is already 2-local. As usual the $\{\bar{\kappa}_j\}$ are natural for subquotient maps.

The map $\{\bar{\kappa}\}\colon K(\mathbb{Z}/2, 0) \to \mathbb{L}_s^o(R)$ defined above is a map of ring spectra. This involves showing that the difference of two maps

$$K(\mathbb{Z}/2, 0) \wedge K(\mathbb{Z}/2, 0) \to \mathbb{L}_s^o(R)$$

is zero. Note that $K(\mathbb{Z}/2, 0) \wedge K(\mathbb{Z}/2, 0)$ is also a ring spectrum and (1.4)(i) shows that the map we are trying to show is zero is actually a module map over this ring spectrum. Hence it suffices to check that it is zero on $M(\mathbb{Z}/2, 0) \wedge M(\mathbb{Z}/2, 0)$ sitting in $K(\mathbb{Z}/2, 0) \wedge K(\mathbb{Z}/2, 0)$. An easy diagram chase using the unit shows that our map vanishes on $S^o \wedge M(\mathbb{Z}/2, 0)$ and on $M(\mathbb{Z}/2, 0) \wedge S^o$. The remaining obstruction is in π_2 of $\mathbb{L}_o^s(R)$, and hence vanishes. A diagram chase now yields the analogue of (1.7):

(1.13)
$$
\begin{array}{ccc}
H_i(\pi_1\,;\mathbb{Z}/2) \otimes H_j(\pi_2\,;\mathbb{Z}/2) & \longrightarrow & H_{i+j}(\pi_1 \times \pi_2\,;\mathbb{Z}/2) \\
\downarrow {\scriptstyle \bar{\kappa}_i \otimes \bar{\kappa}_j} & & \downarrow {\scriptstyle \bar{\kappa}_{i+j}} \\
L_s^i(R\pi_1) \otimes L_s^j(R\pi_2) & \longrightarrow & L_s^{i+j}(R[\pi_1 \times \pi_2])
\end{array}
$$

Finally we come to the reason for introducing the $\{\bar{\kappa}_j\}$. The ring homomorphism

$$h\colon R \to M_2(\mathbb{Z})$$

sending ε to $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is involution-invariant with respect to

$$\beta\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

on $M_2(\mathbb{Z})$. This is just transpose followed by conjugation with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A spectrum level transfer map is defined by the composite

(1.14) $\underline{\text{trf}}\colon \mathbb{L}_o^s(R\pi, \beta, 1) \xrightarrow{\ h_*\ } \mathbb{L}_o^{\bar{x}}(\mathbb{Z}\pi \otimes M_2(\mathbb{Z}), \alpha \otimes \beta, 1) \longrightarrow \mathbb{L}_o^s(\mathbb{Z}\pi, \alpha, -1),$

where $\bar{X}$ is the image of $R^\times \oplus \{\pi^{ab}\}$ in $\bar{K}_1(\mathbb{Z}\pi \oplus M_2(\mathbb{Z}))$. The second map in (1.14) is given by [**7**, 7.1]. The generator of $L_0^s(R)$ is the rank-1 quadratic form given by $\langle \varepsilon \rangle$. It is straightforward to calculate that, under trf_* the form goes to a form with Arf invariant 1. Since $\underline{\text{trf}}$ is compatible with products on the spectrum

level [7, 7.3], the following diagram commutes up to homotopy:

$$B\pi^w \wedge M(\mathbb{Z}/2, 0) \xrightarrow{a_{\pi,w} \wedge \bar{K}} \mathbb{L}^\circ_s(\mathbb{Z}\pi, w) \wedge \mathbb{L}^s_\circ(R, \beta, 1) \longrightarrow \mathbb{L}^{\bar{X}}_\circ(\mathbb{Z}\pi \otimes M_2(\mathbb{Z}), \alpha \otimes \beta, 1)$$

$$\downarrow{1 \wedge \mathrm{trf}} \qquad\qquad\qquad \downarrow{\mathrm{trf}}$$

$$(1.15) \qquad\qquad \mathbb{L}^\circ_s(\mathbb{Z}\pi, w) \wedge \mathbb{L}^s_\circ(\mathbb{Z}, 1, -1) \longrightarrow \mathbb{L}^s_\circ(\mathbb{Z}\pi, \alpha, -1)$$

with the diagonal map $a_{\pi,w} \wedge K$.

THEOREM 1.16. *For any pair (π, w) there is a factorization, commuting with w-preserving group homomorphisms and subquotient maps,*

$$H_j(\pi ; \mathbb{Z}/2) \xrightarrow{\kappa_j} L^s_{j+2}(\mathbb{Z}\pi, w)_{(2)}$$

$$\bar{\kappa}_j \searrow \qquad \nearrow \mathrm{trf}_*$$

$$L^s_j(R\pi, w)$$

Proof. This is an immediate consequence of Example (1.12) and (1.15).

2. The vanishing of the $\mathcal{I}_*$

For any finite group π, we let

$$Cl_1(\pi) = \ker\{\bar{K}_1(\mathbb{Z}\pi) \to \bar{K}_1(\hat{\mathbb{Z}}\pi) \oplus \bar{K}_1(\mathbb{Q}\pi)\}$$

and if $\rho \subseteq \pi$ is a 2-Sylow subgroup let

$$\bar{Cl}(\pi) = \mathrm{im}\{Cl_1(\rho) \to Cl_1(\pi)\}.$$

THEOREM 2.1 [29]. *For any finite group π, the homomorphisms*

$$\mathcal{I}_j \colon H_j(\pi ; \mathbb{Z}_{(2)}) \to L^{\bar{Cl}}_j(\mathbb{Z}\pi)_{(2)}$$

are zero for $j \geqslant 1$.

Notice that $\bar{Cl}(\pi) \subseteq \bar{Y}(\pi)$ where $\bar{Y}(\pi) = \mathrm{im}\{SK_1(\mathbb{Z}\rho) \to SK_1(\mathbb{Z}\pi)\} \subseteq \bar{K}_1(\mathbb{Z}\pi)$ was the decoration used in the statement of Theorem A. Also if π is a finite 2-group, then from [20],

$$(2.2) \qquad\qquad \bar{Y}(\pi)/\bar{Cl}(\pi) \cong SK_1(\hat{\mathbb{Z}}_2\pi).$$

We will use the following result of [9].

THEOREM 2.3. *For any finite 2-group π, the subquotient maps give an injection*

$$L^{\bar{Cl}}_{j+1}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \longrightarrow \bigoplus L^{\bar{Cl}}_{j+1}(\mathbb{Z}[H/N] \to \hat{\mathbb{Z}}_2[H/N]),$$

where the sum is over all basic subquotients of π.

Recall that the basic 2-groups are cyclic, $C(2^k)$, quaternion, $Q(2^k)$, dihedral, $D(2^k)$ for $k > 3$, and semidihedral, $SD(2^k)$.

Since $\mathcal{I}_j$ factors by (1.11) through

$$\tilde{\mathcal{I}}_j \colon H_j(\pi ; \mathbb{Z}_{(2)}) \to L^{\bar{Cl}}_{j+1}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi)_{(2)},$$

we have finished if we can show that $\tilde{\mathcal{J}}_j$ is trivial for $j \geqslant 1$ on the basic 2-groups. Note that we also get a commutative diagram:

$$
\begin{array}{ccc}
H_j(\pi ; \mathbb{Z}_{(2)}) & \xrightarrow{\tilde{\mathcal{J}}_j} & L_{j+1}^{C\bar{l}}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi)_{(2)} \\
\downarrow & & \downarrow r_* \\
H_j(\pi ; \mathbb{Z}/2) & \xrightarrow{\bar{\tilde{\mathcal{J}}}_j} & L_{j+1}^{C\bar{l}}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi ; \mathbb{Z}/2)
\end{array}
$$

where $\bar{\tilde{\mathcal{J}}}_j$ is the mod 2 analogue of $\tilde{\mathcal{J}}_j$.

For π a 2-group we have

$$
L_{j+1}^{C\bar{l}}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \xrightarrow{\approx} L_{j+1}^{X}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi),
$$

which is computed in [**34**, 5.2.2] for the basic 2-groups. Notice that r_*, the reduction mod 2, is injective on the torsion subgroup. Since each basic group contains a unique central element z of order 2, we can analyse $\tilde{\mathcal{J}}_j$ by comparing the relative L-groups under the projection $p: \pi \to \pi/\langle z \rangle$. We now consider each type separately.

(i) $C(2)$: compare with the trivial group;

$$
p_* \oplus \text{res}: L_{j+1}^{X}(\mathbb{Z}[C(2)] \to \hat{\mathbb{Z}}_2[C(2)])
$$
$$
\longrightarrow L_{j+1}^{X}(\mathbb{Z}[e] \to \hat{\mathbb{Z}}_2[e]) \oplus L_{j+1}^{X}(\mathbb{Z}[e] \to \hat{\mathbb{Z}}_2[e])
$$

is injective for all j, whence $\tilde{\mathcal{J}}_j$ and $\bar{\tilde{\mathcal{J}}}_j$ are trivial when $j \geqslant 1$.

(ii) $C(2^k)$, $k > 1$: here

$$
p_*: \text{tors } L_{j+1}^{X}(\mathbb{Z}[C(2^k)] \to \hat{\mathbb{Z}}_2[C(2^k)]) \longrightarrow L_{j+1}^{X}(\mathbb{Z}[C(2^{k-1})] \to \hat{\mathbb{Z}}_2[C(2^{k-1})])
$$

is injective for all j.

(iii) $D(2^k)$: Quillen [**22**] has proved that $H_j(D(2^k) ; \mathbb{Z}/2)$ is generated by elementary abelian subgroups. The quadratic detection theorem [**9**] implies that

$$
L_{j+1}^{X}(\mathbb{Z}[C(2^k)] \to \hat{\mathbb{Z}}_2[C(2^k)])
$$

is detected by $C(2)$ subquotients.

(iv) $Q(2^k)$: recall that $H_j(Q(2^k) ; \mathbb{Z}_{(2)}) = 0$ for $j \equiv 2 \pmod 4$. Also

$$
p_*: L_{j+1}^{X}(\mathbb{Z}[Q(2^k)] \to \hat{\mathbb{Z}}_2[Q(2^k)]) \longrightarrow L_{j+1}^{X}(\mathbb{Z}[D(2^{k-1})] \to \hat{\mathbb{Z}}_2[D(2^{k-1})])
$$

is injective for $j \equiv 0$ or $3 \pmod 4$. Finally $H_j(Q(2^k) ; \mathbb{Z}_{(2)})$ is generated by cyclic subgroups for $j \equiv 1 \pmod 4$.

(v) $SD(2^k)$: here

$$
p_*: \text{tors } L_{j+1}^{X}(\mathbb{Z}[SD(2^k)] \to \hat{\mathbb{Z}}_2[SD(2^k)]) \longrightarrow L_{j+1}^{X}(\mathbb{Z}[D(2^{k-1})] \to \hat{\mathbb{Z}}_2[D(2^{k-1})])
$$

is injective for all j.

3. *The 2-adic L-group*

In this section we will describe the structure of $L_i^{\bar{Y}}(\hat{R}_2\pi, \alpha, u)$ for a finite 2-group. Throughout this paper we let

(3.1) $X = SK_1(A\pi) \subseteq K_1(A\pi)$ for $A = R$ or $\hat{R}_2$,

$Y = R^\times \oplus \pi^{ab} \oplus X$ so $Y/X \cong R^\times \oplus \pi^{ab}$,

$\bar{Y}$ denote the image of Y in $K_1(\mathbb{Z}\pi)$, modulo the image of $K_1(\mathbb{Z}) = \{\pm 1\}$.

The antistructure (A, α, u) will always be a geometric antistructure [**8**, Appendix I]:

$$(3.2) \qquad\qquad \alpha(rg) = w(g)\bar{r}\theta(g^{-1})$$

for all $g \in \pi$ and $r \in R$. Here $w\colon \pi \to \{\pm 1\}$ is a homomorphism and $\theta\colon \pi \to \pi$ is an automorphism with $\theta^2(g) = bgb^{-1}$ for all $g \in \pi$, and $\omega \circ \theta = \omega$. Also we assume that $u = \pm b$, $w(b) = 1$, and $\theta(b) = b$. The Galois automorphism $(\sqrt{5} \to -\sqrt{5})$ is denoted by $\bar{\ }\colon R \to R$.

The antistructure with $\theta = \mathrm{Id}$, w trivial, and unit $u = e$ will be called the *standard antistructure*.

The main result of this section is

THEOREM 3.3. *For the standard antistructure there is an isomorphism*

$$L_i^{\bar{Y}}(\hat{R}_2\pi) \xrightarrow{\ \approx\ } L_i^{\bar{Y}}(\hat{R}_2) \oplus H^i(\pi^{ab}),$$

where the involution on π^{ab} is $g \to g^{-1}$. If $f\colon \pi_1 \to \pi_2$ is a group homomorphism, the isomorphism is natural so that the induced maps fit into a commutative diagram

$$\begin{array}{ccc}
L_i^{\bar{Y}}(\hat{R}_2\pi) & \longrightarrow & L_i^{\bar{Y}}(\hat{R}_2) \oplus H^i(\pi_1^{ab}), \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \mathrm{Id} \oplus f_*} \\
L_i^{\bar{Y}}(\hat{R}_2\pi_2) & \xrightarrow{\ \approx\ } & L_i^{\bar{Y}}(\hat{R}_2) \oplus H^i(\pi_2^{ab})
\end{array}$$

REMARK. We have $L_i^{\bar{Y}}(\hat{R}_2) \cong \mathbb{Z}/2, 0, 0, \mathbb{Z}/2$ for $i \equiv 0, 1, 2,$ or $3 \pmod 4$, see (3.11). As usual in this context, the notation $H^i(\pi^{ab})$ means the Tate cohomology of $\mathbb{Z}/2$ with coefficients in the module π^{ab}. This convention will be used throughout the paper.

Theorem 3.3 has two immediate corollaries.

THEOREM 3.4 (the 2-adic detection theorem). *Let $\{f_k\colon \pi \to C_k, 1 \le k \le r\}$ be a set of projections of π to cyclic groups C_k such that the direct sum $\oplus f_k$ gives an isomorphism $\pi^{ab} \cong \bigoplus_{k=1}^r C_k$. Then the induced map*

$$L_i^{\bar{Y}}(\hat{R}_2\pi) \to \bigoplus_{k=1}^r L_i^{\bar{Y}}(\hat{R}_2 C_k)$$

is an injection.

In § 5 we will need the following remark.

PROPOSITION 3.5. *Let $z \in \pi$ be a central element of order 2 whose image in $H_1(\pi\,;\mathbb{Z}/2)$ is trivial. Then the map $L_{2i-1}^{\bar{Y}}(\hat{R}_2\pi) \to L_{2i-1}^{\bar{Y}}(\hat{R}_2\pi/\langle z \rangle)$ is an isomorphism.*

We will give the proof of Theorem 3.3 after collecting together some needed lemmas. The general method is to understand the round to unround exact sequence [**8**, p. 61, 1.1.6] and then compute the round groups. The following general discussion ought to be in the literature but we cannot find it.

Let (A, β, v) be a ring with antistructure and let $V \subseteq K_1(A)$ be a β-invariant subgroup with $\{\pm v\} \in V$. Then [**7**, Proposition 3.2] says that

$$(3.6) \qquad 0 \longrightarrow L_{2i}^V(A, \beta, v) \longrightarrow L_{2i}^{\bar V}(A, \beta, v) \xrightarrow{rk_i} \mathbb{Z}/2$$

$$\xrightarrow{\partial_i} L_{2i-1}^V(A, \beta, v) \longrightarrow L_{2i-1}^{\bar V}(A, \beta, v) \longrightarrow 0$$

is exact. There is a Rothenberg sequence (due to Wall [**34**]) and we let

$$\tau_{2i-1}: L_{2i-1}^V(A, \beta, v) \to H^1(V)$$

denote the discriminant map.

THEOREM 3.7. *The composite*

$$\mathbb{Z}/2 \xrightarrow{\partial_i} L_{2i-1}^V(A, \beta, v) \xrightarrow{\tau_{2i-1}} H^1(V)$$

sends the generator to the class of $(-1)^i v \in H^1(V)$.

Proof. Using Wall's definition of $L_{2i-1}^V(A, \beta, v)$ and of the map in the Rothenberg sequence, it is easy to chase down the class in H^1 represented by the generator of the $\mathbb{Z}/2$.

In our case this gives

THEOREM 3.8. *Let* $(A\pi, \alpha, u)$ *be a geometric antistructure on* $A\pi$ *for* $A = R$ *or* $\hat R_2$. *Then* $rk_i: L_{2i}^{\bar Y}(A\pi, \alpha, u) \to \mathbb{Z}/2$ *is surjective if and only if*

$$\langle (-1)^i u \rangle \in H^1(R^\times \oplus \pi^{ab})$$

is trivial.

Proof. If $\langle (-1)^i u \rangle \neq 0$ in $H^1(Y/X)$, it certainly cannot be 0 in $H^1(Y)$ either, so one way follows easily from Theorem 3.7. If $\langle (-1)^i u \rangle = 0$ then we can find $w \in R^\times \oplus \pi^{ab}$ such that $(-1)^i u = w\beta^{-1}(w)$. If we scale by w [**8**, p. 74, 2.5.5], we see that the map rk_i above is equivalent to the map

$$rk_0: L_0^{\bar Y}(A\pi, \alpha^w, 1) \to \mathbb{Z}/2.$$

But $\langle \varepsilon \rangle$ is a rank-1 quadratic form in $L_0^{\bar Y}(A\pi, \alpha^w, 1)$.

We now turn to the computation of the round groups. The result here is

THEOREM 3.9. *The groups* $L_i^X(R\pi, \alpha, u)$ *are trivial and the other map in the* L^X *to* L^Y *Rothenberg sequence*

$$L_i^Y(\hat R_i \pi, \alpha, u) \to \hat H^i(R^\times \oplus \pi^{ab})$$

is an isomorphism.

REMARK 3.10. We will not give the full proof in this paper. We will do the case when π is abelian or where the map w is trivial. These two cases suffice for our applications.

In particular, it is now easy to show

PROPOSITION 3.11. *For the trivial group,*

$$L_i^{\bar{Y}}(\hat{R}_2) \cong \mathbb{Z}/2, 0, 0, \mathbb{Z}/2.$$

If C_2^- denotes the antistructure with $u = 1$ and w non-trivial, then

$$L_i^{\bar{Y}}(\hat{R}_2[C_2^-]) \cong \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2$$

for $i \equiv 0$, 1, 2, or 3 (mod 4).

Proof. Use (3.9) to calculate L_i^Y and then use (3.8) to determine the rk_i.

We now turn to the proof of Theorem 3.9. Since $\hat{R}_2\pi$ modulo its radical is $\mathbb{F}_4$ with the non-trivial involution, the L^K groups are trivial by a result of Wall [**34**]. Hence (3.9) is equivalent to

LEMMA 3.12. $H^i(\mathbb{Z}/2 \,; K_1(R\pi)/X) \cong 0$ *for $i \equiv 0$ or 1 (mod 2).*

Proof. First we prove (3.12) assuming that π is abelian. Then $A = \hat{R}_2\pi$ is a commutative, complete local $\hat{R}_2$-algebra: let m denote its maximal ideal. As usual, we have the involution-invariant exact sequence

$$0 \longrightarrow \frac{1 + m^{i-1}}{1 + m^i} \longrightarrow (A/m^i)^\times \xrightarrow{p_i} (A/m^{i-1})^\times \longrightarrow 0.$$

Since $H^*(\mathbb{Z}/2 \,; (A/m)^\times) \cong 0$ (by Hilbert's Theorem 90 and a Herbrand quotient argument) we can assume by induction that $H^*(\mathbb{Z}/2 \,; (A/m^{i-1})^\times) \cong 0$. Also,

$$\frac{1 + m^{i-1}}{1 + m^i} \cong \frac{m^{i-1}}{m^i},$$

and this last group is an $\hat{R}_2$-module. Moreover, the isomorphism preserves the involution, so (since $\varepsilon + \bar{\varepsilon} = 1$),

$$H^*\left(\mathbb{Z}/2 \,; \frac{1 + m^{i-1}}{1 + m^i}\right) \cong 0.$$

Since A is complete, $A^\times = \varprojlim (A/m^i)^\times$ and, since p_i is surjective for all i, $H^*(\mathbb{Z}/2 \,; A^\times) \cong 0$.

In the abelian case, $X = SK_1(\hat{R}_2\pi)$ is trivial so we have finished. If π is non-abelian, we reduce to the abelian case using results of Oliver [**19**]. Here we need to assume w is trivial or else generalize [**19**] along the lines of [**21**]. We stick to the case where w is trivial. Then we have (from [**19**]) the following involution-invariant exact sequence

(3.13) $$0 \to \hat{R}_2^\times \oplus \pi^{ab} \to K_1(\hat{R}_2\pi)/X \to \overline{I(\hat{R}_2\pi)} \to \pi^{ab} \to 0.$$

The involution on $\overline{I(\hat{R}_2\pi)}$ satisfies $\alpha(rx) = \bar{r}\alpha(x)$ so $H^*(\mathbb{Z}/2 \,; \overline{I(\hat{R}_2\pi)}) \cong 0$. The sequence (3.13) is natural for group homomorphisms, and if we use the natural epimorphism $\pi \to \pi^{ab}$ to compare (3.13) for π with (3.13) for π^{ab}, we see that

$$H^*(K_1(\hat{R}_2\pi)/X) \cong H^*(K_1(\hat{R}_2[\pi^{ab}])) \cong 0.$$

This concludes our proof of Theorem 3.9, so we turn to

Proof of Theorem 3.3. We now have the standard antistructure. In this case we have that $\hat{R}_2 \to \hat{R}_2\pi$ and $\hat{R}_2\pi \to \hat{R}_2$ are maps of rings with antistructure.

It follows easily from Theorem 3.9 that

$$L_i^Y(\hat{R}_2\pi) \cong L_i^Y(\hat{R}_2) \oplus H^i(\pi^{ab})$$

and from this and Theorem 3.8, the result follows.

4. *An exact sequence for* $L_i^{\bar{Y}}(R\pi)$

The information we need about the $L_i^{\bar{Y}}(R\pi, \alpha, u)$ comes from the exact sequence

$$(4.1) \quad \cdots \longrightarrow L_i^{\bar{Y}}(R\pi, \alpha, u) \longrightarrow L_i^{\bar{Y}}(\hat{R}_2\pi, \alpha, u) \xrightarrow{\tilde{\psi}_i} L_i^{\bar{Y}}(R\pi \to \hat{R}_2\pi, \alpha, u)$$

$$\longrightarrow L_{i-1}^{\bar{Y}}(R\pi, \alpha, u) \longrightarrow \cdots .$$

There is a similar exact sequence relating the round L-groups, and

$$L_i^Y(R\pi \to \hat{R}_2\pi, \alpha, u) \longrightarrow L_i^{\bar{Y}}(R\pi \to \hat{R}_2\pi, \alpha, u)$$

is an isomorphism.

The two main results of this section concern (4.1) with the *standard antistructure* ($\theta = \mathrm{Id}$, w trivial, $u = 1$). In Theorem 4.5 we compute

$$L_i^{\bar{Y}}(R\pi \to \hat{R}_2\pi)$$

and in Theorem 4.10 we compute $\tilde{\psi}_i$.

PROPOSITION 4.2. *For any geometric antistructure, the composite*

$$L_i^{\bar{Y}}(R\pi \to \hat{R}_2\pi, \alpha, u) \xleftarrow{\approx} L_i^Y(R\pi \to \hat{R}\pi, \alpha, u) \xleftarrow{\approx} L_i^X(R\pi \to \hat{R}_2\pi, \alpha, u)$$

$$\xrightarrow{\approx} L_{i-1}^X(R\pi, \alpha, u) \xrightarrow{\approx} L_{i-1}^S(\mathbb{F}_5\pi, \alpha, u)$$

is an isomorphism.

Proof. Recall $X = SK_1(R\pi)$ or its image in $K_1(\hat{R}_l\pi)$ where $\hat{R}_l$ is the completion at some prime. Since $SK_1(\hat{R}_5\pi) \cong 0$ for π a 2-group [19], $SK_1(R\pi) \to K_1(\mathbb{F}_5\pi)$ is trivial, so we do land in $L_{i-1}^S(\mathbb{F}_5\pi, \alpha, u)$.

Since the antistructure is fixed, we suppress it in the notation. The map

$$L_i^X(R\pi \to \hat{R}_2\pi) \longrightarrow L_{i-1}^S(\mathbb{F}_5\pi),$$

defined above, factors as the composite

$$L_i^X(R\pi \to \hat{R}_2\pi) \xrightarrow{1} L_i^S\left(\prod_{l \neq 2} \hat{R}_l\pi \oplus \mathbb{Q}(\varepsilon)\pi \to \hat{\mathbb{Q}}(\varepsilon)\pi\right)$$

$$2\Big\downarrow$$

$$L_{i-1}^s(\hat{R}_5\pi) \xrightarrow{3} L_{i-1}^S(\mathbb{F}_5\pi)$$

The map 1 is an isomorphism by excision: the map 3 is an isomorphism from the reduction modulo the radical theorem [34].

In the centres of the various simple factors of $\mathbb{Q}\pi$, 2 is the only prime that ramifies. Since the discriminant of $\mathbb{Q}(\varepsilon)$ is 5, 2 and 5 are the only primes that ramify in the centres of the various simple factors of $\mathbb{Q}(\varepsilon)\pi$. In $\mathbb{Q}(\varepsilon)\pi$, all the factors are of type GL or U and in $\hat{R}_l\pi$ for $l \neq 5$, all the factors still have Type GL or U. By a result of Wall [34], all the L^S groups are trivial for GL and U type factors, so the map 2 above is an isomorphism.

Since all finite division algebras are fields, the general theory in [8] and [9] shows

PROPOSITION 4.3. *For all* i, $L_i^S(\mathbb{F}_5\pi, \alpha, u) \cong \prod L_i^S(\mathbb{F}_r, \alpha, \pm 1)$ *where the product is taken over the invariant factors of* $\mathbb{F}_5\pi$, *and* $\mathbb{F}_r$ *denotes the centre of the rth factor.*

A similar result holds for the other two terms in the L^S to L^K Rothenberg sequence for $\mathbb{F}_5\pi$, and the whole sequence for $\mathbb{F}_5\pi$ is the product of the ones for the various $\mathbb{F}_r$.

With (4.3) established, we turn to

LEMMA 4.4. *If* α *is a non-trivial,*

$$L_i^S(\mathbb{F}_5, \alpha, \pm 1) \cong L_i^K(\mathbb{F}_5, \alpha, \pm 1) \cong H^i(\mathbb{F}_5^\times) \cong 0.$$

If α *is the identity, and* $\mathbb{F}_r$ *is an extension of* $\mathbb{F}_5$ *then*

$$0 \to L_i^K(\mathbb{F}_r, 1, 1) \to H^i(\mathbb{F}_r^\times) \to L_{i-1}^S(\mathbb{F}_r, 1, 1) \to 0$$

is exact, $H^i(\mathbb{F}_r^\times) \cong \mathbb{Z}/2$,

$$L_i^K(\mathbb{F}_r, 1, 1) \cong \mathbb{Z}/2, \ \mathbb{Z}/2, \ 0, \ 0, \quad and \quad L_i^S(\mathbb{F}_r, 1, 1) \cong 0, \ \mathbb{Z}/2, \ \mathbb{Z}/2, \ 0$$

for $i \equiv 0, 1, 2, $ *or* $3 \pmod 4$.

Proof. Wall [34] shows that the map $L_i^S \to L_i^K$ is trivial in the above cases. The groups L_i^K are computed in [8, p. 90, 3.3.1], for α non-trivial, and [8, p. 83, 3.2.1], for α the identity. The Tate cohomology calculations are easy since $\mathbb{F}_r^\times$ is cyclic and, if α is non-trivial, $H^1(\mathbb{F}_r^\times) \cong 0$ by Hilbert's Theorem 90.

In the proof to follow, we explain the notion of the *type* of a factor of $\mathbb{Q}\pi$.

THEOREM 4.5. *For the standard antistructure,*

$$L_{i+1}^{\bar{Y}}(R\pi \to \hat{R}_2\pi) \cong (\mathbb{Z}/2)^{s_i},$$

where $s_1 = s_2$ *is a number of Type* O *factors in* $\mathbb{Q}\pi$, *and* $s_3 = s_0$ *is a number of Type* Sp *factors in* $\mathbb{Q}\pi$.

Before beginning the proof of Theorem 4.5, we give a general discussion of the relation between types of factors in $\mathbb{Q}\pi$ and types of factors in $\mathbb{F}_5\pi$. We let (α, u) be an arbitrary geometric antistructure. Recall [8] that

$$\mathbb{Z}[\tfrac{1}{2}]\pi \cong \prod \mathrm{End}_{\Delta_i}(L_i),$$

where Δ_i is a $\mathbb{Z}[\frac{1}{2}]$-maximal order in one of the following division algebras:

$$\mathbb{Q}, \quad \mathbb{Q}(\zeta_{2^k}), \quad k \geqslant 2; \quad \mathbb{Q}(\zeta_{2^k} + \bar{\zeta}_{2^k}), \quad k \geqslant 3;$$

$$\mathbb{Q}(\zeta_{2^k} - \bar{\zeta}_{2^k}), \quad k \geqslant 4; \quad \text{or} \quad \left(\frac{-1, -1}{\mathbb{Q}(\zeta_{2^k} + \bar{\zeta}_{2^k})}\right), \quad k \geqslant 2.$$

Clearly,

$$\mathbb{Q}\pi \cong \prod M_{n_i}(\mathbb{Q} \otimes_{\mathbb{Z}[\frac{1}{2}]} \Delta_i) \quad \text{and} \quad \mathbb{F}_5\pi \cong \prod M_{n_i}(\mathbb{F}_5 \otimes \Delta_i).$$

A geometric antistructure begins as an antistructure on $\mathbb{Z}\pi$. Factors of $\mathbb{Q}\pi$ that are flipped (so called GL type) give rise to GL factors of $\mathbb{F}_5\pi$.

We also showed in [8] that any invariant factor $\mathrm{End}_{\Delta_i}(L_i)$ is quadratic Morita equivalent to $(\Delta_i, \alpha, \pm 1)$: hence the corresponding factor in $\mathbb{F}_5\pi$ is quadratic Morita equivalent to $(\mathbb{F}_5 \otimes \Delta_i, 1 \otimes \alpha, \pm 1)$.

Invariant factors of $\mathbb{Q}\pi$ for which α is non-trivial on the centre are called Type U; if α is the identity on the centre and the unit is 1 (respectively -1), the factor is said to be of Type O (respectively Sp).

Since 5 splits in $\mathbb{Q}(\zeta_{2^k})$ for $k \geqslant 2$, but is inert in $\mathbb{Q}(\zeta_{2^k} \pm \bar{\zeta}_{2^k})$, $\mathbb{F}_5 \otimes \Delta_i$ is a field for $\Delta_i = \mathbb{Z}[\frac{1}{2}][\zeta_{2^k} \pm \bar{\zeta}_{2^k}]$ but it splits into two fields for $\Delta_i = \mathbb{Z}[\frac{1}{2}][\zeta_{2^k}]$. When Δ_i is the $\mathbb{Z}[\frac{1}{2}]$-maximal order in

$$\left(\frac{-1, -1}{\mathbb{Q}(\zeta_{2^k} + \bar{\zeta}_{2^k})}\right),$$

we have

$$\mathbb{F}_5 \otimes \Delta_i \cong M_2(\mathbb{F}_5 \otimes \mathbb{Z}[\zeta_{2^k} + \bar{\zeta}_{2^k}])$$

and this is quadratic Morita equivalent to $\mathbb{F}_5 \otimes \mathbb{Z}[\zeta_{2^k} + \bar{\zeta}_{2^k}]$ with no type change. Hence a Type O or Sp factor of $\mathbb{Q}\pi$ gives rise to a single factor of $\mathbb{F}_5\pi$ of the same type unless the centre of the factor is $\mathbb{Q}(\zeta_{2^k})$, with $k \geqslant 2$, in which case we get two factors of $\mathbb{F}_5\pi$ of the same type.

Proof of Theorem 4.5. Theorem 4.5 follows from Proposition 4.3, Lemma 4.4, the above discussion, and the next lemma.

LEMMA 4.6. *For the standard antistructure, the centre determines the type:*

$\mathbb{Q}$ *and* $\mathbb{Q}(\zeta_{2^k} + \bar{\zeta}_{2^k})$ *for* $k \geqslant 3$ *are Type* O;

$\left(\frac{-1, -1}{\mathbb{Q}(\zeta_{2^k} + \bar{\zeta}_{2^k})}\right)$, *for* $k \geqslant 2$ *are Type* Sp;

and the others are Type U.

Proof and remark. This result is well known. For more complicated antistructures the centre does not determine the type, but in [8, Appendix I], we gave one way to determine the type of the factors. The proof of Theorem 4.5 then outlines how to compute $L_i^Y(R\pi \to \hat{R}_2\pi, \alpha, u)$ for any geometric antistructure.

Next we turn to the calculation of the $\tilde{\psi}_i$ maps. Our first result, Corollary 4.8

below, reduces the calculation of the maps

$$\psi_i\colon\ L_i^Y(\hat{R}_2\pi,\ \alpha)\ \longrightarrow\ L_i^Y(R\pi\to\hat{R}_2\pi,\ \alpha,\ u)$$

to a K-theory calculation. But first we remark

LEMMA 4.7. *The following two composites are equal:*

$$L_i^Y(\hat{R}_2\pi,\ \alpha,\ u)\ \xrightarrow{\ \psi_i\ }\ L_i^Y(R\pi\to\hat{R}_2\pi,\ \alpha,\ u)\ \xleftarrow{\ \cong\ }\ L_i^X(R\pi\to\hat{R}_2\pi,\ \alpha,\ u)$$
$$\longrightarrow\ L_{i-1}^X(R\pi,\ \alpha,\ u)$$

and

$$L_i^Y(\hat{R}_2\pi,\ \alpha,\ u)\to H^i(Y/X)\to L_{i-1}^X(R\pi,\ \alpha,\ u).$$

Proof. The two composites are the two ways of constructing the Mayer–Vietoris boundary map for the fibre square of spectra

$$\begin{array}{ccc}
\mathbb{L}_0^X(R\pi,\ \alpha,\ u) & \longrightarrow & \mathbb{L}_0^X(\hat{R}_2\pi,\ \alpha,\ u)\\
\downarrow & & \downarrow\\
\mathbb{L}_0^Y(R\pi,\ \alpha,\ u) & \longrightarrow & \mathbb{L}_0^Y(\hat{R}_2\pi,\ \alpha,\ u)
\end{array}$$

COROLLARY 4.8. *The following diagram commutes:*

$$\begin{array}{ccc}
L_i^Y(\hat{R}_2\pi,\ \alpha,\ u)\ \xrightarrow{\ \psi_i\ }\ L_i^Y(R\pi\ \longrightarrow\ \hat{R}_2\pi,\ \alpha,\ u)\ \longrightarrow\ L_{i-1}^S(\mathbb{F}_5\pi,\ \alpha,\ u)\\
\downarrow\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\uparrow\\
\hat{H}^i(Y/X)\ \xrightarrow{\qquad\qquad\qquad r_i\qquad\qquad\qquad}\ H^i(K_1(\mathbb{F}_5\pi))
\end{array}$$

where r_i is the map induced by the obvious map

$$r\colon\ Y/X = R^\times \oplus \pi^{ab} \to K_1(\mathbb{F}_5\pi).$$

Proof. The result is clear from (4.2), (4.7), and the naturality of Rothenberg sequences.

Rather than compute the map in Tate cohomology induced by r, we will compute the map induced by

$$\hat{r}\colon\ \{\pm 1\}\oplus\pi^{ab}\subseteq R^\times\oplus\pi^{ab}\to K_1(\mathbb{F}_5\pi)\to K_1(\mathbb{F}_5\otimes\Delta_i)$$

when Δ_i, as before, is $\mathbb{Z}[\tfrac{1}{2}]$-maximal order in the division algebra associated to a Type O or Sp factor of $\mathbb{Q}\pi$.

LEMMA 4.9. *If Δ_i is non-commutative, $\hat{r}$ is trivial, and hence so is the induced map on Tate cohomology. If $\Delta_i\cong\mathbb{Z}[\tfrac{1}{2}][\zeta_{2^k}\pm\bar{\zeta}_{2^k}]$ the map induced on H^0 is trivial.*

Proof. Clearly $\hat{r}$ factors through $K_1(\Delta_i)$. If Δ_i is quaternionic, $K_1(\Delta_i)$ is isomorphic to totally positive units in the centre. Since this group is torsion-free we have finished.

In the other cases, the torsion in $K_1(\Delta_i)$ is $\{\pm 1\}$. Since $\mathbb{F}_5\otimes\Delta$ is a field,

$K_1(\mathbb{F}_5 \otimes \Delta)$ is cyclic of order $5^d - 1$ and here has order divisible by 4. Since the involutions are trivial the map induced on H^0 is trivial (and the map induced on H^1 is an isomorphism).

Finally we compute the maps $\bar{\psi}_i$ in sequence (4.1).

THEOREM 4.10. *For the standard antistructure,*

$$\bar{\psi}_i \colon L_i^{\bar{Y}}(\hat{R}_2\pi) \longrightarrow L_i^{\bar{Y}}(R\pi \to \hat{R}_2\pi)$$

is trivial for $i \equiv 0$, 1, or 2 and monic for $i \equiv 3$ (mod 4).

Proof. For the standard antistructure,

$$H^i(\{\pm 1\} \oplus \pi^{ab}) \to \hat{H}^i(R^\times \oplus \pi^{ab})$$

is surjective. From Lemmas 4.6, 4.9, and Corollary 4.8 we can compute r_i. This, together with Proposition 4.2 and Lemma 4.3, shows that $\bar{\psi}_i$ is trivial for $i \equiv 0$, 1, or 2. By Proposition 3.2 we see that $\bar{\psi}_i$ is trivial for $i \equiv 0$, 1, or 2.

If $i \equiv 3$, we can compute that for $\pi = e$ or C_2, $\bar{\psi}_3$ is an isomorphism. The 2-adic detection theorem, (3.4), and naturality complete the proof.

PROPOSITION 4.11. *For $i \equiv 0$, 1, 2, 3 (mod 4),*

(i) $L_i^{\bar{Y}}(R) \cong \mathbb{Z}/2,\ \mathbb{Z}/2,\ 0,\ 0,$

(ii) $L_i^{\bar{Y}}(R[C_2^+]) \cong (\mathbb{Z}/2)^2,\ (\mathbb{Z}/2)^3,\ \mathbb{Z}/2,\ 0.$

Proof. Apply (3.11) and (4.10).

5. *A detection theorem for $L_i^{\bar{Y}}(R\pi)$*

The goal of this section is to reduce the calculation of the groups $L_i^{\bar{Y}}(R\pi)$ for π a finite 2-group to a few special cases. First we recall from (4.5) that $L_i^{\bar{Y}}(R\pi \to \hat{R}_2\pi)$ is a direct sum of terms corresponding to the irreducible rational representations of π. We let $L_i^{\bar{Y}}(R\pi \to \hat{R}_2\pi ; \rho)$ be the factor associated to ρ.

Associated to each irreducible rational representation ρ of π there are a division algebra D_ρ and a simple factor of $\mathbb{Q}\pi$: note that $L_i^{\bar{Y}}(R\pi \to \hat{R}_2\pi ; \rho)$ depends only on D_ρ.

We also recall the relationship [9] between basic groups and the irreducible rational representations of π. Each basic group (and D_8) has a unique irreducible faithful rational representation. For each irreducible rational representation ρ of π, there is a subquotient $N \triangleleft H \subseteq \pi$ such that

(i) H/N is isomorphic to a basic group,

(ii) if ρ_0 denotes the pull-back to H of the faithful irreducible rational representation of H/N, then $\rho_0|^\pi = \rho$ and $D_\rho = D_{\rho_0}$.

Neither N nor H need be unique, but H/N is and in particular $H/N \cong \{e\}$ if and only if ρ is trivial.

Our first lemma explains why we want D_8 in addition to the basic dihedrals.

LEMMA 5.1. *If ρ is an irreducible rational representation of π such that $D_\rho \cong \mathbb{Q}$*

and such that ρ does not factor through π^{ab}, then there exists a subquotient $N \triangleleft H \subseteq \pi$ with $H/N \cong D_8$ such that $\rho_0 \vert^\pi = \rho$, where ρ_0 is the pull-back to H of the faithful irreducible rational representation of D_8.

Proof. By the above discussion we can find a subquotient $N_0 \triangleleft H_0$ with $H_0/N_0 \cong C_2$ such that $\psi \vert^\pi = \rho$ where ψ is the faithful irreducible rational representation of C_2. Since ρ does not factor through π^{ab}, $H_0 \neq \pi$ and so we can find a subgroup $H \subseteq \pi$ with H_0 index 2 in H. Furthermore, by [9], we can assume that $\psi \vert^H = \rho_0$ is irreducible.

Hence $\rho_0 \vert_{H_0} = \psi + \psi'$ where $\psi \neq \psi'$ by the Frobenius reciprocity theorem, and $\psi'(g) = \psi(tgt^{-1})$ for all $g \in H_0$ and some fixed $t \in H - H_0$. Hence $tN_0 t^{-1} \neq N_0$. We set $N = N_0 \cap tN_0 t^{-1}$ and note N has index 2 in N_0, so H/N has order 8.

Since $tN_0 t^{-1} \neq N_0$, H/N is not abelian and so is either D_8 or Q_8. Note that N_0/N, $tN_0 t^{-1}/N$, and $(H_0 \cap tH_0 t^{-1})/N$ are three distinct C_2 in H/N, so $H/N \cong D_8$.

We need one more lemma on representations.

LEMMA 5.2. *Let ρ be an irreducible rational representation of π. Suppose there is a subquotient $N \triangleleft H \subseteq \pi$, with $H/N \cong D(2^k)$, for $k \geq 3$, so that if ρ_0 is the faithful irreducible rational representation of $D(2^k)$ pulled back to H, $\rho_0 \vert^\pi = \rho$ and $D_{\rho_0} \cong D_\rho$. Let ψ be an irreducible rational representation of H which factors through $(H/N)^{ab}$. Then ψ^π and ρ have no common constituent.*

Proof. Notice that ψ has dimension 1 since $(H/N)^{ab} \cong C_2 \times C_2$, but ρ_0 has dimension greater than 1. Hence $\dim(\psi^\pi) = (\dim \psi)[\pi : H] < \dim \rho$, so the result is clear.

Our next step is to identify the needed constituents of $L_i^{\bar{Y}}(R\pi \to \hat{R}_2 \pi)$.

LEMMA 5.3. *The sequence*

$$0 \longrightarrow \sum L_{i+1}^{\bar{Y}}(R\pi \to \hat{R}_2 \pi ; \rho) \longrightarrow L_i^{\bar{Y}}(R\pi) \longrightarrow L_i^{\bar{Y}}(R\pi^{ab}) \longrightarrow 0$$

is exact where the sum runs over the irreducible rational representations of π which do not factor through π^{ab}.

Proof. From [9], we know that

$$L_{i+1}^{\bar{Y}}(R\pi \to \hat{R}_2 \pi) \cong \sum L_{i+1}^{\bar{Y}}(R\pi \to \hat{R}_2 \pi ; \rho) \oplus L_{i+1}^{\bar{Y}}(R\pi^{ab} \to \hat{R}_2 \pi^{ab}).$$

The map $L_{i+1}^{\bar{Y}}(\hat{R}_2 \pi) \to L_{i+1}^{\bar{Y}}(\hat{R}_2 \pi^{ab})$ is an isomorphism (Theorem 3.3) and the map $\bar{\psi}_{i+1} : L_{i+1}^{\bar{Y}}(\hat{R}_2 \pi) \to L_{i+1}^{\bar{Y}}(R\pi \to \hat{R}_2 \pi)$ is either monic or trivial (Theorem 4.10). The required result is a simple diagram chase.

Let $\mathcal{S}_i$ run over the subquotients $N \triangleleft H \subseteq \pi$ such that, if $i \equiv 0$ or 3 (mod 4), we have one quaternionic subquotient for each irreducible symplectic representation (and it can be induced from H/N as in the discussion preceding (5.1)). Similarly, if $i \equiv 1$ or 2 (mod 4), we take one dihedral subquotient for each irreducible orthogonal representation of degree at least 2.

THEOREM 5.4. *Let π be a finite 2-group. With the standard antistructure on $R\pi$, the map*

$$L_i^{\bar{Y}}(R\pi) \to L_i^{\bar{Y}}(R[\pi^{ab}]) \oplus \sum \{L_i^{\bar{Y}}(R[H/N]): N \lhd H \subseteq \mathscr{S}_i\}$$

is injective.

Proof. If $i \neq 2$, then $\bar{\psi}_{i+1}$ is trivial and the proof follows easily. If $i = 2$, $\bar{\psi}_3$ is monic, and indeed the composite

$$L_3^{\bar{Y}}(\hat{R}_2\pi) \to L_3^{\bar{Y}}(R\pi \to \hat{R}_2\pi) \to L_3^{\bar{Y}}(R\pi^{ab} \to \hat{R}_2\pi^{ab})$$

is an injection (see the proof of Theorem 4.10). Consider the commutative diagram:

$$
\begin{array}{ccc}
\sum L_3^{\bar{Y}}(R\pi \to \hat{R}_2\pi \; ; \rho) & \longrightarrow & L_2^{\bar{Y}}(R\pi) \\
\downarrow{\scriptstyle 1} & & \downarrow \\
0 \longrightarrow \sum L_3^{\bar{Y}}(\hat{R}_2[H/N]) \longrightarrow \sum L_3^{\bar{Y}}(R[H/N] \to \hat{R}_2[H/N]) \longrightarrow & & \sum L_2^{\bar{Y}}(R[H/N]) \\
\downarrow{\scriptstyle 2} & & \\
\sum L_3^{\bar{Y}}(R[(H/N)^{ab}] \to \hat{R}_2[(H/N)^{ab}]) & &
\end{array}
$$

where the sums run over the sub-quotients H/N in $\mathscr{S}_2$: one for each irreducible orthogonal representation ρ of π which does not factor through π^{ab}.

The middle row is exact and the vertical map 1 is injective by [9, Theorem 7.1]. Furthermore, by Lemma 5.3 and [9, Lemma 4.5(ii)], the composite of the maps 1 and 2 is trivial. The required result is now a diagram chase.

6. *Determining the $\bar{\kappa}_i$ for certain 2-groups*

To finish the proof of Theorem A, we need to compute the $\bar{\kappa}_i$. By the detection Theorem 5.4 it is enough to do this for abelian, dihedral, and quaternionic 2-groups. The results are summarized in Theorem 6.8.

First we point out two useful consequences of (1.13):

(6.1) $$\bar{\kappa}_*: H_*(C_2 \; ; \mathbb{Z}/2) \to L_*^{\bar{Y}}(R[C_2])$$

is a ring homomorphism where the (graded) ring structure on both sides is induced by the multiplication map $m: C_2 \times C_2 \to C_2$. Note that with this product, $H_*(C_2 \; ; \mathbb{Z}/2)$ is an exterior algebra on generators $a_0 = 1$ and a_i in dimensions $i = 2^l$ for $l \geq 0$.

(6.2) For any central $C_2 = \langle T \rangle$ in a 2-group π,

$$\bar{\kappa}_*: H_*(\pi \; ; \mathbb{Z}/2) \to L_*^{\bar{Y}}(R\pi)$$

is an $H_*(C_2 \; ; \mathbb{Z}/2)$-module homomorphism. The group $L_0^{\bar{Y}}(R[C_2]) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by the rank-1 forms $\langle \varepsilon \rangle$ and $\langle \varepsilon T \rangle$ acts on $L_*^{\bar{Y}}(R\pi)$ by 'scaling', so acts trivially on elements in the image of the round group $L_*^{Y}(R\pi)$.

PROPOSITION 6.3. *The non-zero $\bar{\kappa}_i$ for C_2^+ or C_2^- are given by*

(i) $\bar{\kappa}_0$ *and $\bar{\kappa}_i$ for C_2^+ when $i = 2^l$ with $l \geq 0$,*

(ii) $\bar{\kappa}_i$ for C_2^- when $i = 2^{l+2} - 2$ or $i = 2^l - 1$ with $l \geq 0$,

(iii) for C_2^+, if $l \geq 3$ then $\bar{\kappa}_{2^l}(a_{2^l}) = \bar{\kappa}_4(s_*(a_{2^l})) = \bar{\kappa}_4(a_4) = \langle \varepsilon T \rangle - \langle \varepsilon \rangle$.

Proof. This is a calculation based on the twisting diagram [8, p. 53] for the inclusion $R \to R[C_2]$:

$$L_0(R) \quad L_0(R[C_2])$$

(6.4)

The L-groups necessary for the top and bottom rows in this diagram are calculated in (3.11) and (4.11); the middle row is forced.

We start with the fact (from § 1) that

$$\bar{\kappa}_0(a_0) = \langle \varepsilon \rangle \in \mathrm{Im}\{L_0^{\bar{Y}}(R) \to L_0^{\bar{Y}}(R[C_2^\pm])\}$$

is non-zero and deduce the others from (6.4) and the following diagrams:

(6.5)
$$
\begin{array}{ccc}
H_i(C_2; \mathbb{Z}/2) & \xrightarrow{\bar{\kappa}_i} & L_i^{\bar{Y}}(R[C_2^+]) \\
{\scriptstyle \cap \phi} \downarrow & & \searrow \\
H_{i-1}(C_2; \mathbb{Z}/2) & \xrightarrow{\bar{\kappa}_{i-1}} L_{i-1}^{\bar{Y}}(R[C_2^-]) & \longrightarrow L_i^{\bar{Y}}(R \to R[C_2^+])
\end{array}
$$

Notice that the L-groups (and the maps between them) in (6.5) also appear in (6.4). The map $\cap \phi$ is the cap product with the non-trivial class ϕ in $H^1(C_2; \mathbb{Z}/2)$. The second diagram is similar:

(6.6)
$$
\begin{array}{ccc}
H_i(C_2; \mathbb{Z}/2) & \xrightarrow{\bar{\kappa}_i} & L_i^{\bar{Y}}(R[C_2^-]) \\
\downarrow & & \searrow \\
\sum\limits_{l \geq 0} H_{i-1-(2^{l+2}-4)}(C_2; \mathbb{Z}/2) & \xrightarrow{\bar{\kappa}_*} L_{i-1}^{\bar{Y}}(R[C_2^+]) & \longrightarrow L_i^{\bar{Y}}(R \to R[C_2^-])
\end{array}
$$

where the left vertical map is the cap product with $\sum \cap (\phi^{2^{l+2}-3})$, so the surgery obstructions spread over many lower components in this case.

To derive these diagrams we combine the cohomological formula for the surgery obstruction with a homotopy commutative diagram of spectra:

$$
\begin{array}{ccc}
B\pi^w \wedge \mathbb{L}^0 & \xrightarrow{a_{\pi,w}} & \mathbb{L}^0(\mathbb{Z}\pi, w) \\
{\scriptstyle t' \wedge 1} \nearrow & & \downarrow \\
\sum B\pi^{w\phi} \wedge \mathbb{L}^0 & & \\
{\scriptstyle \sum a_{\pi,w\phi}} \searrow & & \downarrow \\
\sum \mathbb{L}^0(\mathbb{Z}\pi, w\phi) & \xrightarrow{t} & \mathbb{L}^0(\mathbb{Z}\rho \to \mathbb{Z}\pi, w)
\end{array}
$$

relating the pre-assembly maps (1.1) with the transfer maps induced by the line bundle associated to an index 2 subgroup $\rho \subseteq \pi$ (the map t is defined as in [**26**, p. 694]). On homotopy groups this diagram just expresses the relationship between the symmetric signature of a closed manifold and a codimension-1 submanifold.

With the aid of these diagrams it is easy to verify the results for $\bar{\kappa}_i$ if $0 \le i \le 4$, and then prove the rest by induction. For (6.3)(iii) note that since $\bar{\kappa}_4$ is non-trivial it must be in the kernel of the projection

$$L_0^{\bar{Y}}(R[C_2]) \to L_0^{\bar{Y}}(R).$$

The results for π cyclic (the oriented case only will be considered from now on) follow directly from

PROPOSITION 6.7. *If π is cyclic, then for both $H_*(\pi \,; \mathbb{Z}/2)$ and $L_*^{\bar{Y}}(R\pi)$ the inclusion $C_2 \to \pi$ induces an isomorphism in even dimensions and the projection $\pi \to C_2$ induces an isomorphism in odd dimensions.*

Proof. For homology this is well-known and for L-theory it is an easy exercise using § 4.

Now consider abelian groups. In this case $L_i^{\bar{Y}}(R\pi)$ is detected by a set of cyclic quotients for $i \equiv 2 \pmod 4$, that is, the result of (3.4) holds for $R\pi$. For $\pi = C_2 \times C_2$ the possible non-zero products into L_2 are restricted by (1.13), (6.2), and (6.3). In fact since

$$\bar{\kappa}_{i+4}(a_i \otimes a_4) = \bar{\kappa}_i(a_i) \otimes \bar{\kappa}_4(a_4) = \bar{\kappa}_i(a_i) \otimes (\langle \varepsilon T \rangle - \langle \varepsilon \rangle)$$

and the images of positive-dimensional $\bar{\kappa}_i$ are in the round L-groups, these products are all zero. The remaining possibly non-zero product is $\bar{\kappa}_2(a_1 \otimes a_1)$. The non-triviality of the Morgan–Pardon example forces the product to be non-zero (this can easily be checked using another twisting diagram).

Since the mod 2 homology of a dihedral group is generated by that of its elementary abelian subgroups [**22**], we only need to remark that, if $C_2 \to D(2^k)$ is the inclusion of the centre, then the induced composite

$$H_2(C_2 \,; \mathbb{Z}/2) \longrightarrow H_2(D(2^k) \,; \mathbb{Z}/2) \xrightarrow{\bar{\kappa}_i} L_2^{\bar{Y}}(R[D(2^k)])$$

is non-trivial.

A quaternion group π has centre $C_2 = \langle T \rangle$ and $H_{4i}(C_2 \,; \mathbb{Z}/2) \to H_{4i}(\pi \,; \mathbb{Z}/2)$ is an isomorphism. Furthermore, the product

$$H_{4i}(C_2 \,; \mathbb{Z}/2) \times H_j(\pi \,; \mathbb{Z}/2) \to H_{4i+j}(\pi \,; \mathbb{Z}/2)$$

is surjective for all $i \ge 0$ and $j = 1$, 2, or 3. Therefore if $i > 0$ and $4i + j \ne 2^l$, the homology classes in the domain of $\bar{\kappa}_i$ are products involving the generator of $H_{4i}(C_2 \,; \mathbb{Z}/2)$. We apply (6.2), (6.3)(iii), and note that again the images of $\bar{\kappa}_j$ for $1 \le j \le 3$ lie in the round group $L_j^Y(R\pi)$, so the product with $\langle \varepsilon T \rangle - \langle \varepsilon \rangle$ is zero. To prove that $\bar{\kappa}_{2^l} \ne 0$ for the quaternion groups, we check (via a twisting diagram) that the map induced by the inclusion

$$L_0^{\bar{Y}}(R[C_2]) \to L_0^{\bar{Y}}(R[Q(2^k)])$$

is injective.

The above results give

THEOREM 6.8. *Let π be an abelian, dihedral, or quaternion 2-group. Then*

(i) $\bar{\kappa}_1$ *and* $\bar{\kappa}_2$ *are injective for all such* π,

(ii) $\bar{\kappa}_3$ *is injective for* π *quaternion and zero for* π *abelian or dihedral,*

(iii) $\bar{\kappa}_{2^{r+2}}$ *is injective for* $r \geqslant 0$ *if* π *is cyclic or quaternion, factors through cyclic quotients for* π *abelian, and factors through* $\pi^{ab} = C_2 \times C_2$ *for* π *dihedral,*

(iv) $\bar{\kappa}_i = 0$ *for* $i > 4$, *when* $i \neq 2^{r+2}$,

(v) $\bar{\kappa}_{2^{r+2}} = \bar{\kappa}_4 \circ s_r$ *for* $r \geqslant 0$ *and* $\bar{\kappa}_i$ *vanishes on the image of integral homology for* $i \geqslant 4$.

7. *Final remarks on the* κ_i

We begin by determining the image of $\kappa_3^{\bar{Y}}$. For $\pi = Q(2^k)$ the answer is in [**3**]: $H_3(\pi ; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and the image of $\kappa_3^{\bar{Y}}$ is the unique $\mathbb{Z}/2$ in $L_1'(\mathbb{Z}\pi)$ which vanishes in $L_1^p(\mathbb{Z}\pi)$.

THEOREM 7.1. *The map* $\kappa_3^{\bar{Y}}$ *factors as the composite*

$$ H_3(\pi ; \mathbb{Z}/2) \longrightarrow \bigoplus_{\rho_0 \lhd \rho \in \mathscr{S}_3} H_3(\rho/\rho_0 ; \mathbb{Z}/2) \xrightarrow{\lambda} L_1^{\bar{Y}}(\mathbb{Z}\pi), $$

where λ *is a natural monomorphism defined in the proof.*

Proof. We let $\lambda = \mathrm{trf}_3 \circ \bar{\lambda}$ and define $\bar{\lambda}$. By the detection Theorem 5.4,

$$ \begin{array}{ccc} H_3(\pi ; \mathbb{Z}/2) & \xrightarrow{\bar{\kappa}_3} & L_3^{\bar{Y}}(R\pi) \\ \downarrow & & \downarrow \\ \oplus H_3(\rho/\rho_0 ; \mathbb{Z}/2) & \longrightarrow & \oplus L_3^{\bar{Y}}(R[\rho/\rho_0]) \end{array} $$

commutes and determines $\bar{\kappa}_3$. From §6 we see that the image of $\bar{\kappa}_3$ for a quaternion group is the unique element in the kernel of

$$ L_3^{\bar{Y}}(R[Q(2^k)]) \to L_3^{\bar{Y}}(\hat{R}_2[Q(2^k)]). $$

Hence it comes from a unique element in the relative group. The map

$$ L_0^{\bar{Y}}(R\pi \to \hat{R}_2\pi) \longrightarrow \oplus L_0^{\bar{Y}}(R[\rho/\rho_0] \to \hat{R}_2[\rho/\rho_0]) $$

is naturally split so we have finished.

Finally, we remark that the coboundary map δ in the Tate cohomology sequence arising from ([**19**], compare with (3.13))

$$ 0 \to Wh'(\hat{\mathbb{Z}}_2\pi) \to \overline{I(\hat{\mathbb{Z}}_2\pi)} \to \pi^{ab} \to 0 $$

can be used to describe the '2-adic' κ_2 and κ_4. More precisely, the composite

$$ H_2(\pi ; \mathbb{Z}/2) \xrightarrow{\bar{\kappa}_2} L_2^{\bar{Y}}(R\pi) \longrightarrow L_2^{\bar{Y}}(\hat{R}_2\pi) \cong H^0(\pi^{ab}) $$

fits into the following commutative diagrams:

(7.2)

$$
\begin{array}{ccccc}
H_2(\pi\,;\mathbb{Z}/2) & \xrightarrow{\ \beta\ } & H^0(\pi^{ab}) & \xrightarrow{\ \delta\ } & H^1(Wh'(\hat{\mathbb{Z}}_2\pi)) \\
\downarrow{\scriptstyle \kappa_2} & & & & \downarrow \\
L_0'(\mathbb{Z}\pi) & & \xrightarrow{\hspace{4cm}} & & L_0'(\hat{\mathbb{Z}}_2\pi)
\end{array}
$$

and

(7.3)

$$
\begin{array}{ccccccc}
H_4(\pi\,;\mathbb{Z}/2) & \xrightarrow{\ s_*\ } & H_2(\pi\,;\mathbb{Z}/2) & \xrightarrow{\ \beta\ } & H^0(\pi^{ab}) & \xrightarrow{\ \delta\ } & H^1(Wh'(\hat{\mathbb{Z}}_2\pi)) \\
{\scriptstyle \kappa_4}\searrow & & & & & \swarrow & \\
& L_2'(\mathbb{Z}\pi) & & \xrightarrow{\hspace{3cm}} & & L_2'(\hat{\mathbb{Z}}_2\pi) &
\end{array}
$$

Note that β may be identified with the integral Bockstein, and δ is explicit since

$$
H^1(Wh'(\hat{\mathbb{Z}}_2\pi)) \cong \frac{\{\langle g\rangle \in \pi^{ab}|\ \langle g\rangle^2 = 1\}}{\{\langle g\rangle|\ g\sim g^{-1}\}}
$$

so that δ is the natural projection map from

$$
H^0(\pi^{ab}) \cong \{\langle g\rangle \in \pi^{ab}|\ \langle g\rangle^2 = 1\}.
$$

Since $H^1(Wh'(\hat{\mathbb{Z}}_2\pi))$ injects into L_0' or L_2', this shows that κ_2 is non-zero for π whenever $H^1(Wh'(\hat{\mathbb{Z}}_2\pi))$ is non-zero. This also proves Theorem B.

PROPOSITION 7.4. *If π is cyclic or quaternion, then κ_2^s and κ_4^s are zero.*

Proof. Since $L^s = L'$ for π cyclic or quaternion [20], the statement for κ_*^s follows from the one for κ_*'. Since the torsion in $L'_{2n}(\mathbb{Z}\pi)$ is detected by transfer to the trivial group [34, 3.3.3 and 5.2.4], the result follows.

Appendix. The assembly map

We are going to outline the proof that the surgery obstruction map factors through a spectrum level assembly map. Factorizations of this sort were first due to Quinn [23] and later reformulated in a more algebraic manner by Ranicki. The proof is essentially contained in Ranicki's paper [24] and his book [26]. We will give a detailed outline of the necessary results as a guide to the reader of these sources.

Let X be an n-dimensional geometric Poincaré complex with first Stiefel–Whitney class denoted by w and let $T(X, w)$ denote the set of surgery problems into (X, w). If this set is non-empty, we have the surgery obstruction map

(A.1) $\lambda\colon T(X, w)\to L_n(\mathbb{Z}\pi_1(X), w)$

described in the Introduction. For convenience we suppose that ∂X is empty.

We first summarize some of the ideas in the algebraic theory of surgery. Ranicki's notion of bordism of algebraic symmetric and quadratic Poincaré complexes [26] can be expanded in the standard way to give simplicial sets and even spectra. A convenient reference for this process is Nicas's thesis [17] where the Quinn spectra are described in detail and the following result is proved.

THEOREM 1 (Quinn). *For every pair* (π, w) *consisting of a group with an orientation there are spectra* $\mathbb{L}_0(\mathbb{Z}\pi, w)$ *whose homotopy groups are the geometric surgery obstruction groups appearing in* (A.1).

The construction of algebraic L-spectra by Ranicki and the proof of the analogous result proceeds in a similar way. The simplices are modelled on algebraic quadratic Poincaré n-ads [24, p. 285] and the main point is to check that the resulting simplicial spaces satisfy the Kan extension condition. This is implied by a 'glueing' construction for a collection of n-ads along compatible boundary pieces (generalizing [26, § 1.7]). After verifying this condition, it follows that the homotopy group in dimension n is just the bordism group of closed Poincaré n-complexes modulo homotopy equivalence. This was shown to be the geometric L-group in [25].

More generally, quadratic L-spectra are defined for any ring with anti-structure (and this is the main advantage of the algebraic theory for our purposes). These spectra have pairings which reflect the usual tensor product pairings of complexes.

In § 1 we introduced some notation and stated the Quinn–Ranicki factorization for the map σ_0. By definition (0.4) this map is the composite

$$[X, G/TOP] \xrightarrow{\ v\ } T(X, w) \xrightarrow{\ \lambda_0\ } L_n(\mathbb{Z}[\pi_1(X)], w),$$

where v is the usual action map of $[X, G/TOP]$ on a base-point $(h_0, \hat{h}_0)$ for $T(X, w)$, and

$$\lambda_0(h_1, \hat{h}_1) = \lambda(h_1, \hat{h}_1) - \lambda_0(h_0, \hat{h}_0).$$

Therefore the factorization is given by the following result.

THEOREM 2. *Fix a base point* $(h_0, \hat{h}_0)$ *in* $T(X, w)$. *The following diagram commutes:*

$$
\begin{array}{ccc}
[X, G/TOP] & \xrightarrow{\ \gamma_0\ } & H_n(x^w ; \mathbb{L}_0) \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle A_{\pi,w} \circ c_*} \\
T(X, w) & \xrightarrow{\ \lambda_0\ } & L_n(\mathbb{Z}\pi_1 X, w)
\end{array}
$$

Proof. The proof will be broken up for convenience into several steps. The key results we use are that Ranicki has identified the surgery obstruction as the bordism class of an algebraic quadratic Poincaré complex, and that his chain complex constructions are natural.

Step 1. The following diagram commutes:

$$
\begin{array}{ccc}
T(X, w) & \xrightarrow{\ c_0\ } & \Omega_{n+1}^{N,P}(X, w) \\
& {\scriptstyle \lambda_0}\searrow & \downarrow{\scriptstyle \sigma_*} \\
& & L_n(\mathbb{Z}\pi_1 X, w)
\end{array}
$$

where $\Omega^{N,P}(X^w)$ denotes the (normal space, Poincaré) bordism group of (X, w).

The map γ_0 sends $(h_1, \hat{h}_1)$ to $(W_1 \cup_X -W_0, M_1 \cup_X -M_0)$, where $h_i\colon M_i \to X$ and W_i is the mapping cylinder of h_i. The map σ_* is defined on p. 618 of [26, Proposition 7.4.1]. If $(h, \hat{h})\colon M \to X$ is a surgery problem with mapping cylinder W, then

$$\lambda(h, \hat{h}) = \sigma_*(W, M \cup -X) \in L_n(\mathbb{Z}\pi, w).$$

This follows from the commutativity of the following diagram:

$$
\begin{array}{ccc}
T\pi(\nu_W)^* & \xrightarrow{\ G\ } & \Sigma^\infty(\bar{W}/\bar{M} \cup -\bar{X}) \\
\Big\downarrow{\scriptstyle\approx} & & \Big\downarrow{\scriptstyle\approx} \\
\Sigma^{\infty+1}(\bar{X}_+) & \xrightarrow{\ F\ } & \Sigma^{\infty+1}(\bar{M})
\end{array}
$$

where G is from [26, p. 621] and F is from [26, p. 35].

Step 2. Next we define a map

$$[X, G/TOP] \xrightarrow{\ \Gamma_0\ } H_{n+1}(X^w ; \Omega^{N,P})$$

following [24, pp. 293–296]. On p. 294, Ranicki defines

$$\Gamma\colon G/TOP \to \Omega^\infty MS(G/TOP)$$

and we consider the composite with Poincaré duality

$$[X, G/TOP] \xrightarrow{\ \Gamma\ } [X, \Omega^\infty MS(G/TOP)] = H^0(X ; MS(G/TOP))$$

$$\cong H_n(X^w ; MS(G/TOP)),$$

using the *MSTOP*-module structure and the *MSTOP*-orientation class coming from $\hat{h}_0$. To define the map Γ_0, use the fact that

$$MS(G/TOP) = \Sigma^{-1}\Omega^{N,STOP},$$

and compose with the natural map

$$H_n(X^w ; MS(G/TOP)) \cong H_{n+1}(X^w ; \Omega^{N,STOP}) \to H_{n+1}(X^w ; \Omega^{N,P}).$$

Step 3. The following diagram commutes:

$$
\begin{array}{ccc}
[X, G/TOP] & \xrightarrow{\ \Gamma_0\ } & H_{n+1}(X^w ; \Omega^{N,P}) \\
\Big\downarrow{\scriptstyle\upsilon} & & \Big\downarrow{\scriptstyle\tau_*} \\
T(X, w) & \xrightarrow{\ c_0\ } & \Omega^{N,P}(X^w)
\end{array}
$$

where τ_* is induced by the following composite of maps of spectra:

$$X^w \wedge \Omega^{N,P} \to \Omega^P(X^w) \wedge \Omega^{N,P} \to \Omega^{N,P}(X^w),$$

and the last map is the map induced on the spectrum level by the obvious Cartesian product on the simplices. For a more detailed discussion of how to do such products, consult Step 4. The proof of the commutativity finishes on the bottom of p. 296 of [24].

Step 4. Compatibility of the geometric and algebraic products. The following diagram commutes:

$$\begin{CD}
\Omega^P(X^w) \wedge \Omega^{N,P}(Y^v) @>>> \mathbb{L}^0(\mathbb{Z}[\pi_1 X], w) \wedge \Sigma\mathbb{L}_0(\mathbb{Z}[\pi_1 Y], v) \\
@VVV @VVV \\
\Omega^{N,P}(X^w \wedge Y^v) @>>> \Sigma\mathbb{L}_0(\mathbb{Z}[\pi_1(X \times Y)], w \times v)
\end{CD}$$

There is an analogous Ω^P to $\mathbb{L}^0$-theory diagram which also commutes. The horizontal maps are just the spectrum level maps induced by taking the underlying symmetric (respectively hyper-quadratic, symmetric) chain complex on each simplex. The vertical maps are given by products.

Proof of Step 4. The maps are discussed in [**24**, p. 287]. The only point to clarify is how to do the products. On the simplicial set level we can take the Cartesian product of simplices. This defines a bi-simplicial set.

We fatten up $\Omega^{N,P}(X^w \wedge Y^v)$ and $\mathbb{L}_0(\mathbb{Z}[\pi_1(X \times Y)], w \times v)$ to be bi-simplicial by taking spaces or complexes over $\Delta^k \times \Delta^m$. The bi-simplicial version of the diagram commutes on the simplex level. Then we take the diagonal simplicial set. It is a well-known result that the diagonal of the bi-simplicial smash product is homotopy equivalent to the smash product.

The only point left to consider is why the diagonal of the *bi-simplicial* version of $\Omega^{N,P}(X^w \wedge Y^v)$ or $\mathbb{L}_0(\mathbb{Z}[\pi_1(X \times Y)], w \times v)$ is homotopy equivalent to the original simplicial version. Since we have mapping cylinders in both algebra and geometry, this is a standard argument. We have lifted this proof from some unpublished notes of Ranicki.

REMARK. The above diagram with P replaced by *STOP* also commutes. The left-hand vertical map is then surjective, which shows that any surgery problem can be decomposed into a sum of products. Furthermore, the surgery obstructions for products are given algebraically as products, so Theorem 2 can be viewed as reducing the general surgery problem to products.

Step 5. The final result follows by combining the diagram in Step 3 with π_{n+1} applied to the following commutative diagram:

$$\begin{CD}
X^w \wedge \Omega^{N,P} @>>> X^w \wedge \Sigma\mathbb{L}_0 \\
@VVV @VVV \\
\Omega^{N,P}(X^w) \wedge \Omega^{N,P} @>>> \mathbb{L}^0(\mathbb{Z}[\pi_1 X], w) \wedge \Sigma\mathbb{L}_0 \\
@VVV @VVV \\
\Omega^{N,P} @>>> \Sigma\mathbb{L}^0(\mathbb{Z}[\pi_1 X], w)
\end{CD}$$

(see Step 4).

REMARK. We have not been very careful in our notation to keep track of Whitehead torsions, but if the Poincaré spaces are simple, our L spectra are the

simple ones. If our Poincaré spaces are finite and have their torsions in some fixed subgroup of the Whitehead group, then Theorem 2 is still true if our L spectra are the ones with torsions in this subgroup.

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