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Generalized splitting theorems

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In (5), we and Fred Cohen gave some quite general splitting theorems. These described how to decompose the suspension spectra of certain filtered spaces $C\mathbf{X}$ as wedges of the suspension spectra of their successive filtration quotients $D_q\mathbf{X}$. The spaces $C\mathbf{X}$ were of the form $C_r \times X_r/(\sim)$ for suitable sequences of spaces $\{C_r\}$ and $\{X_r\}$, and the construction $C\mathbf{X}$ was intended to be a reworking in 'proper generality' of the constructions introduced in (9).

We no longer believe that sufficient generality was achieved in (5). The original motivation concerned iterated loop spaces $\Omega^n\Sigma^n X$, where X is path connected. Consider a cofibration $i: A \to X$, where A is also connected. Let $\partial: X/A \overset{\simeq}{\leftarrow} C(i) \to \Sigma A$ be the standard map and consider the fibre $F_n(X, A)$ of $\Omega^{n-1}\Sigma^{n-1}(\partial)$ $(n \geqslant 1)$. By a slight elaboration of the argument in (9), §6, which gives complete details when X is the cone on A, there is a commutative diagram

$$\begin{array}{ccccc} C_nA & \longrightarrow & E_n(X,A) & \longrightarrow & C_{n-1}(X/A) \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \tilde{\alpha}_n} & & \downarrow{\scriptstyle \alpha_{n-1}} \\ \Omega^n\Sigma^nA & \longrightarrow & F_n(X,A) & \longrightarrow & \Omega^{n-1}\Sigma^{n-1}(X/A) \end{array}$$

where the bottom row is the canonical fibration. By (9), 7·3, the top row is a quasifibration. Since α_{n-1} and α_n are weak equivalences, by (9), 6·1, $\tilde{\alpha}_n$ is also a weak equivalence. As pointed out to us by Joe Neisendorfer, the methods of (5) can be generalized to obtain a stable splitting of the filtered space $E_n(X, A)$. However, $E_n(X, A)$ is not of the form $C\mathbf{X}$, but rather of the form $Y_r/(\sim)$, where $\{Y_r\}$ is a suitable sequence of spaces which do not split as products.

A quite similar relative James construction $M(X, A)$ was introduced by Gray (7). By (7), 2·13, this fits into a commutative diagram

$$\begin{array}{ccccc} MA & \longrightarrow & M(X,A) & \longrightarrow & X/A \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tilde{\beta}} & & \| \\ \Omega\Sigma A & \longrightarrow & F_1(\partial) & \longrightarrow & X/A \end{array}$$

By (7), §2 or (9), p. 59, the top row is a quasifibration, hence $\tilde{\beta}$ is a weak equivalence since β is a weak equivalence. Again, $\Sigma M(X, A)$ splits, and the methods of (5) apply after appropriate generalization.

We introduce 'Λ-arrays' $\mathbf{Y}$ in section 1 and a coalescence functor Γ from Λ-arrays to spaces in section 2. We give a general procedure for constructing combinatorial maps $\Gamma\mathbf{Y} \to C\mathbf{X}$ in section 3 and generalize the splitting theorems of (5) in section 4. The general constructions are probably of greater interest than the given applications. In particular, our methods lead to many more natural combinatorial maps relating function spaces than have yet been exploited. The idea is so simple and intuitive that the reader is quite likely to see his own quite different applications. It is by now apparent that the use of combinatorial approximations of function spaces is one of the most powerful tools in the homotopy theorist's kit. Our new combinatorial spaces and maps are bound to increase the range and flexibility of this tool.

We wish to express our deep thanks to Fred Cohen and Joe Neisendorfer for their ideas and stimulation. It is only at their insistence that they are not listed as co-authors of this paper.

An appendix corrects the cofibration conditions in (5), (6) and (10).

1. *Λ-arrays*

Let $\mathscr{U}$ be the category of compactly generated weak Hausdorff spaces and $\mathscr{T}$ the category of non-degenerately based spaces in $\mathscr{U}$.

Recall that Λ denotes the category of finite based sets $\mathbf{n} = \{0, 1, \ldots, n\}$ and based injections; the basepoint of $\mathbf{n}$ is 0. We need an observation and a bit of notation before we can define our main objects of study.

LEMMA 1·1. *The category Λ has pullbacks. A square*

$$\begin{array}{ccc} \mathbf{p} & \longrightarrow & \mathbf{s} \\ \downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbf{r} & \xrightarrow{\phi} & \mathbf{t} \end{array}$$

is a pullback if and only if $p = |\mathrm{Im}\,\phi \cap \mathrm{Im}\,\psi - \{0\}|$.

Proof. Given $\phi: \mathbf{r} \to \mathbf{t}$ and $\psi: \mathbf{s} \to t$, consider the set

$$\pi = \{(a,b) | \phi(a) = \psi(b)\} \subset \mathbf{r} \times \mathbf{s}.$$

Order its elements, starting with 0th element $(0,0)$. The ordering specifies a bijection $\mathbf{p} \to \pi$, where p is the cardinality of $\pi - \{(0,0)\}$, and the projections $\pi \to \mathbf{r}$ and $\pi \to \mathbf{s}$ induce projections displaying $\mathbf{p}$ as the required pullback in Λ. The last statement should now be clear.

Notations 1·2. For $\phi: \mathbf{r} \to \mathbf{s}$ in Λ, let $\Sigma_\phi \subset \Sigma_s$ denote the sub-group consisting of those permutations τ such that $\tau(b) \in \mathrm{Im}\,\phi$ if $b \in \mathrm{Im}\,\phi$. Such a τ satisfies $\tau\phi = \phi\sigma$ for a uniquely determined $\sigma \in \Sigma_r$, and $\tau \to \sigma$ specifies a homomorphism $\Sigma_\phi \to \Sigma_r$.

Definition 1·3. A Λ-array $\mathbf{Y}$ is a collection of unbased spaces Y_ϕ indexed on the morphisms ϕ of Λ together with maps

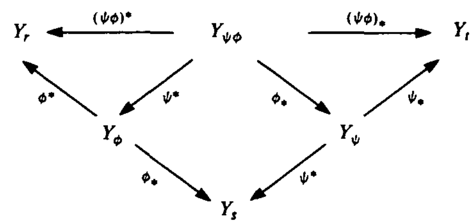
$$Y_\phi \xleftarrow{\psi^*} Y_{\psi\phi} \xrightarrow{\phi_*} Y_\psi$$

for each composable pair of morphisms (ϕ, ψ) such that the following properties are satisfied. Write Y_r for the space indexed on the identity morphism of $\mathbf{r}$ and note that, for $\phi: \mathbf{r} \to \mathbf{s}$, we are given maps

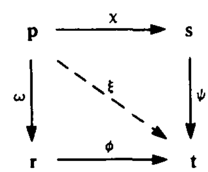
$$Y_r \xleftarrow{\phi^*} Y_\phi \xrightarrow{\phi_*} Y_s.$$

(1) If ϕ (resp ψ) is an isomorphism (that is, a permutation), then ϕ_* (resp ψ^*) is a homeomorphism; if ϕ (resp ψ) is an identity morphism, then ϕ_* (resp ψ^*) is an identity map.

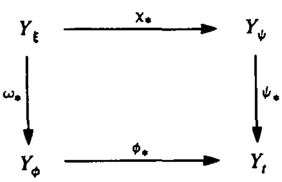
(2) For $\phi: \mathbf{r} \to \mathbf{s}$ and $\psi: \mathbf{s} \to \mathbf{t}$, the following diagram commutes and its square is a pullback:



(3) If the square (with common composite denoted ξ)



is a pullback in Λ, then the following square is a pullback:



(4) For $\phi: \mathbf{r} \to \mathbf{s}$, the map $\phi_*: Y_\phi \to Y_s$ is a Σ_ϕ-cofibration. Here, for $\tau \in \Sigma_\phi$ with $\tau\phi = \phi\sigma$, the action of τ on Y_ϕ is specified to be the composite

$$Y_\phi \xrightarrow{(\tau^*)^{-1}} Y_{\tau\phi} = Y_{\phi\sigma} \xrightarrow{\sigma_*} Y_\phi;$$

an action of Σ_s on Y_s is obtained by specialization to the identity morphism of $\mathbf{s}$, and ϕ_* is necessarily Σ_ϕ-equivariant.

A Λ-array is said to be Σ-free if each Y_s is Σ_s-free; it follows that each Y_ϕ is Σ_ϕ-free.

Those who prefer a more categorical description are invited to contemplate the relationship between this definition and exercise 3 of (8), p. 223.

Since we shall make heavy use of them, we recall the following notions from (5).

Definition 1.4. A coefficient system is a contravariant functor $\mathscr{C}: \Lambda \to \mathscr{U}$ such that $\mathscr{C}_0$ is a point. A Λ-space is a covariant functor $\mathbf{X}: \Lambda \to \mathscr{U}$ which preserves pullbacks and is such that X_0 is a point and each $\phi: X_r \to X_s$ is a Σ_ϕ-cofibration. Since $\mathbf{0}$ is an initial object in Λ, it follows that X takes values in $\mathscr{T}$.

In (5) we erroneously omitted the pullback condition (which is automatic for Π-spaces, to which we usually restricted ourselves) and misdefined Σ_ϕ; see the appendix for discussion. The standard example of a Λ-space is $\{X^r\}$ for a based space X. The standard example of a coefficient system is $\mathscr{C}(Z) = \{F(Z, r)\}$ for an unbased space Z, where $F(Z, r)$ is the configuration space of ordered r-tuples of distinct points of Z. Compare (5), 1·6 and 1·9. The following example makes it clear that the notion of a Λ-array really does generalize the context of (5).

Example 1·5. Let $\mathscr{C}$ be a coefficient system and let $\mathbf{X}$ be a Λ-space. Define a Λ-array $\mathbf{Y} = \mathbf{Y}(\mathscr{C}, \mathbf{X})$ to consist of the spaces $Y_\phi = \mathscr{C}_s \times X_r$ for $\phi: \mathbf{r} \to \mathbf{s}$ and the following maps for $\psi: \mathbf{s} \to \mathbf{t}$:

$$\mathscr{C}_s \times X_r \xleftarrow{\psi\times 1} \mathscr{C}_t \times X_r \xrightarrow{1\times\phi} \mathscr{C}_t \times X_s.$$

The pullback square postulated in (2) is just

$$\begin{array}{ccccc} & & \mathscr{C}_t \times X_r & & \\ & \swarrow{\scriptstyle \psi\times 1} & & \searrow{\scriptstyle 1\times\phi} & \\ \mathscr{C}_s \times X_r & & & & \mathscr{C}_t \times X_s \\ & \searrow{\scriptstyle 1\times\phi} & & \swarrow{\scriptstyle \psi\times 1} & \\ & & \mathscr{C}_s \times X_s & & \end{array}$$

The pullback square postulated in (3) is the product of $\mathscr{C}_t$ with a pullback square given by our assumption that $\mathbf{X}$ preserves pullbacks. We write $\mathbf{Y}(\mathscr{C}, X)$ when $\mathbf{X} = \{X^r\}$ for a based space X. We write $\mathbf{Y}(Z, \mathbf{X})$ when $\mathscr{C} = \mathscr{C}(Z)$ for an unbased space Z.

We give several procedures for generating further examples.

Examples 1·6. Let $\mathbf{Y}$ be a Λ-array.

(i) If $X \in \mathscr{U}$, define a Λ-array $\mathbf{Y} \times X$ with

$$(\mathbf{Y} \times X)_\phi = Y_\phi \times X$$

and with the evident maps.

(ii) If $X \in \mathscr{T}$, define a Λ-array $\mathbf{Y}^+ \wedge X$ with

$$(\mathbf{Y}^+ \wedge X)_\phi = Y_\phi^+ \wedge X = Y_\phi \times X / Y_\phi \times \{*\}.$$

(iii) If $\mathscr{C}$ is a coefficient system, define a Λ-array $\mathscr{C} \times \mathbf{Y}$ with

$$(\mathscr{C} \times \mathbf{Y})_\phi = \mathscr{C}_s \times Y_\phi \quad \text{for} \quad \phi: \mathbf{r} \to \mathbf{s}$$

and with maps (for $\psi: \mathbf{s} \to \mathbf{t}$)

$$\mathscr{C}_s \times Y_\phi \xleftarrow{\psi\times\psi^*} \mathscr{C}_t \times Y_{\psi\phi} \xrightarrow{1\times\phi_*} \mathscr{C}_t \times Y_\psi.$$

(iv) By symmetry, if $\mathbf{X}$ is a Λ-space, define a Λ-array $\mathbf{Y} \times \mathbf{X}$ with

$$(\mathbf{Y} \times \mathbf{X})_\phi = Y_\phi \times X_r \quad \text{for} \quad \phi: \mathbf{r} \to \mathbf{s}$$

and define a quotient Λ-array $\mathbf{Y}^+ \wedge \mathbf{X}$ with

$$(\mathbf{Y}^+ \wedge \mathbf{X})_\phi = Y_\phi^+ \wedge X_r = Y_\phi \times X_r / Y_\phi \times \{*\}.$$

Definition 1.7. Let $\mathbf{A}$ and $\mathbf{Y}$ be Λ-arrays. We say that $\mathbf{A}$ is a subarray of $\mathbf{Y}$ if there is a map $i: \mathbf{A} \to \mathbf{Y}$ of Λ-arrays such that each $i_\phi: A_\phi \to \mathbf{Y}_\phi$ is a closed inclusion. Note that the square of (2) for $\mathbf{A}$ will be a pullback provided that both of the following squares are pullbacks:

$$\begin{array}{ccccc} A_\phi & \xleftarrow{\psi^*} & A_{\psi\phi} & \xrightarrow{\phi_*} & A_\psi \\ \downarrow{\scriptstyle i_\phi} & & \downarrow{\scriptstyle i_{\psi\phi}} & & \downarrow{\scriptstyle i_\psi} \\ Y_\phi & \xleftarrow{\psi^*} & Y_{\psi\phi} & \xrightarrow{\phi_*} & Y_\psi \end{array}$$

Similarly, the square of (3) for $\mathbf{A}$ will be a pullback provided that the right square just displayed is a pullback for all (ϕ, ψ).

Subarrays of appropriate $\mathbf{Y}(\mathscr{C}, X)$ appear naturally in the study of iterated loop spaces, where they lead to the relative approximations discussed in the introduction.

Example 1·8. Let $\mathscr{C}_n$ be the little n-cubes operad ((9), p. 30) and let (X, A) be an NDR-pair in $\mathscr{T}$, with $* \in A$. Define a subarray $\mathbf{Y} = \mathbf{Y}(\mathscr{C}_n, X, A)$ of $\mathbf{Y}(\mathscr{C}_n, X)$ as follows. For $\phi: \mathbf{r} \to \mathbf{s}$, Y_ϕ consists of those points

$$(\langle c_1, \ldots, c_s \rangle, x_1, \ldots, x_r) \in \mathscr{C}_{n,s} \times X^r$$

such that whenever $x_i \notin A$, the intersection in J^n, $J = (0, 1)$, of the sets

$$(c_j'(0), 1) \times c_j''(J^{n-1}) \quad \text{and} \quad c_k(J^n)$$

is empty for all $k \neq j$, where $j = \phi(i)$ and where $c_j = c_j' \times c_j'': J \times J^{n-1} \to J^n$. In fact, we have inclusions of subarrays

$$\mathbf{Y}(\mathscr{C}_n, A) \subset \mathbf{Y}(\mathscr{C}_n, X, A) \subset \mathbf{Y}(\mathscr{C}_n, X).$$

A closely related example leads to the relative James construction.

Example 1·9. Let $\mathscr{M}$ be the operad with $\mathscr{M}_j = \Sigma_j$ ((9), p. 19). For (X, A) as above, define a subarray $\mathbf{Y} = \mathbf{Y}(\mathscr{M}, X, A)$ of $\mathbf{Y}(\mathscr{M}, X)$ by letting Y_ϕ consist of those points

$$(\sigma, x_1, \ldots, x_r) \in \mathscr{M}_s \times X^r$$

such that $x_i \notin A$ implies $\sigma^{-1}\phi(i) = s$. This is precisely the image of $\mathbf{Y}(\mathscr{C}_1, X, A)$ under the morphism

$$\mathbf{Y}(\mathscr{C}_1, X) \to \mathbf{Y}(\mathscr{M}, X)$$

of Λ-arrays induced by $\epsilon: \mathscr{C}_1 \to \mathscr{M}$ (see (9), pp. 34 and 59).

There is also a configuration space analog of Example 1·8.

Example 1.10. Define a subarray $\mathbf{Y} = \mathbf{Y}(J^n, X, A)$ of $\mathbf{Y}(J^n, X)$ as follows. For $\phi: \mathbf{r} \to \mathbf{s}$, Y_ϕ consists of those points

$$(\langle c_1, \ldots, c_s \rangle, x_1, \ldots, x_r) \in F(J^n, s) \times X^r$$

such that whenever $x_i \notin A$, $c_k \notin (c_j', 1) \times \{c_j''\}$ for $k \neq j$, where $j = \phi(i)$ and where

$$c_j = (c_j', c_j'') \in J \times J^{n-1} = J^n.$$

Centrepoint projection specifies a Σ_ϕ-homotopy equivalence

$$g_\phi: \mathbf{Y}(\mathscr{C}_n, X, A)_\phi \to \mathbf{Y}(J^n, X, A)_\phi,$$

and these maps give a morphism g of Λ-arrays.

2. *The spaces* ΓY

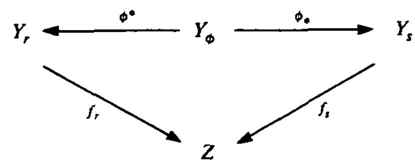
We give a coalescence functor Γ from Λ-arrays to spaces. The discussion parallels that of (5), §2.

Definition 2·1. Let $\mathbf{Y}$ be a Λ-array. Define

$$\Gamma\mathbf{Y} = \coprod_{r \geq 0} Y_r/(\sim),$$

where the equivalence relation $\sim$ is that generated by the elementary relations $x < y$ if $x \in Y_r$, $y \in Y_s$, and there exist $\phi: \mathbf{r} \to \mathbf{s}$ in Λ and $z \in Y_\phi$ such that $x = \phi^*(z)$ and $y = \phi_*(z)$. Let $F_s\Gamma\mathbf{Y}$ be the image of $\coprod_{0 \leqslant r \leqslant s} Y_r$ and give $F_s\Gamma\mathbf{Y}$ the quotient topology. Then give $\Gamma\mathbf{Y}$ the topology of the union of the $F_s\Gamma\mathbf{Y}$.

Observe that a collection of maps $f_r: Y_r \to Z$ extends to a map $f: \Gamma\mathbf{Y} \to Z$ if and only if the following diagram commutes for each injection $\phi: \mathbf{r} \to \mathbf{s}$:



Categorically, $\Gamma\mathbf{Y}$ is the coequalizer in the diagram

$$\coprod_\phi Y_\phi \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_r Y_r \longrightarrow \Gamma\mathbf{Y},$$

where α and β have restrictions $\phi_*: Y_\phi \to Y_s$ and $\phi^*: Y_\phi \to Y_r$.

The following lemma will lead to the verification that the spaces $\Gamma\mathbf{Y}$ are well-behaved colimits of sequences of cofibrations. Its proof will display the role played by the pullbacks in our specification of a Λ-array. We need some notations.

Notations 2·2. Let $\mathbf{Y}$ be a Λ-array. Define $Z_\phi = Y_\phi/\Sigma_\phi$ and $Z_r = Y_r/\Sigma_r$. Observe that we obtain induced maps

$$Z_r \xleftarrow{\phi^*} Z_\phi \xrightarrow{\phi_*} Z_s$$

such that ϕ_* is a cofibration. Let $\sigma_q: \mathbf{r-1} \to \mathbf{r}$ be the qth ordered injection, $0 \leqslant q < r$ (so that $q+1 \notin \operatorname{Im}\sigma_q$), and define

$$\partial Z_r = \bigcup_{q=0}^{r-1} \operatorname{Im}\sigma_{q*} \subset Z_r \quad \text{and} \quad \mathring{Z}_r = Z_r - \partial Z_r.$$

It follows from (1), app. 2·7, that $\partial Z_r \to Z_r$ is a cofibration. (This entails replacing G by Σ_r in the cited result and contemplating the relationship between the intersections it deals with and the pullbacks of definition 1·3 (3).) Finally, define

$$D_q\mathbf{Y} = Z_q/\partial Z_q.$$

LEMMA 2·3. *The natural map* $Y_q \to F_r\Gamma\mathbf{Y}$ *factors through a map* $\pi_q: Z_q \to F_r\Gamma Y$, $0 \leqslant q \leqslant r$. *For an element* $\alpha \in F_r\Gamma\mathbf{Y}$, *there exists a unique* $q \leqslant r$ *and a unique* $\bar{x} \in \mathring{Z}_q$ *such that* $\pi_q(\bar{x}) = \alpha$.

Proof. If $y \in Y_q$ and $\tau \in \Sigma_q$, then $y < \tau y$ since

$$\tau^*(\tau^*)^{-1}(y) = y \quad \text{and} \quad \tau_*(\tau^*)^{-1}(y) = \tau y.$$

This implies the first statement, and of course $\tau y < y$ also holds. Let q be minimal such that $\pi_q(\bar{x}) = \alpha$ for some $\bar{x} \in Z_q$. We claim first that $\bar{x} \in \mathring{Z}_q$. Let $\bar{x}$ be the image of $x \in Y_q$. If $\bar{x} \in \partial Z_q$, then $x = \phi_*(z)$ for some $\phi: \mathbf{p} \to \mathbf{q}$ and $z \in Y_\phi$ with $p < q$. Thus $\phi^*(z) < x$, and this contradicts the minimality of q. It remains to prove uniqueness, and it clearly suffices to verify that $x \sim x''$ implies $x < x''$. If $x' < x$, then, by the arguments just given, we must have that x' differs from x by a permutation and thus that $x < x'$. Thus suppose that $x < x'$ and that either $x' < x''$ or $x'' < x'$. We shall verify that $x < x''$ under either hypothesis. Inductively, this will verify the desired implication and so complete the proof. Assume that

$$x = \phi^*(y) \quad \text{and} \quad x' = \phi_*(y) \quad \text{for} \quad \phi: \mathbf{q} \to \mathbf{q} \quad \text{and} \quad y \in Y_\phi.$$

Case 1. Assume that

$$x' = \psi^*(z) \quad \text{and} \quad x'' = \psi_*(z) \quad \text{for} \quad \psi: \mathbf{q}' \to \mathbf{q}'' \quad \text{and} \quad z \in Y_\psi.$$

Let $\zeta = \psi \circ \phi$. Then the following diagram is a pullback:

$$\begin{array}{ccc} Y_\zeta & \xrightarrow{\phi_*} & Y_\psi \\ {\scriptstyle \psi^*}\downarrow & & \downarrow{\scriptstyle \psi^*} \\ Y_\phi & \xrightarrow{\phi_*} & Y_{q'} \end{array}$$

Choose $w \in Y_\zeta$ such that $\psi^*(w) = y$ and $\phi_*(w) = z$. Then

$$x = \phi^*(y) = \phi^*\psi^*(w) = \zeta^*(w) \quad \text{and} \quad x'' = \psi_*(z) = \psi_*\phi_*(w) = \zeta_*(w).$$

Case 2. Assume that

$$x' = \psi_*(z) \quad \text{and} \quad x'' = \psi^*(z) \quad \text{for} \quad \psi: \mathbf{q}'' \to \mathbf{q}' \quad \text{and} \quad z \in Y_\psi.$$

Construct a pullback in Λ (with common composite ξ):

$$\begin{array}{ccc} \mathbf{p} & \xrightarrow{\chi} & \mathbf{q}'' \\ {\scriptstyle \imath}\downarrow & \searrow^{\xi} & \downarrow{\scriptstyle \psi} \\ \mathbf{q} & \xrightarrow{\phi} & \mathbf{q}' \end{array}$$

Then the following diagram is a pullback:

$$\begin{array}{ccc} Y_\xi & \xrightarrow{\chi_*} & Y_\psi \\ {\scriptstyle \imath_*}\downarrow & & \downarrow{\scriptstyle \psi_*} \\ Y_\phi & \xrightarrow{\phi_*} & Y_{q'} \end{array}$$

Choose $w \in Y_\xi$ such that $\imath_*(w) = y$ and $\chi_*(w) = z$. Then

$$x' = \psi_*(z) = \psi_*\chi_*(w) = \xi_*(w)$$

and thus $x < x'$ and $x' > \xi^*(w) \in Y_p$. By the minimality of q, we must have $p = q$. Thus $\imath$ must be a permutation. By virtue of the element of choice in the construction of pullback diagrams in Λ (see Lemma 1·1), we may choose our original pullback so that $\imath$ is the identity. Then $\imath_*$ is the identity and $w = y$. Consider $\psi^*: Y_\phi \to Y_\chi$. This makes sense since $\phi = \psi \circ \chi$. Let $v = \psi^*(y) \in Y_\chi$. Then

$$x = \phi^*(y) = \chi^*\psi^*(y) = \chi^*(v) \quad \text{and} \quad x'' = \psi^*(z) = \psi^*\chi_*(y) = \chi_*\psi^*(y) = \chi_*(v).$$

With this description of points, it is a simple matter to check the validity of the following inductive description of $\Gamma\mathbf{Y}$.

Proposition 2·4. *$\Gamma\mathbf{Y}$ is a well-defined space in $\mathscr{U}$ and in $\mathscr{T}$ if Y_0 has a non-degenerate basepoint. For $r \geqslant 1$, the following diagram is a pushout*:

$$\begin{array}{ccc} \partial Z_r & \xrightarrow{d_r} & F_{r-1}\Gamma\mathbf{Y}, \\ \cap & & \downarrow \\ Z_r & \longrightarrow & F_r\Gamma\mathbf{Y} \end{array}$$

where $d_r(\sigma_{q*}z) = \pi_{r-1}(\sigma_q^* z)$ for $0 \leqslant q < r$, $\pi_{r-1}: Z_{r-1} \to F_{r-1}\Gamma\mathbf{Y}$. The inclusions $F_{r-1}\Gamma\mathbf{Y} \to F_r\Gamma\mathbf{Y}$ are thus cofibrations, and

$$F_r\Gamma\mathbf{Y}/F_{r-1}\mathbf{Y} \cong Z_r/\partial Z_r = D_q\mathbf{Y}.$$

The essential point is that d_r is well defined.

Examples 2·5. (i) If $\mathbf{Y} = \mathbf{Y}(\mathscr{C}, \mathbf{X})$ for a coefficient system $\mathscr{C}$ and Λ-space $\mathbf{X}$, we write $\Gamma\mathbf{Y} = C\mathbf{X}$. We write CX when $\mathbf{X} = \{X^r\}$ for a based space X and $C(Z, \mathbf{X})$ when $\mathscr{C} = \mathscr{C}(Z)$ for an unbased space Z. These spaces were the objects of study in (5).

(ii) For a space X, $\Gamma(\mathbf{Y} \times X) \simeq (\Gamma\mathbf{Y}) \times X$; if X is based,

$$(\mathbf{Y}^+ \wedge X) \cong (\Gamma\mathbf{Y})^+ \wedge X.$$

Here $D_q(\mathbf{Y}^+ \wedge X) \cong (D_q\mathbf{Y}) \wedge X$ for $q > 0$.

(iii) If $\mathbf{A}$ is a subarray of $\mathbf{Y}$, then $\Gamma\mathbf{A}$ is a closed subspace of $\Gamma\mathbf{Y}$.

(iv) If $\mathbf{Y} = \mathbf{Y}(\mathscr{C}_n, X, A)$ as in Example 1·8, then $\Gamma\mathbf{Y}$ is the space $E_n(X, A)$ introduced and studied in (9), §6.

(v) If $\mathbf{Y} = \mathbf{Y}(\mathscr{M}, X, A)$ as in Example 1·9, then $\Gamma\mathbf{Y}$ is the relative James construction $M(X, A)$ of Gray (7) and (9), p. 59.

(vi) If $\mathbf{Y} = \mathbf{Y}(J^n, X, A)$ as in Example 1·10, then $\Gamma\mathbf{Y}$ is a space equivalent to $E_n(X, A)$ which we shall denote $E(J^n, X, A)$.

We have the following invariance statement, which generalizes (5), 2·6 and 2·7.

Lemma 2·6. *Let $f: \mathbf{Y} \to \mathbf{Y}'$ be a map of Λ-arrays. If each $f_\phi: Z_\phi \to Z'_\phi$ is a (weak) equivalence, then $\Gamma f: \Gamma\mathbf{Y} \to \Gamma\mathbf{Y}'$ is a (weak) equivalence. If $\mathbf{Y}$ and $\mathbf{Y}'$ are Σ-free and each $f_\phi: Y_\phi \to Y'_\phi$ is a weak equivalence, then each $f_\phi: Z_\phi \to Z'_\phi$ is a weak equivalence.*

Proof. The second statement holds by an obvious covering space argument. In the presence of cofibrations, pushouts and colimits of (weak) equivalences are (weak) equivalences, hence it suffices to check that f_r restricts to a (weak) equivalence

$$\partial Z_r \to \partial Z'_r$$

for $r \geqslant 1$. Since $\sigma_{q*}: Z_{r-1} \to \sigma_{q*}Z_{r-1}$ is a homeomorphism, it suffices to observe that an inductive pushout argument shows that f_r restricts to a (weak) equivalence

$$\bigcup_{0 \leqslant i \leqslant q} (\sigma_{i*}Z_{r-1}) \to \bigcup_{0 \leqslant i \leqslant q} (\sigma_{i*}Z'_{r-1}), \quad (0 \leqslant q < r).$$

The relevant intersections are easily interpreted as spaces $\phi_* Z_\phi$ by use of the pullback condition of definition 1·3 (3).

3. *Combinatorial maps* $\Gamma\mathbf{Y} \to C\mathbf{X}$

Our splitting theorems are based on the use of appropriate 'James maps'. The definition of these maps is only one application, albeit the most important one, of a quite general framework for the construction of combinatorial maps $\Gamma\mathbf{Y} \to C\mathbf{X}$. The 'Segal maps' exploited in our study of the Kahn–Priddy theorem (2, 3) also fit into this framework.

Definition 3·1. An ordered functor $F: \Lambda \to \Lambda$ is a covariant functor which preserves pullbacks. By an easy exercise in the use of pullbacks in Λ, it follows that $r < s$ implies $F(\mathbf{r}) \leqslant F(\mathbf{s})$, with equality if and only if $F(\mathbf{r}) = F(\mathbf{s})$ for all $r < s$. That is, F is either constant on objects or eventually strictly increasing.

The characterization of pullbacks in Λ given in Lemma 1·1 makes it a simple matter to verify whether or not a given functor $\Lambda \to \Lambda$ is ordered.

Example 3·2. For $q \geqslant 0$, define an ordered functor $J_q: \Lambda \to \Lambda$ as follows. On objects, $J_q(\mathbf{r}) = \mathbf{0}$ if $r < q$ and $J_q(\mathbf{r}) = (\mathbf{r}-\mathbf{q}, \mathbf{q})$ if $r \geqslant q$, where $(r-q, q)$ is the binomial coefficient. On morphisms $\phi: \mathbf{r} \to \mathbf{s}$, $J_q(\phi) = \bar{\phi}$ as specified in (5), 3·1. To review, let R be the set of ordered injections $\mathbf{q} \to \mathbf{r}$, let S be the set of ordered injections $\mathbf{q} \to \mathbf{s}$, and give R and S the reverse lexicographic ordering. This choice of ordering yields identifications of R and S with the positive elements of $(\mathbf{r}-\mathbf{q}, \mathbf{q})$ and $(\mathbf{s}-\mathbf{q}, \mathbf{q})$. Map R to S by sending $\psi: \mathbf{q} \to \mathbf{r}$ to the composite

$$\mathbf{q} \xrightarrow{\tau} \mathbf{q} \xrightarrow{\psi} \mathbf{r} \xrightarrow{\phi} \mathbf{s},$$

where τ is the unique permutation such that $\phi\psi\tau$ is ordered. Via our identifications, this function $R \to S$ specifies $\bar{\phi}$. Observe that J_1 is the identity functor I. By convention, J_0 is the constant functor at the object $\mathbf{1}$.

Example 3·3. The category Λ has an evident wedge sum $\vee: \Lambda \times \Lambda \to \Lambda$ and smash product $\wedge: \Lambda \times \Lambda \to \Lambda$ ((2), 2·1), and these are easily seen to preserve pullbacks. If F and F' are ordered functors $\Lambda \to \Lambda$, then so are the composite $F' \circ F$, the wedge sum $F \vee F' = \vee \circ (F \times F') \circ \Delta$, and the smash product $F \wedge F' = \wedge \circ (F \times F') \circ \Delta$. Let E_n be the constant functor at the object $\mathbf{n}$. Then the n-fold wedge sum of F with itself is $F \wedge E_n$. Define $K_q = J_q \wedge E_{q!}$ and $S_q = I \wedge E_{q!}$ for $q \geqslant 0$.

The J_q lead to James maps. The S_q lead to Segal maps and the K_q lead to maps suitable for the analysis of certain composites of Segal maps and James maps. This analysis proves the Kahn–Priddy theorem. We shall say no more about this application here, but the connection will be obvious to readers of (2, 3).

The following definition makes sense by comparison of definitions 1·3 and 3·1.

Definitions 3·4. Let $F: \Lambda \to \Lambda$ be an ordered functor and let $\mathbf{Y}$ be a Λ-array. Define a new Λ-array $F^*\mathbf{Y}$ by letting $(F^*\mathbf{Y})_\phi = Y_{F(\phi)}$, with structural maps

$$Y_{F(\phi)} \xleftarrow{F(\psi)^*} Y_{F(\psi\phi)} \xrightarrow{F(\phi)_*} Y_{F(\psi)}.$$

Observe that the inclusion $\amalg Y_{F(r)} \to \amalg Y_r$ induces a map $\Gamma F^*\mathbf{Y} \to \Gamma\mathbf{Y}$ which is natural in $\mathbf{Y}$.

We are interested in constructing combinatorial maps $\Gamma\mathbf{Y} \to C\mathbf{X}$ associated to a given ordered functor $F: \Lambda \to \Lambda$. It suffices to construct maps $\mathbf{Y} \to F^*\mathbf{Y}(\mathscr{C}, \mathbf{X})$ of Λ-arrays since we can then apply Γ and compose with the map

$$\Gamma F^*\mathbf{Y}(\mathscr{C}, \mathbf{X}) \to \Gamma\mathbf{Y}(\mathscr{C}, \mathbf{X}) = C\mathbf{X}$$

given by the previous definition. The following symmetric pair of definitions specifies what is needed.

Definition 3·5. An F-system for a Λ-array $\mathbf{Y}$ is a coefficient system $\mathscr{C}$ and a sequence of maps $\xi_r\colon Y_r \to \mathscr{C}_{F(r)}$ such that the following diagram commutes for $\phi\colon \mathbf{r} \to \mathbf{s}$ in Λ:

$$\begin{array}{ccccc} Y_r & \xleftarrow{\phi^*} & Y_\phi & \xrightarrow{\phi_*} & Y_s \\ \downarrow{\scriptstyle \xi_r} & & & & \downarrow{\scriptstyle \xi_s} \\ \mathscr{C}_{F(r)} & & \xleftarrow{\quad F(\phi) \quad} & & \mathscr{C}_{F(s)} \end{array}$$

Definition 3·6. An F-space for a Λ-array $\mathbf{Y}$ is a Λ-space $\mathbf{X}$ and a sequence of maps $\pi_r\colon Y_r \to X_{F(r)}$ such that the following diagram commutes for $\phi\colon \mathbf{r} \to \mathbf{s}$ in Λ:

$$\begin{array}{ccccc} Y_r & \xleftarrow{\phi^*} & Y_\phi & \xrightarrow{\phi_*} & Y_s \\ \downarrow{\scriptstyle \pi_r} & & & & \downarrow{\scriptstyle \pi_s} \\ X_{F(r)} & & \xrightarrow{\quad F(\phi) \quad} & & X_{F(s)} \end{array}$$

An enjoyable diagram chase gives the following result.

Lemma 3·7. *Given an F-system $(\mathscr{C}, \xi)$ and an F-space $(\mathbf{X}, \pi)$ for a Λ-array $\mathbf{Y}$, the maps*

$$(\xi_s \phi_*, \pi_r \phi^*)\colon Y_\phi \to \mathscr{C}_{F(s)} \times X_{F(r)}$$

specify a morphism $\mathbf{Y} \to F^* Y(\mathscr{C}, \mathbf{X})$ of Λ-arrays and so induce a map $\Gamma\mathbf{Y} \to C\mathbf{X}$.

This observation provides a remarkably versatile tool for the construction of maps between function spaces. Even when $\mathbf{Y}$ has the form $C'\mathbf{X}'$, these maps are much more general than could be obtained from sequences of maps relating $\mathscr{C}'$ and $\mathscr{C}$ and $\mathbf{X}'$ and $\mathbf{X}$ separately.

In practice, the construction of F-systems and of F-spaces is quite asymmetric. We next describe extra structure on Λ-arrays which ensures the existence of J_q-spaces $(D_q \mathbf{Y}, \pi)$ for all $q \geq 0$. It was to obtain this structure that Π-spaces were introduced in (5). Recall that Π is the category of finite based sets $\mathbf{n}$ and based functions $\phi\colon \mathbf{r} \to \mathbf{s}$ such that $\phi^{-1}(j)$ has at most one element for $1 \leq j \leq s$; Λ is the subcategory of those ϕ such that $\phi^{-1}(0) = \{0\}$. An injection $\phi\colon \mathbf{r} \to \mathbf{s}$ determines a projection $\phi^{-1}\colon \mathbf{s} \to \mathbf{r}$ via $\phi^{-1}\phi(a) = a$ and $\phi^{-1}(b) = 0$ if $b \notin \operatorname{Im}\phi$. The following observation will clarify our definitions and their relationship to the definitions of (5).

Lemma 3·8. *Consider the following two diagrams, where ϕ, ψ, ω, and ι are all injections:*

$$\begin{array}{ccc} \mathbf{p} & \xrightarrow{\iota} & \mathbf{q} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \omega} \\ \mathbf{r} & \xrightarrow{\phi} & \mathbf{s} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{p} & \xrightarrow{\iota} & \mathbf{q} \\ \uparrow{\scriptstyle \psi^{-1}} & & \uparrow{\scriptstyle \omega^{-1}} \\ \mathbf{r} & \xrightarrow{\phi} & \mathbf{s} \end{array}$$

Assume the first diagram commutes. Then the second diagram commutes if and only if the first diagram is a pullback.

Definitions 3·9. Let **Y** be a Λ-array.

(i) **Y** has projections if for every pullback diagram

$$\begin{array}{ccc} \mathbf{p} & \xrightarrow{\iota} & \mathbf{q} \\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\omega} \\ \mathbf{r} & \xrightarrow{\phi} & \mathbf{s} \end{array}$$

in Λ, there is a map $(\psi, \omega)^*: Y_\phi \to Y_\iota$ such that the following diagram commutes:

$$\begin{array}{ccccc} Y_r & \xleftarrow{\phi^*} & Y_\phi & \xrightarrow{\phi_*} & Y_s \\ \Big\downarrow{\scriptstyle(\psi,\psi)^*} & & \Big\downarrow{\scriptstyle(\psi,\omega)^*} & & \Big\downarrow{\scriptstyle(\omega,\omega)^*} \\ Y_p & \xleftarrow{\iota^*} & Y_\iota & \xrightarrow{\iota_*} & Y_q \end{array}$$

When $p = r$, so that ψ is a permutation, we require

$$(\psi, \omega)^* = \omega^* \psi_*^{-1}: Y_\phi = Y_{\omega\iota\psi^{-1}} \to Y_{\omega\iota} \to Y_\iota.$$

(ii) **Y** is a Π-array if for every composable pair of injections (ϕ, ψ), there is a map $\psi^*: Y_\phi \to Y_{\phi\psi}$ such that the following diagram commutes, where $\phi\psi = \omega\iota$ is a pullback relation as in (i):

$$\begin{array}{ccccc} Y_r & \xleftarrow{\phi^*} & Y_\phi & \xrightarrow{\phi_*} & Y_s \\ \Big\downarrow{\scriptstyle\psi^*} & & \Big\downarrow{\scriptstyle\psi^*} & & \Big\downarrow{\scriptstyle\omega^*} \\ Y_\psi & \xleftarrow{\phi^*} & Y_{\phi\psi} = Y_{\omega\iota} & \xrightarrow{\iota_*} & Y_\omega \end{array}$$

(Here the ψ^* and ω^* are new maps, while the remaining maps are given by Definition 1·3.) When ψ is a permutation, we require

$$\psi^* = \psi_*^{-1}: Y_\phi = Y_{\phi\psi\psi^{-1}} \to Y_{\phi\psi},$$

and it is useful to think of ψ^* as ψ_*^{-1} in general; then the right square above translates under the lemma to a functoriality diagram with respect to morphisms in Π.

An immediate diagram chase gives the following result

LEMMA 3·10. *If* **Y** *is a* Π*-array, then* **Y** *has projections*

$$(\psi, \omega)^* = \omega^* \psi^*: Y_\phi \to Y_{\phi\psi} = Y_{\omega\iota} \to Y_\iota.$$

A simple computation from the combinatorics of (5), 3·1 gives the following result, which may be viewed as a generalization of (5), 3·2.

LEMMA 3·11. *If the* Λ*-array* **Y** *has projections, then* $(D_q\mathbf{Y}, \pi)$ *is a* J_q*-space, where*

$$\pi_r: Y_r \to (D_q\mathbf{Y})^{(r-q,q)}$$

has ith coordinate the composite of the projection

$$(\psi_i, \psi_i)^*: Y_r \to Y_q$$

induced by the ith ordered injection $\psi_i: \mathbf{q} \to \mathbf{r}$ *and the natural quotient map*

$$Y_q \to Z_q \to D_q\mathbf{Y}.$$

Example 3·12. A Π-space is a functor $\mathbf{X}: \Pi \to \mathscr{U}$ whose restriction to Λ is a Λ-space. If $\mathscr{C}$ is a coefficient system and $\mathbf{X}$ is a Π-space, then $\mathbf{Y}(\mathscr{C}, \mathbf{X})$ is a Π-array. For $\psi: \mathbf{p} \to \mathbf{r}$ and $\phi: \mathbf{r} \to \mathbf{s}$,

$$\psi^* = 1 \times \psi^{-1}: \mathscr{C}_s \times X_r \to \mathscr{C}_s \times X_p.$$

Example 3·13. If $\mathbf{Y}$ has projections or is a Π-array, then the same is true for all of the constructions on $\mathbf{Y}$ in Example 1·6 provided that $\mathbf{X}$, where used, is a Π-space.

Example 3·14. The subarrays of Examples 1·8–1·10 inherit structures of Π-array from their ambient Π-arrays.

In sum, the theory only needs Λ-arrays with projections, but we know of no examples which are not actually Π-arrays. We shall state our results for Π-arrays, but they will all apply to Λ-arrays with projections.

4. *Splitting theorems*

To prove splitting theorems for Π-arrays $\mathbf{Y}$, we need only construct appropriate J_q-systems $(\mathscr{C}, \xi)$ and parrot the arguments of (5).

We adopt the convention $D_0\mathbf{Y} = Y_0^+$ to give $D_0\mathbf{Y}$ a basepoint. (Alternatively, we could assume that Y_0 and $\Gamma\mathbf{Y}$ have basepoints.)

The following homological splitting theorem generalizes [5, 4·10].

THEOREM 4·1. *Let* $\mathbf{Y}$ *be a* Π*-array and let* G *be an Abelian group. For all* $r \geqslant 1$ *(including* $r = \infty$*),*

$$H_*(F_r\Gamma\mathbf{Y}; G) \cong \sum_{q=0}^{r} \tilde{H}_*(D_q\mathbf{Y}; G).$$

These isomorphisms are natural in $\mathbf{Y}$ *and* G *and commute with Bockstein operations.*

Proof. Let $\mathscr{N}$ be the coefficient system with each $\mathscr{N}_r$ a point. For any ordered functor $F: \Lambda \to \Lambda$, the unique maps $Y_r \to \mathscr{N}_{F(r)}$ specify an F-system in the sense of Definition 3·5. Taking $F = J_q$, we obtain James maps $j_q: \Gamma\mathbf{Y} \to ND_q\mathbf{Y}$ for $q \geqslant 0$, where N is the infinite symmetric product functor. Here j_0 carries $\Gamma\mathbf{Y}$ to Y_0. The rest of the argument is precisely like that of (5), pp. 477–497. As there, we make use of Π-arrays $\mathbf{Y}^+ \wedge X$ (compare Examples 1·6 (ii), 2·5 (ii), and 3·13) to pass from the case with each Y_r connected and $G = Z$ to the general case. Note that $D_0(\mathbf{Y}^+ \wedge X) = (D_0\mathbf{Y}) \wedge X$ under our convention $D_0\mathbf{Y} = Y_0^+$.

The following homotopical splitting theorem generalizes (5), 8·2. We write Σ^∞ for the suspension spectrum functor (rather than Q_∞ as in (5)).

THEOREM 4·2. *Let* $\mathbf{Y}$ *be a* Σ*-free* Π*-array. For all* $r \geqslant 1$ *(including* $r = \infty$*), there is a natural isomorphism in the stable category*

$$\tilde{k}_r: \Sigma^\infty(F_r\Gamma\mathbf{Y})^+ \to \bigvee_{q=0}^{r} \Sigma^\infty D_q\mathbf{Y}.$$

Moreover, $\tilde{k}_r$ *is the sum over* q *of restrictions of James–Hopf maps*

$$h_q^s: \Sigma^\infty(\Gamma\mathbf{Y})^+ \to \Sigma^\infty D_q\mathbf{Y}.$$

To prove this, we need a canonical way of obtaining a J_q-system $(\mathscr{C}, \xi_q)$ for $\mathbf{Y}$. We follow the ideas of (5), §5. We have a quotient map $\pi: Y_q \to Z_q = Y_q/\Sigma_q$ and we have m ordered injections $\psi_i: \mathbf{q} \to \mathbf{r}$, where $m = (r-q, q)$. Define $\xi_{q,r}: Y_r \to Z_q^m$ by

$$\xi_{qr}(y) = \mathop{\times}_{i} \pi(\psi_i, \psi_i)^*(y), \quad (\psi_i, \psi_i)^*: Y_r \to Y_q.$$

This gives a J_q-system, but it has the defect that the receiving coefficient system given by the powers of Z_q is not separated. We remedy this by fiat.

Definition 4·3. $\mathbf{Y}$ is said to be separated if the maps $\xi_{qr}: Y_r \to Z_q^m$ take values in the configuration space $F(Z_q, m)$ for all q and r. Taking $\mathscr{C} = \mathscr{C}(Z_q)$ in Definition 3·5 and Lemma 3·7, there result canonical James maps

$$j_q: \Gamma\mathbf{Y}^+ \to C(Z_q, D_q\mathbf{Y})$$

for all $q \geq 0$; here j_0 carries $\Gamma\mathbf{Y}$ to the subspace

$$Y_0 \times Y_0 \subset F_1 C(Y_0, Y_0^+) = (Y_0 \times Y_0)^+.$$

For all q, the disjoint basepoint adjoined to $\Gamma\mathbf{Y}$ serves to make j_q a based map.

Of course, when $\mathbf{Y} = \mathbf{Y}(\mathscr{C}, \mathbf{X})$ as in (5), Z_q is unnecessarily large. Here we can first project $\mathscr{C}_r \times X_r$ to $\mathscr{C}_r$ and then apply the construction just given (provided that $\mathscr{C}$ is separated). This replaces $Z_r = \mathscr{C}_r \times_{\Sigma_r} X_r$ by $\mathscr{B}_r = \mathscr{C}_r/\Sigma_r$. From this point, the proof of Theorem 4·2 is exactly the same as that of (5), 8·2, except that $\mathscr{B}_r$ there is replaced by Z_r here. In particular, we exploit Example 1·6 (iii) to replace general Σ-free Π-arrays $\mathbf{Y}$ by the separated Π-arrays $\tilde{\mathbf{Y}} = \mathscr{C}(R^\infty) \times \mathbf{Y}$. Lemma 2·6 ensures that the projection $\tilde{\mathbf{Y}} \to \mathbf{Y}$ induces weak equivalences $\Gamma\tilde{\mathbf{Y}} \to \Gamma\mathbf{Y}$ and $D_q\tilde{\mathbf{Y}} \to D_q\mathbf{Y}$.

We conclude by discussing the relative splitting theorems promised in the introduction, and we assume given a cofibration $A \subset X$ of nondegenerately based spaces. While A and X need not be connected, we only obtain implications for function spaces if they are.

THEOREM 4·4. *For all $r \geq 1$ (including $r = \infty$), there is a natural homotopy equivalence*

$$\tilde{k}_r: \Sigma F_r M(X, A) \to \bigvee_{q=1}^{r} \Sigma(A^{[q]} \wedge X).$$

Moreover, $\tilde{k}_r$ is the sum over q of restrictions of James–Hopf maps

$$h_q = \tilde{j}_q: \Sigma M(X, A) \to \Sigma(A^{[q]} \wedge X).$$

Proof. $\mathbf{Y}(\mathscr{M}, X, A)_r \subset \mathscr{M}_r \times X^r$. Projecting to $\mathscr{M}_r$ and using the James system of (5), 4·3, we obtain a J_q-system. Alternatively, as in (5), §3, we can forget about $\mathscr{M}$ and ignore equivariance. In any case, we obtain James maps

$$j_q: M(X, A) \to D_q\mathbf{Y}(\mathscr{M}, X, A) = A^{[q]} \wedge X,$$

and the rest of the argument is precisely like that of (5), pp. 474–476.

The function space implication is clear from the introduction, as is the implication of the stable splittings of $E_n(X, A)$ and $E(J^n, X, A)$ obtained by application of Theorem 4·2. Of course, the latter spaces are equivalent by Lemma 2·6 (compare Examples 1·8, 1·10, 2·5, and 3·14). For concrete applications, it is important to know just how far their stable James–Hopf maps desuspend, and use of $E(J^n, X, A)$ gives better results on this question. The precise analysis is exactly as in (5), pp. 480–481. The James maps for $E(J^n, X, A)$ are obtained from the J_q-system derived by including

$$\mathbf{Y}(J^n, X, A)_r \quad \text{in} \quad F(J^n, r) \times X^r,$$

projecting to $F(J^n, r)$, applying the canonical maps

$$\xi_{qr}: F(J^n, r) \to (B(J^n, q))^m,$$

where $$B(J^n, q) = F(J^n, q)/\Sigma_q,$$

of (5), p. 479, and then embedding $B(J^n, q)$ in R^t for the smallest possible t. The resulting James maps

$$j_q: E(J^n, X, A) \to C(R^t, D_q(J^n, X, A))$$

lead to James–Hopf maps $\Sigma^t E(J^n, X, A) \to \Sigma^t D_q(J^n, X, A)$ as in (5), 5·5. We studied the relevant embedding question for $B(J^n, 2)$ in (5), 5.8. As observed by Cohen (4), the fundamental theorem of algebra implies that $B(J^2, n)$ embeds in R^{2n}.

Appendix

Due to a misreading of Boardman and Vogt (1), app. 2·7, for which the first author accepts all blame, the cofibration conditions are mis-stated in (5), (6) and (10). For an injection $\phi: \mathbf{r} \to \mathbf{s}$, there are three groups of permutations $\tau: \mathbf{s} \to \mathbf{s}$ one might consider, namely those with $\tau(b) = b$ for $b \in \operatorname{Im}\phi$, those with $\tau(b) = b$ for $b \notin \operatorname{Im}\phi$, and those with $\tau(b) \in \operatorname{Im}\phi$ if $b \in \operatorname{Im}\phi$ and thus also $\tau(b) \notin \operatorname{Im}\phi$ if $b \notin \operatorname{Im}\phi$. As in this paper, it is the third and largest group which should be taken as Σ_ϕ in (10) 1·2 (3). In (5), 1·8, the group denoted Σ_s should consist of those permutations which fix the set of letters in s, but in any case the present definition, (1·4), of a Λ-space is to be preferred. (It is easily checked that the definitions of Π-spaces here and there are equivalent.) These changes make our references to (1) correct but result in no further changes in these papers; in particular, the whiskering construction of (10), app. B, works to arrange the present more stringent equivariant cofibration condition.

In (6), 1·1 (i), the inclusion of $\bigcap_{i \in s} \mathscr{A}_{r+1} \tau^i$ in $\mathscr{C}_{r+1}$ should be required to be a Σ_s-cofibration, where $\Sigma_s = \{\sigma \mid$ if $i \in s$ and $\tau^i \sigma = \rho \tau^j$ with $\rho \in \Sigma_r$, then $j \in s\} \subset \Sigma_{r+1}$ and $s \subset \{0, 1, \ldots, r\}$. The conditions (6), 1·1 (i.a) and (i.b), clearly still imply this condition since they imply that the intersection is empty unless $s = \{i\}$. As the main examples all proceeded by verification of the latter conditions, the correction is of only pedantic interest and results in no further changes.

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