Contemporary Mathematics
Volume 19, 1983

The Homology of Function Spaces

F.R. Cohen and L.R. Taylor[1]

In this note we compute the homology of certain function spaces. Specifically, if X and Y are based spaces, we wish to study $H_*(\mathrm{Map}(X,Y);F_p)$, where Map(X,Y) is the space of based maps in the compact-open topology, and where F_p is the prime field of characteristic p, p = 0 or a prime.

In the 1950's, K. Borsuk [2], S. Mardešić [9], and J.C. Moore [10] studied the case $Y = S^n$. In particular, Moore computed the above homology groups through a range under some mild assumptions on X. We complete Moore's result by computing $H_*(\mathrm{Map}(X,S^m);F_p)$ for all * and for most of Moore's spaces X.

We can also compute $H_*(\mathrm{Map}(X,Y);F_p)$ for somewhat more complicated Y (e.g. Y a wedge of spheres) but nothing like a general result has emerged. In particular we pose

Problem: Compute $H_*(\mathrm{Map}(X,\Sigma^m Y);F_p)$ for m>> dimension X. Is the answer a functor of $H_*(X;F_p)$ and $H_*(Y;F_p)$?

To do our calculations we return to an old idea. If $X_0 \subset X_1 \subset \ldots \subset X_n = X$ is a filtration of X by NDR pairs (X_i, X_{i-1}) , we get a family of fibrations

$$\mathrm{Map}(X_i/X_{i-1},Y) \longrightarrow \mathrm{Map}(X_i,Y) \longrightarrow \mathrm{Map}(X_{i-1},Y) \quad .$$

Federer [6] produced his spectral sequence by using the cellular filtration on the CW complex X and applying π_* to the resulting family of fibrations. We originally modified this filtration slightly and then showed that the resulting Serre spectral sequences were orientable and collapsed. The actual answer was then easily worked out.

The method of showing that these fibrations behave so well is to study the map

$$\phi\colon \mathrm{Map}(X,Y) \longrightarrow \mathrm{Map}(X,Q(Y)) \quad .$$

We get results because Q(Y) is an ∞-loop space and this makes some calculations rather easy. Indeed, they become so easy that we never need to formally intro-

---

1980 Mathematics Subject Classification. 54C35.

[1]Both authors are partially supported by the N.S.F.; the first author was partially supported by the Alfred P. Sloan Foundation.

duce any filtration on X at all. We also use Spanier's function space approach to S-duality,[11], to identify Map(X,Q(Y)) in many cases with Q(Z) for some space Z. In particular, if $Y = S^r$, Z is a Spanier-Whitehead r-dual to X.

We introduce some notation. For any space X define

$$d(X) \leq r \quad \text{if} \quad H^*(X;\mathbb{Z}) = 0 \quad \text{for all } * > r$$

and

$$b(X) \leq s \quad \text{if} \quad \tilde{H}_*(X;\mathbb{Z}) = 0 \quad \text{for all } * < s \ .$$

Define $d_p(X)$ and $b_p(X)$ similarly but using mod-p homology and cohomology instead of the groups with $\mathbb{Z}$ coefficients. Set

$$\ell(X) = d(X) - b(X) + 1 \quad \text{and} \quad \ell_p(X) = d_p(X) - b_p(X) + 1 \ .$$

Let $\beta_i(X) = \dim_{F_p} H_i(X;F_p)$. We will use $\Omega^i Y$ to denote $\mathrm{Map}(S^i,Y)$ and then $(\Omega^i Y)^{\beta_i(X)}$ will denote the Cartesian product of $\beta_i(X)$ copies of $\Omega^i Y$. We say that a space A has finite type if $H_i(A;\mathbb{Z})$ is finitely-generated for all i.

In the statement below of our main theorem we have fixed p and an integer r. We will require that our space X satisfy

$i)_r$: X is a connected CW complex of finite type with $d(X) \leq r$ and with $d_p(X) + \ell_p(X) \leq r$.

We will require that Y satisfy

$ii)_r$: Y is an r-connected CW complex of finite type: furthermore we require

a) the map $\Omega^j Y \longrightarrow \Omega^j Q(Y)$ is a mod-p homology monomorphism for $0<j<r$

and

b) if (B,C) is any pair such that B and C satisfy $i)_r$ and so that the inclusion is a mod-p homology monomorphism, then the map $\mathrm{Map}(B,Q(Y)) \longrightarrow \mathrm{Map}(C,Q(Y))$ is a mod-p homology epimorphism.

We can now state

Main Theorem: Fix p and r. Assume that X satisfies $i)_r$ and that Y satisfies $ii)_r$. Then, as vector spaces,

$$H_*(\mathrm{Map}(X,Y);F_p) \simeq \bigotimes_{i=1}^{r} H_*((\Omega^i Y)^{\beta_i(X)};F_p) \ .$$

With slightly stronger hypotheses we have

Addendum: If X is a p-local co-H space, or if Y is a p-local H space, Map(X,Y) is itself a p-local H space. Assume either that X is a p-local co-H space with $b_p(X) > 1$ or that Y is a p-local H space such that the map $Y \longrightarrow Q(Y)$ is a p-local H map. Then, if X satisfies $i)_r$ and Y satisfies $ii)_r$, $H_*(\mathrm{Map}(X,Y);F_p)$ is an associative, commutative ring. Furthermore, if $H_*(\Omega^i Y;F_p)$ is a free commutative ring for all i, $b_p(X) \leq i \leq d_p(X)$, then $H_*(\mathrm{Map}(X,Y);F_p)$ is also a free commutative ring. The isomorphism in the main theorem is an isomorphism of rings.

Remark: Moore's results [10] apply to spaces more general than CW complexes. The main theorem can also be applied to more general spaces. The technique is to replace the spaces one has by appropriate CW complexes without changing the homology of the mapping space. We choose not to do this here as the details would lead us too far afield.

Remark: If X is a co-H space with an exponent, Map(X,Y) is an H space with an exponent. One of the techniques in [4] for showing that certain spaces have exponents is to split these spaces into products of "simpler" pieces which are known to have exponents and whose homology is known. The main theorem provides many new examples.

Here are some explicit spaces to which our results apply.

Example 1: Let $Y = S^{2n+1}$. Then Y satisfies $ii)_{2n}$. Assume that X satisfies $i)_{2n}$. If $p \neq 2$, then Y is a p-local H space with $Y \longrightarrow Q(Y)$ a p-local H map. The main theorem applies so

$$H_*(\mathrm{Map}(X,S^{2n+1});F_p) \simeq \bigotimes_{i=1}^{2n} H_*((\Omega^i S^{2n+1})^{\beta_i(X)};F_p)$$

as vector spaces. If $p \neq 2$, then the two sides are isomorphic as algebras.

Example 2: Let $Y = S^{n+1}$ and let p be even (i.e. 0 or 2). Then Y satisfies $ii)_n$. Assume X satisfies $i)_n$. Again the main theorem applies.

Example 3: If $Y = \Omega S^{n+1}$ then Y satisfies $ii)_{n-1}$. Let X satisfy $i)_{n-1}$ and let X be a suspension. The main theorem applies to show

$$H_*(\mathrm{Map}(X,S^{n+1});F_p) \simeq \bigotimes_{i=1}^{n} H_*((\Omega^i S^{n+1})^{\beta_i(X)};F_p)$$

as algebras.

Example 4: If $Y = \Omega(S^{n_1} \vee \dots \vee S^{n_t})$ then Y satisfies $ii)_r$ for $r+2 = \min\{n_1,\dots,n_t\}$. Let X be a suspension which satisfies $i)_r$. The main theorem applies so

$$H_*(\mathrm{Map}(X, \bigvee_{j=1}^{t} S^{n_j});F_p) \simeq \bigotimes_{i=1}^{r} H_*((\Omega^i(\bigvee_{j=1}^{t} S^{n_j}))^{\beta_i(X)};F_p)$$

as algebras: $H_*(\Omega^i(\bigvee_{j=1}^{t} S^{n_j});F_p)$ can be computed via the Hilton-Milnor theorem [12].

Example 5: Let $Y = S^{2n+1}\{p^\ell\}$ be the fibre of the degree p^ℓ map, $p^\ell: S^{2n+1} \longrightarrow S^{2n+1}$. Then Y satisfies $ii)_{2n-1}$ if p is odd. If X satisfies $i)_{2n-1}$

$$H_*(\mathrm{Map}(X,S^{2n+1}\{p^\ell\});F_p) \simeq \bigotimes_{i=1}^{2n-1} H_*((\Omega^i S^{2n+1}\{p^\ell\})^{\beta_i(X)};F_p)$$

as vector spaces.

We conclude this section with a problem. Suppose that $Y = S^{2n+1}$ and $d(X) + \ell(X) \leq 2n$. Then X has a (2n+1)-dual, say Z. The proof of the main theorem together with example 1 will give that the natural map

$$\psi: Q(Z) \longrightarrow \mathrm{Map}(X,QS^{2n+1})$$

is a mod-p homology isomorphism. Thus the structure of the homology of $\mathrm{Map}(X,QS^{2n+1})$ as a Hopf algebra over the Steenrod algebra is reasonably well understood [3]. Furthermore, the map

$$\phi: \mathrm{Map}(X,S^{2n+1}) \longrightarrow \mathrm{Map}(X,QS^{2n+1})$$

is a monomorphism in mod-p homology.

What is the image of ϕ_* in terms of the usual generators for $H_*(Q(Z);F_p)$? We remark that the calculations in this paper would be of more use if one could decide this question.

§2: The proof of the main theorem.

In what follows we have fixed a p, an r, and a CW complex Y satisfying $ii)_r$. With this data fixed, consider the following statement, T(X), for a space X.

$$T(X): \quad H_*(\mathrm{Map}(X,Y);F_p) \simeq \bigotimes_{i=1}^{r} H_*((\Omega^i Y)^{\beta_i(X)};F_p) \text{ and the map}$$

$$\phi: \mathrm{Map}(X,Y) \longrightarrow \mathrm{Map}(X,Q(Y)) \text{ is a mod-p homology monomorphism.}$$

Our goal is to prove that for any CW complex X satisfying $i)_r$, T(X) is true. Recall the following.

Lemma 2.1: Let X be a CW complex and fix an integer k. Then we can find a CW complex X_0 and a map $i: X_0 \longrightarrow X$ such that

a) $\pi_* X_0 \longrightarrow \pi_* X$ is onto for $* \leq k$

b) $H_*(X_0;Z) \longrightarrow H_*(X;Z)$ is an isomorphism for $* \leq k$

c) $H_*(X_0;Z) = 0$ for $* > k$

d) the dimension of X_0 as a CW complex is at most k+1.

Proof: Let $B \subset X$ be a k-skeleton. Then $H_k(B;Z)$ is free abelian and $H_k(B;Z) \longrightarrow H_k X;Z)$ is onto. Hence the kernel of this map is free abelian so we can attach some k+1 cells to kill this kernel precisely. The resulting complex, X_0, has a map $i: X_0 \longrightarrow X$ with all the desired properties. //

We will need the following result several times in the sequel.

Lemma 2.2: Let X satisfy $i)_r$ and let Y satisfy $\check{ii})_r$. Then Map(X,Y) is $(b(Y) - d(X))$-connected and of finite type. Furthermore, if $\tilde{H}_*(X;F_p) = 0$, then $\tilde{H}_*(\mathrm{Map}(X,Y);F_p) = 0$ also.

Proof: The connectivity result is a direct consequence of Federer's spectral sequence [6]. In our case, Map(X,Y) is at least simply-connected. The last two results are mod-C arguments: E_2 of the Federer spectral sequence is of finite type, hence so is $\pi_*(\mathrm{Map}(X,Y))$; E_2 of the Federer spectral sequence is torsion prime to p, hence $\pi_*(\mathrm{Map}(X,Y))$ is torsion prime to p. Each of these homotopy results implies the respective homology result.//

The key lemma we need in our proof is

Lemma 2.3: Let $X_3 \xrightarrow{i} X_2 \xrightarrow{g} X_1$ be a cofibre sequence of CW complexes for which the cofibration, i, is a mod-p homology monomorphism. If X_2 satisfies $i)_r$ and if X_3 satisfies $i)_{r-1}$, then X_1 satisfies $i)_r$. If X_1, X_2, and X_3 satisfy $i)_r$ and if $T(X_1)$ and $T(X_3)$ are true, then $T(X_2)$ is also true.

Proof: It is no trouble to show that X_1 satisfies $i)_r$ so let us consider the map of fibrations

$$\begin{array}{ccc}
\mathrm{Map}(X_1,Y) & \xrightarrow{\phi_1} & \mathrm{Map}(X_1,Q(Y)) \\
\downarrow g^* & & \downarrow g^*_\infty \\
\mathrm{Map}(X_2,Y) & \xrightarrow{\phi_2} & \mathrm{Map}(X_2,Q(Y)) \\
\downarrow i^* & & \downarrow i^*_\infty \\
\mathrm{Map}(X_3,Y) & \xrightarrow{\phi_3} & \mathrm{Map}(X_3,Q(Y))
\end{array}$$

First we note that the Serre spectral sequence in mod-p homology for the right-hand fibration collapses: by $ii)_r$, i^*_∞ is a mod-p homology epimorphism and i^*_∞ is an ∞-loop map, so the Serre spectral sequence is a spectral sequence of algebras.

Now suppose $T(X_1)$ is true. Since both ϕ_1 and g^*_∞ are mod-p homology monomorphisms, so is g^*. Thus the Serre spectral sequence in mod-p cohomology collapses and so it does also in mod-p homology. Since $E^2 = E^\infty$,

$E^2 \approx \bigotimes_{i=1}^{r} H_*((\Omega^i Y)^{\beta_i(X_2)};F_p)$ by our assumptions on $T(X_1)$ and $T(X_3)$. //

We can combine 2.1 and 2.3 as follows: we can assume, without loss of generality, that X is $(b_p(X) - 1)$-connected. To see this, construct the X_0 in 2.1 for $k = b_p(X) - 1$. Then apply 2.3 to $X_0 \xrightarrow{i} X \to X/X_0$ (after making i into a cofibration). Note $\tilde{H}_*(X_0;F_p) = 0$ for all *, so $\tilde{H}_*(\mathrm{Map}(X_0,Y);F_p) = 0$ for all * by lemma 2.2. Hence $T(X_0)$ is true and X_0 satisfies $i)_{b_p(X)}$. Since $b_p(X) \le r - 1$, we have that T(X) is true if $T(X/X_0)$ is.

We will now prove T(X) by induction on $s(X) = \sum_{i=1}^{\infty} \beta_i(X)$.

If X satisfies $i)_r$ and if $s(X) = 0$, then lemma 2.2 shows $\tilde{H}_*(\mathrm{Map}(X,Y);F_p) = 0$ for all * so T(X) is clearly true.

If $s(X) > 0$ and (as we may assume) X is $(b_p(X)-1)$-connected, we can find a cofibre sequence $S^t \xrightarrow{i} X \longrightarrow X_1$ (with $t=b_p(X)$) for which i is a mod-p homology monomorphism. Since X satisfies $i)_r$, lemma 2.3 applies. Since $s(X_1) = s(X) - 1$ we are done: T(X) is true for any complex satisfying $i)_r$.

We next consider our algebra results. If Y is a p-local H space or if X is a p-local co-H space then Map(X,Y) is a p-local H space. Moreover, the map ϕ: $\mathrm{Map}(X,Y) \longrightarrow \mathrm{Map}(X,Q(Y))$ is a p-local H map in both cases of the addendum.

The space Map(X,Q(Y)) always has a homotopy associative, homotopy commutative H space structure coming from its ∞-loop space structure. If X is a p-local co-H space the two p-local H space structures on Map(X,Q(Y)) coincide [12] p.126. Since ϕ is a mod-p homology monomorphism, we see that the Pontrjagin ring, $H_*(\mathrm{Map}(X,Y);F_p)$, is associative and commutative.

We need some notation. If A is an algebra, let V(A) denote the module of indecomposables. (<u>The more usual notation, QA, might be confusing</u>.) If Z is a p-local H space, let V(Z) be $V(H_*(Z;F_p))$: $V(Z)^\beta$ denotes the direct sum of β copies of V(Z).

Now $H_*(\Omega^i Y;F_p)$ is an algebra for $i \geq 1$. If Y is also a p-local H space, recall that the two H space structures on $\Omega^i Y$ are homotopic. Hence $V(\Omega^i Y)$ is unambiguous. Consider the following statement

$$A(X)\colon H_*(\mathrm{Map}(X,Y);F_p) \text{ is a free associative, commutative algebra with } V(\mathrm{Map}(X,Y)) \simeq \bigoplus_{i=1}^{r} V(\Omega^i Y)^{\beta_i(X)} .$$

To prove that A(X) is true in the relevant cases, we again induct on s(X). There are no difficulties if the H space structure on Map(X,Y) comes from Y. If X is a p-local co-H space there are two points to check before the proof can be completed. First note that we can assume X is $(b_p(X)-1)$-connected: since $X_0 \longrightarrow X \rightarrow X \vee X \rightarrow X/X_0 \vee X/X_0$ is null-homotopic, X/X_0 is again a p-local co-H space. Next note that if $t=b_p(X)$ we can choose $i\colon S^t \longrightarrow X$ so that the cofibre of i has smaller s and is still a p-local co-H space. (This uses $b_p(X)>1$.)

§3. Some sufficient conditions for $ii)_r$.

We still need some examples of spaces Y satisfying $ii)_r$. We will assume throughout this section that Y has the homotopy type of a CW complex and that the basepoint of Y is non-degenerate. We will also consider p to be fixed.

There are two parts to $ii)_r$ and it is convenient to treat the two parts separately. They are stated below as $M_r(Y)$ and $E_r(Y)$.

$M_r(Y)$: Y is of finite type and r-connected the maps $\Omega^j Y \longrightarrow \Omega^j Q(Y)$ induce monomorphisms in mod-p homology for each j, $0 \le j \le r$.

$E_r(Y)$: Y is of finite type and r-connected if (B,C) is any CW pair such that B and C satisfy $i)_r$ and such that the inclusion is a mod-p homology monomorphism, then the map

$$\mathrm{Map}(B,Q(Y)) \longrightarrow \mathrm{Map}(C,Q(Y))$$

is a mod-p homology epimorphism.

We begin with some remarks.

Lemma 3.1: If $M_r(Y)$ is true, so is $M_{r-1}(\Omega Y)$.

Proof: There is an evaluation map $\Sigma\Omega Y \longrightarrow Y$. Furthermore $Q(Y) = \Omega Q(\Sigma Y)$ and the composite $\Omega^{j+1} Y \longrightarrow \Omega^j Q(\Omega Y) = \Omega^{j+1} Q(\Sigma\Omega Y) \longrightarrow \Omega^{j+1} Q(Y)$ is the $(j+1)^{st}$ loop of the inclusion $Y \longrightarrow Q(Y)$.//

Lemma 3.2: If $E_r(\Sigma Y)$ is true, so is $E_{r-1}(Y)$.

Proof: $\mathrm{Map}(X,Q(Y)) = \mathrm{Map}(X,\Omega Q(\Sigma Y)) = \mathrm{Map}(\Sigma X,Q(\Sigma Y))$.//

Lemma 3.3: Let Y_1 be of finite type and r-connected. Moreover, assume Y_1 is a mod-p retract of Y_2. Then, if $M_r(Y_2)$ is true, so is $M_r(Y_1)$.

Proof: We have $i: Y_1 \longrightarrow Y_2$ and $r: Y_2 \longrightarrow Y_1$ so that $r \circ i$ induces an isomorphism on $H_*(Y_1;F_p)$. But then so does $\Omega^j r \circ \Omega^j i$ so the result is an easy diagram chase.//

Lemma 3.4: Let Y_1 be of finite type and r-connected. Furthermore, assume Y_1 is a stable mod-p retract of Y_2. Then, if $E_r(Y_2)$ is true, so is $E_r(Y_1)$.

Proof: This time we know that $Q(Y_1)$ is a mod-p retract of $Q(Y_2)$. Since $d(X) \le r$, $\mathrm{Map}(X,Q(Y_1))$ is a mod-p retract of $\mathrm{Map}(X,Q(Y_2))$. This can be seen from the Federer spectral sequence. Our result is now a diagram chase.//

The next two results on wedges and products require some restrictions. Thus we pause for

Definition 3.5: Let I be an index set and let $\{Y_\alpha\}$ $\alpha\varepsilon I$ be a collection of spaces. We say this collection has strong finite type if $\bigvee_{\alpha\varepsilon I} Y_\alpha$ has finite type. We say the collection is r-connected if each Y_α is.

Remarks: If the collection has strong finite type then so does each Y_α but not conversely. For each n >0 all but finitely many of the Y_α must have $b(Y_\alpha) > n$: this condition plus each Y_α being of finite type implies the collection has strong finite type.

Lemma 2.2 shows that if $\{Y_\alpha\}$ has strong finite type and is r-connected, and if X satisfies $i)_r$, then $\{\mathrm{Map}(X,Y_\alpha)\}$ has strong finite type and is at least 1-connected.

Finally, let $\{Y_\alpha\}$ have strong finite type with all but finitely many Y_α simply-connected. Then the map $\underset{\alpha}{\times} Y_\alpha \longrightarrow \prod_\alpha Y_\alpha$ is a weak equivalence where $\prod_\alpha Y_\alpha$ denotes the Cartesian product and $\underset{\alpha}{\times} Y_\alpha$ denotes the weak product (the subset of $\prod_\alpha Y_\alpha$ with all but finitely many coordinates being the respective basepoints). The result is clear since $\pi_*(\prod_\alpha Y_\alpha)$ = direct product $\pi_*(Y_\alpha)$ and $\pi_*(\underset{\alpha}{\times} Y_\alpha)$ = direct sum $\pi_*(Y_\alpha)$. Now we can prove

Lemma 3.6: Suppose $\{Y_\alpha\}$ is a collection of strong finite type and that each $M_r(Y_\alpha)$ is true. Then $M_r(\prod_\alpha Y_\alpha)$ and $M_r(\underset{\alpha}{\times} Y_\alpha)$ are true.

Proof: We have a map $Q(\prod_\alpha Y_\alpha) \longrightarrow \prod_\alpha Q(Y_\alpha)$ and we know that $\Omega^j \prod_\alpha Z_\alpha = \prod_\alpha \Omega^j Z_\alpha$. The following diagram commutes

$$\begin{tikzcd}
\Omega^j(\prod_\alpha Y_\alpha) \arrow[r] \arrow[d, "1"] & \prod_\alpha \Omega^j Y_\alpha \arrow[d, "2"] & \underset{\alpha}{\times} \Omega^j Y_\alpha \arrow[l, "4"'] \arrow[d, "3"] \\
\Omega^j Q(\prod_\alpha Y_\alpha) \arrow[r] & \prod_\alpha \Omega^j Q(Y_\alpha) & \underset{\alpha}{\times} \Omega^j Q(Y_\alpha) \arrow[l, "5"']
\end{tikzcd}$$

Applying the Künneth formula to $\underset{\alpha}{\times}$ (but not to $\prod_\alpha$) we see that the map labeled 3 is a mod-p homology monomorphism.

Since $\{Q(Y_\alpha)\}$ has strong finite type if $\{Y_\alpha\}$ does, 4 and 5 are $H_*(\ ;F_p)$ isomorphisms. Hence 1 is an $H_*(\ ;F_p)$ monomorphism.//

Lemma 3.7: Suppose $\{Y_\alpha\}$ is a collection of strong finite type and suppose that each $E_r(Y_\alpha)$ is true. Then $E_r(\bigvee_\alpha Y_\alpha)$ is also true.

Proof: Since $Q(\bigvee_\alpha Y_\alpha) = \prod_\alpha Q(Y_\alpha)$ we have $\mathrm{Map}(X,Q(\bigvee_\alpha Y_\alpha)) = \prod_\alpha \mathrm{Map}(X,Q(Y_\alpha))$. Since $\{\mathrm{Map}(X,Q(Y_\alpha))\}$ has strong finite type if X satisfies $i)_r$, $\mathrm{Map}(X,Q(\bigvee_\alpha Y_\alpha))$ and $\underset{\alpha}{\times}\mathrm{Map}(X,Q(Y_\alpha))$ are weak equivalent. The result follows easily from these remarks.//

Our next result produces lots of examples which satisfy E_r.

Proposition 3.8: Let Y be simply-connected and of finite type. Suppose $b(Y) - \ell(Y) \geq r$. Then $E_r(Y)$ is true.

Proof: Since $\ell(Y) \geq 1$, Y is r-connected. The result will follow easily once we can compute $\mathrm{Map}(X,Q(Y))$, so we begin to describe the answer. The first step is to replace X by a space which more accurately reflects the mod-p properties of X. First note that by lemmas 2.1 and 2.2 we can mod out by a p-acyclic complex, X_0, so that $\mathrm{Map}(X/X_0,Q(Y)) \longrightarrow \mathrm{Map}(X,Q(Y))$ is a p-local weak equivalence. Note X/X_0 is $(b_p(X)-1)$-connected.

We can also find $\hat{X} \subset X/X_0$ so that $d(\hat{X}) = d_p(X/X_0)$ and again $\mathrm{Map}(X/X_0,Q(Y)) \text{——} \mathrm{Map}(\hat{X},Q(Y))$ is a p-local weak equivalence. If we have a map $h\colon X_1 \longrightarrow X_2$ we can find $\bar{h}\colon X_1/X_{10} \longrightarrow X_2/X_{20}$ and $\hat{h}\colon \hat{X}_1 \longrightarrow \hat{X}_2$ so that

$$\begin{tikzcd}
\mathrm{Map}(X_2,Q(Y)) \arrow[r, "\hat{h}^*"] & \mathrm{Map}(X_1,Q(Y)) \\
\mathrm{Map}(X_2/X_{20},Q(Y)) \arrow[r] \arrow[u] \arrow[d] & \mathrm{Map}(X_1/X_{10},Q(Y)) \arrow[u] \arrow[d] \\
\mathrm{Map}(X_2,Q(Y)) \arrow[r, "h^*"] & \mathrm{Map}(X_1,Q(Y))
\end{tikzcd}$$

commutes.

Thus, in verifying $E_r(Y)$, it suffices to demonstrate the required epicity under the further assumption that B and C satisfy $\hat{i})_r$, where we say that a space satisfies $\hat{i})_r$ if it satisfies $i)_r$ and if X is $(b(X)-1)$-connected with $d(X) + \ell(X) \leq r$.

Now if X satisfies $\hat{i})_r$ there exists a spectrum called the 0-Spanier-Whitehead dual to X, denoted X^*. It follows from the Freudenthal suspension theorem that $\Sigma^r X^*$ is the suspension spectrum of a CW complex.

The Freudenthal suspension theorem also gives that an r-connected complex of finite type with $b(Y) - \ell(Y) \geq r$ is an r-fold suspension. Hence $X^* \wedge Y$ can be taken to be a CW complex.

From Spanier-Whitehead duality we have a map $X \wedge (X^* \wedge Y) \longrightarrow Q(Y)$ which we can adjoint to get $\psi\colon X^* \wedge Y \longrightarrow \mathrm{Map}(X,Q(Y))$.

We remark that ψ is natural in X and Y. Given $h\colon X_1 \longrightarrow X_2$ we get a stable map $h^*\colon X_2^* \longrightarrow X_1^*$ and the Freudenthal suspension theorem gives that there is a real map $\Sigma^r X_2^* \longrightarrow \Sigma^r X_1^*$ realizing h . Hence we get a map $h^* \wedge 1\colon X_2^* \wedge Y \longrightarrow X_1^* \wedge Y$. If $g\colon Y_2 \longrightarrow Y_1$ is an r-fold suspension it is easy to define $h^* \wedge g\colon X_2^* \wedge Y \longrightarrow X_1^* \wedge Y$ and

$$\begin{tikzcd}
X_2^* \wedge Y_2 \arrow[r, dash, "\psi"] \arrow[d, "h^* \wedge g"] & \mathrm{Map}(X_2,Q(Y_2)) \arrow[d, "h^* \circ g^{\#}"] \\
X_1^* \wedge Y_1 \arrow[r, dash, "\psi"] & \mathrm{Map}(X_1,Q(Y_1))
\end{tikzcd}$$

commutes up to homotopy. For g to be an r-fold suspension, the Freudenthal suspension theorem requires $b(Y_2) \leq b(Y_1)$.

Consider the statement $T(X,Y)$.

$T(X,Y)$: $\psi\colon X^* \wedge Y$ —— $\mathrm{Map}(X,Q(Y)$ adjoints to an equivalence
$\psi_\infty\colon Q(X^* \wedge Y)$—— $\mathrm{Map}(X,Q(Y))$ using the ∞-loop space structure
in $\mathrm{Map}(X,Q(Y))$.

Our first goal is to prove that $T(X,Y)$ is true if X satisfies $\hat{i})_r$ and if Y is simply-connected of finite type with $b(Y) - \ell(Y) \geq r$.

The proof resembles the proof of the main theorem so much that we will only give a sketch. It suffices to show that ψ_∞ induces a mod-p homology isomorphism for all p. We will induct on $s(X^* \wedge Y)$.

If $s(X^* \wedge Y) = 1$, X and Y are mod-p spheres. For spheres the result is clear and by lemma 2.2 the result follows for mod-p homology spheres.

If $s(X^* \wedge Y) > 1$ we can find $i\colon S^m \longrightarrow X$ or $j\colon S^n \longrightarrow Y$ so that the induced map in mod-p homology is a monomorphism. If i exists, let X_0 be S^m and let Y_0 be Y: if i does not exist (i.e. $s(X^*)=1$) let X_0 be X and let Y_0 be the cofibre of j. Note that $Y \longrightarrow Y_0$ is always an r-fold suspension. Then

$$\begin{array}{ccc} Q(X^*\wedge Y) & \longrightarrow & Map(X,Q(Y)) \\ \downarrow & & \downarrow \\ Q(X_0^*\wedge Y_0) & \longrightarrow & Map(X_0,Q(Y_0)) \end{array}$$

commutes. Since $s(X_0^*\wedge Y_0) < s(X^*\wedge Y)$ we can assume by induction that the bottom map is a mod-p homology isomorphism. The mod-p homology Serre spectral sequence for the left-hand fibration collapses by direct calculation. Hence it also collapses for the right-hand fibration.

The fibres are $Q(X_1^*\wedge Y_1)$ and $Map(X_1,Q(Y_1))$ respectively for a suitable choice of X_1 and Y_1. The ψ_∞ on the fibres commutes with the ψ_∞ on the total spaces. Since $s(X_1^*\wedge Y_1) < s(X^*\wedge Y)$ we have our result.

Once we know that ψ_∞ is an equivalence the proposition is easy to prove. Since we may assume that B and C satisfy $\dagger)_r$, it suffices to recall that if $i:B \longrightarrow C$ is a mod-p homology monomorphism, $i:B^* \longrightarrow C^*$ is a mod-p homology epimorphism. This proves 3.8.//

Remark: The reader may well wonder how we have reduced Spanier's function space approach to S-duality to such trivial manipulations. What we use periodically is the work of Dyer-Lashof [5] and Araki-Kudo[1]. Life is a great deal easier when one knows that $H_*(Q(Z);F_p)$ is a functor of $H_*(Z;F_p)$ with the new classes being given by operations which are natural with respect to ∞-loop maps.

There exist spaces, such as $J(Y) = \Omega\Sigma Y$, which do not have finite ℓ but which are stably equivalent to a wedge of spaces which do. Precisely, let us consider spaces Y with the property that there is a collection $\{Y_\alpha\}$ of strong finite type which is r-connected so that $b(Y_\alpha) - \ell(Y_\alpha) \geq r$ for each $\alpha\varepsilon I$ and so that there is an equivalence in the stable category between Y and $\underset{\alpha}{\vee} Y_\alpha$. Any space with this property will be called an I_r-space.

Example 3.9: If Y is an I_r-space so is J(Y).

Proof: $J(Y) \underset{S}{\simeq} \underset{k=1}{\vee} Y^{[k]}$ where $Y^{[k]}$ denotes the k-fold smash. This is due to James[8]. If Y is an I_r-space, so is $Y^{[k]}$ by direct calculation.//

Example 3.10: Let Y be an I_r-space and suppose A is an I_0-space. Let $i:A \longrightarrow Y$ be a map, and let J(Y,A) denote the fibre of the map $Y/A \longrightarrow \Sigma A$ in the Barratt-Puppe sequence for i. Then J(Y,A) is an I_r-space.

Proof: B. Gray [7] has shown that $J(Y,A) \underset{S}{\simeq} \underset{k=0}{\vee} Y\wedge A^{[k]}$ (where $A^{[0]} = S^0$). Again the result follows from a short calculation.//

Corollary 3.11: $E_r(Y)$ is true for any I_r-space Y.

Proof: By 3.7 and 3.8 $E_r(\vee_\alpha Y_\alpha)$ is true. By 3.4, $E_r(Y)$ is also true.//

Remark: As concrete examples we have: for the Moore space $P^r(p^k)$, E_{r-2} is true ($P^r = S^{r-1} \cup e^r$); $E_{r-2}(\Omega P^{r+1})$ is true; if $S^r\{p^k\} \longrightarrow S^r \xrightarrow{p^k} S^r$ is a fibration, $E_{r-2}(S^r\{p^k\})$ is true.

Corollary 3.11 can be used to get many examples of spaces for which E_r is true. Some of these spaces have interesting retracts so, by 3.4, E_r is true for them also. The situation with regard to M_r is much less satisfactory. Lemma 3.6 deals satisfactorily with products and 3.1 allows us to loop down, but these are usually not useful procedures. Worse still is our substitute for proposition 3.8. We have

<u>Proposition 3.12</u>: Let Y be an r-connected mod-p homology sphere of dimension m+1. If pm is even, $M_r(Y)$ is true: $M_{r-1}(\Omega Y)$ is always true.

Proof: The first part is a direct calculation from [3]. Note that if pm is odd, $M_r(Y)$ is false.

If pm is even, $M_{r-1}(\Omega Y)$ is true by 3.1. If pm is odd, p is odd and (m+1) is even. Hence, at p, $\Omega Y \simeq S^m \times \Omega S^{2m+1}$: since $M_{m-1}(S^m)$ and $E_{2m}(\Omega S^{2m+1})$ are both true, it follows that $E_{m-1}(\Omega S^{m+1})$ is true and hence so is $M_{r-1}(\Omega Y)$.//

The Hilton-Milnor theorem can be used to push on a bit.

<u>Lemma 3.13</u>: Let $\{Y_\alpha\}$ be an r-connected collection of strong finite type. For any $L \in I^n$ let $Y_L = Y_{\alpha_1} \wedge \ldots \wedge Y_{\alpha_m}$. Suppose $M_r(J(Y_L))$ is true for all $L \in I^n$ and all n. Then $M_r(J(\vee_\alpha Y_\alpha))$ is true.

Proof: The Hilton-Milnor theorem [12] says that $J(\vee_\alpha Y_\alpha) = \times J(Y_L)$ as L runs over a specified set of sequences. The result follows easily from 3.6 once one checks that the collection $\{Y_L\}$ one acquires is of strong finite type.//

<u>Remarks</u>: Corollary 3.11 and proposition 3.12 give us examples 1,2 and 3 in the introduction. Example 4 follows from 3.11, 3.12, and 3.13. Example 5 follows from 3.11, 3.12 and

<u>Lemma 3.14</u>: Let $i: A \longrightarrow Y$ be an r-fold suspension and suppose $M_r(J(A))$ and $M_r(Y/A)$ are true. Then, if i induces a monomorphism in mod-p homology, $M_r(J(Y,A))$ is also true.

Proof:We have the fibration $J(A) \longrightarrow J(Y,A) \xrightarrow{\beta} Y/A$ and a map $J(Y,A) \longrightarrow J(Y)$ so that the composite $J(A) \longrightarrow J(Y,A) \longrightarrow J(Y)$ is just $J(i)$. Since i is an r-fold suspension and a mod-p homology monomorphism,

$\Omega^j J(A) \longrightarrow \Omega^j J(Y,A)$ is a mod-p homology monomorphism for $0 \le j \le r$. (This uses that $H_*(\Omega^n \Sigma^n X; F_p)$ is functorially determined by $H_*(X; F_p)$.) Hence the mod-p homology Serre spectral sequences for the fibrations

$\Omega^j J(A) \longrightarrow \Omega^j J(Y,A) \longrightarrow \Omega^j Y/A$ collapse.

Let F denote the homotopy theoretic fibre of the map $Q\beta$. We have a map

$Q(J(Y,A)) \longrightarrow Q(J(Y))$ and, just as above, $Q(J(A)) \longrightarrow F \longrightarrow Q(J(Y,A))$ is a mod-p homology monomorphism. This fails to prove triviality of the Serre spectral sequence since $H_*(F;F_p)$ is much larger than $H_*(Q(J(A));F_p)$. So we turn to the map $Q(J(Y,A)) \longrightarrow Q(Y/A)$. There is a map $Y \longrightarrow J(Y,A)$ so that the composite $Y \longrightarrow J(Y,A) \longrightarrow Y/A$ is the collapse map, which is onto in mod-p homology. Hence $Q(J(Y,A)) \longrightarrow Q(Y/A)$ is onto in mod-p homology. Since $Y \longrightarrow Y/A$ is an r-fold suspension, $\Omega^j Q(J(Y,A)) \longrightarrow \Omega^j Q(Y/A)$ is a mod-p homology epimorphism for $0 \leq j \leq r$ so the mod-p homology Serre spectral sequences for these fibrations collapse.

If $M_r(J(A))$ and $M_r(Y/A)$ are both true, it is an easy diagram chase to show that $M_r(J(Y,A))$ is true.//

Remark: We can apply example 3.10 and lemma 3.14 to show that the following space satisfies $ii)_{r-2}$. Let $\xi: S^{2k+r} \longrightarrow S^{r+k+1}$ be any map with rp even. Then the fibre of ξ satisfies $ii)_{r-2}$.

BIBLIOGRAPHY

1.S. Araki and T. Kudo, Topology of H_n-spaces and H-squaring operations, Mem. Fac. Sci. Kyusyu Univ. Ser. A 10 (1956), 85-120.

2.K. Borsuk, Concerning the homological structure of the functional space $S_m{}^X$, Fund. Math. 39 (1952), 25-37.

3.F.R. Cohen, T.J. Lada, and J.P. May, The homology of iterated loop spaces, Lecture Notes in Math. no. 533, Springer, 1976.

4.F. R. Cohen, J.C. Moore, and J.A. Neisendorfer, The double suspension and exponents of the homotopy groups of spheres, Ann. of Math. (2) 110 (1979), 549-565.

5.E. Dyer and R.K. Lashof, Homology of iterated loop spaces, Amer. J. Math. 84 (1962), 35-88.

6.H. Federer, A study of function spaces by spectral sequences, Trans. Amer. Math. Soc. 82 (1956), 340-361.

7.B. Gray, On the homotopy groups of mapping cones, Proc. Adv. Study Inst. Alg. Top. (1970), 104-142, Aarhus, Denmark.

8.I. M. James, Reduced product spaces, Ann. of Math. (2) 62 (1955), 170-197.

9.S. Mardesic, Sur l'homologie de l'espace fonctionnel $S_m{}^X$ et la structure homologique de X, C.R. Acad.Sci. Paris 242 (1956), 1112-1114.

10.J.C. Moore, On a theorem of Borsuk, Fund. Math. 43 (1956), 195-201.

11.E.H. Spanier, Function spaces and duality, Ann. of Math. (2) 70 (1959), 338-378.

12.G.W. Whitehead, Elements of homotopy theory, Graduate Texts in Math. no. 61, Springer, 1978.

M.I.T., Department of Mathematics,
Cambridge, MA 02139

University of Notre Dame, Dept. of Math
Notre Dame, IN 46556