SURGERY GROUPS AND INNER AUTOMORPHISMS

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The object of this paper is to investigate the map induced on surgery groups by an inner automorphism of the group. To describe our results, let \( \omega : G \to \mathbb{Z}_2 \) be a homomorphism. \( L_n(G, \omega) \) is the \( n \)th Wall group. This notation is ambiguous as there are Wall groups \( L_n^\mathbb{Z} \) for simple homotopy equivalence, \( L_n^h \) for homotopy equivalence, and there are other possibilities (see section 1 for a thorough discussion). \( L_n \) denotes any of these groups.

\( \text{Aut}(G, \omega) \) denotes the group of automorphisms of \( G \) which preserve \( \omega \). There is a homomorphism \( \gamma : \text{Aut}(G, \omega) \to \text{Aut}(L_n(G, \omega)) \). Finally there is a standard homomorphism from \( \mathbb{Z}_2 \) to the automorphism group of any abelian group. If the group is written additively, \(-1 \in \mathbb{Z}_2 \) just goes to multiplication by \(-1\).

Theorem 1:

\[
\begin{array}{ccc}
G & \xrightarrow{\omega} & \text{Aut}(G, \omega) \\
& \searrow & \downarrow \\
& & \text{Aut}(L_n(G, \omega)) \\
& \swarrow & \nearrow \\
\mathbb{Z}_2 & \xrightarrow{\omega} & \text{Aut}(L_n(G, \omega))
\end{array}
\]

commutes.

Corollary 1.1: If \( G \to \mathbb{Z}_2 \) is trivial, then any inner automorphism induces the identity on \( L_n(G, \omega) \).

Corollary 1.2: If \( \text{Center}(G) \to \mathbb{Z}_2 \) is onto, then \( L_n(G, \omega) \) is a \( \mathbb{Z}_2 \)-vector space, and any inner automorphism induces the identity.

Here are examples of non-trivial actions on \( L_n(G, \omega) \). Let \( \alpha : \mathbb{Z} \to \text{Aut}(\mathbb{Z}) \iso \mathbb{Z}_2 \) be onto, and let \( K \) be the semi-direct product of \( \mathbb{Z} \) and \( \mathbb{Z} \) by \( \alpha \). There is a homomorphism \( \omega : K = \mathbb{Z} \times_\alpha \mathbb{Z} \to \mathbb{Z}_2 \). By Wall [5], page 171, \( L_1(K, \omega) \iso \mathbb{Z} \) and \( L_2(K, \omega) \iso \mathbb{Z} \oplus \mathbb{Z}_2 \). Any element \( k \in K \) such that \( \omega(k) = -1 \) gives an inner automorphism of \( K \).
which does not induce the identity on $L_1$ and $L_2$, $K \times Z \times Z$ with the obvious $\omega$ has inner automorphisms which are the identity on neither $L_1$, $L_2$, $L_3$, nor $L_4$.

Cappell and Shaneson [1] have defined surgery groups $\Gamma_n(\mathcal{F})$, where $\mathcal{F} : Z[G] \rightarrow A'$ is an epimorphism of rings with involution. Any $g \in G$ induces an automorphism of $\mathcal{F}$ by acting on $Z[G]$ via conjugation by $g$ and on $A'$ via conjugation by the image of $g$ in $A'$. This gives a homomorphism $G \rightarrow \text{Aut}(\mathcal{F})$. The $\Gamma_n$ are functors, so there is a homomorphism $\text{Aut}(\mathcal{F}) \rightarrow \text{Aut}(\Gamma_n(\mathcal{F}))$. As with Wall groups, $\Gamma_n$ denotes any of the possible torsions or projective classes which can be used to manufacture Cappell-Shaneson groups.

Theorem 2:

$$
\begin{array}{ccc}
G & \rightarrow & \text{Aut}(\mathcal{F}) \\
\downarrow w & & \downarrow \\
Z_2 & \rightarrow & \text{Aut}(\Gamma_n(\mathcal{F}))
\end{array}
$$

commutes.

There are also relative Wall and $\Gamma$ groups associated to any groupoid of finite type. If $\mathcal{H}$ is the groupoid, there is given a homomorphism of groupoids $\omega : \mathcal{H} \rightarrow Z_2$. $\text{Aut}(\mathcal{H}, \omega)$ will denote the automorphism group of the groupoid whose elements preserve $\omega$. Each element of $\mathcal{H} \rightarrow Z_2$ gives a collection of automorphisms, $\mathcal{r} G$, where $\mathcal{r} G$ is just the inner automorphism on $G$ given by the component of $G$ in the product.

$$(\mathcal{H}) \subseteq \mathcal{H} \rightarrow Z_2$$

is the subgroup such that $\{\mathcal{r} G\} \in \text{Aut}(\mathcal{H})$. $(\mathcal{H}) \subseteq \mathcal{H}$ is the subgroup such that $\omega(g_{G_1}) = \omega(g_{G_2})$ where $G_1, G_2 \in \mathcal{H}$ are arbitrary and $g_0$ is the component of $G$ in the product.

Conjecture:

$$
\begin{array}{ccc}
(\mathcal{H}) \subseteq Z_2 & \rightarrow & \text{Aut}(\mathcal{H}, \omega) \\
\downarrow w & & \downarrow \\
Z_2 & \rightarrow & \text{Aut}(\Gamma_n(\mathcal{H}, \omega))
\end{array}
$$

commutes.

There is a similar conjecture for $\Gamma$ groups.

Almost all that I can prove is

Theorem 3: The conjecture is true for the Wall groups $L_{2k}, L_{2k+1}^h$ and $L_{2k+1}^h$ if $\mathcal{H}$ is a pair. I know nothing about relative $\Gamma$ groups.

Remarks: The obvious versions of Corollaries 1.1 and 1.2 are
valid for Theorems 2 and 3, and all are proved by simple diagram chasing.

Notice that the groups are not assumed to be finitely generated or finitely presented.

Section 1: Preliminaries.

Wall and $\Gamma$ groups can actually be defined for any rings with involution, not just integral group rings. Even Wall and $\Gamma$ groups are obtained by a Grothendieck construction from modules with quadratic form plus additional structure. If $A$ is a ring with involution, and, if $A \subset K_0(A)$ is a subgroup invariant under the induced involution, then we can define Wall groups $L^{A}_{2k}(A, -)$ by insisting that our module be a projective module whose image in $\tilde{K}_0$ is contained in $A$. If $B \subset K_1(A)$ is invariant under the induced involution, and, if we insist that our modules be free and based, then we can define Wall groups $L^{B}_{2k}(A, -)$ by insisting that the torsion of the adjoint map lie in $B$.

Even $\Gamma$ groups can also be defined. Given $\mathcal{F} : \Lambda \to \Lambda'$ a local epimorphism (see [1]) we have quadratic modules over $\Lambda$ with nice properties when tensored over $\Lambda'$. If $A \subset K_0(A')$ and $B \subset K_1(A')$ are invariant subgroups, we can define $\Gamma^{A}_{2k}(\mathcal{F})$ and $\Gamma^{B}_{2k}(\mathcal{F})$ by insisting in the first case that our modules when tensored are projective with class in $A$ and in the second case that our modules are free and based over $A$ and the adjoint map when tensored over $A'$ has torsion in $B$ (this uses lemma 1.2 of [1]).

Hopefully the above motivates considering $Q_{\mathcal{K}}(\Lambda, -)$, the category whose objects are $\xi \in [F, \Lambda, \mu]$ where $F$ is a right $\Lambda$-module; $\lambda : F \times F \to \Lambda$ is a map; $\mu : F \to A/(x^{\pm 1})^{K_{2k}}$ is a map; and the five relations of Wall [5] page 45 are satisfied (with $\chi_n(x) = 0$). Morphisms are just $\Lambda$-module homomorphisms which preserve this extra structure. $Q_{\mathcal{K}}(\Lambda, -)$ is a similar category: the only difference is that now we require $F$ to be free and based. The first two paragraphs of this section just say that Wall and $\Gamma$ groups, with any torsions, etc. are constructed.
naturally from various subcategories of $Q_k$ and $Q_k^t$. We remark that any map of rings with involution induces a functor on $Q_k$ and on $Q_k^t$.

Odd Wall and $\Gamma$ groups are more tricky. $L_{2k+1}^B$ can be defined as a quotient of $U(A;B)$, where $U(A;B) \subseteq U(A)$ is the subgroup of the infinite unitary group over $A$ whose elements are those matrices in $U(A)$ with torsions in $B$. $L_{2k+1}^A$ can be defined directly (Novikov [4]) or by a theorem of Farrell-Wagoner which equates $L_{2k+1}^B$ to $L_{2k}$ of some other ring. We will use this latter method.

$\Gamma_{2k+1}^B$ can be defined as those elements of $L_{2k+1}^B(A')$ which come from elements in $U(A)$ with a special property. One might define $\Gamma_{2k+1}^A = \{ x \in L_{2k+1}^A(A') | 2x \text{ is in the image of } K_{1}(A') \rightarrow K_1(A') \rightarrow \frac{A}{\mathcal{F}} \rightarrow L_{2k+1}(A') \rightarrow \Pi_{2k+1}(A') \}$. I know of no use for groups like $\Gamma^A$, so the definition is not very important.

As is usual with surgery groups, the proof divides into an even and an odd case.

Section 2: The Even Case.

Let $A$ be a ring with involution $-\dagger$, and suppose $u$ is a unit of $A$ such that $u^{-1}u = \pm 1$. It follows that $u^{-1}u = u^{-1}u$. There is an isomorphism of rings with involution $r_u : A \rightarrow A$ given by $r_u(x) = u^{-1}xu$. Any such map induces functors $u : Q_k(A, -) \rightarrow Q_k(A, -)$ and $u : Q_k^t(A, -) \rightarrow Q_k^t(A, -)$.

On the category $Q_k$ (or $Q_k^t$) there is a functor $c$ such that $c([P, \lambda, \mu]) = [P, -\lambda, -\mu]$. Note $c \circ c$ is the identity.

Theorem: Let $\xi \in Q_k(A, -)$. Then $u(\xi)$ is naturally isomorphic to $\{ \xi \text{ if } u^{-1}u = 1 \}$ or $c(\xi)$ if $u^{-1}u = -1$. If $\xi \in Q_k^t(A, -)$ then the torsion of the equivalence is in $\{ u \}$, where $\{ u \}$ is the subgroup of $K_1(A)$ generated by the unit $u$.

proof: Under $u$, the module $P$ goes to $P \otimes A = P^H$, where $A$ is made into a $A$-module using $r_u$. Let $i : P \rightarrow P^H$ be the natural map. Then $i$ is $r_u$-linear, that is, $x \cdot a = i(i^{-1}(x) \cdot uau^{-1})$ where $x$ is the product in $P^H$ and $\cdot$ is the product in $P$. $u(\xi)(x, y) = r_u(\lambda(i^{-1}(x), i^{-1}(y)))$ and
\[ u_\mu(x) = r_u^k \mu(1^{-1}(x)) \text{ where } r_u^k \text{ is the map induced on } \Lambda/\{x+(-1)^k x\} \text{ by } r_u. \text{ If } P \text{ is based, use } 1 \text{ to base } P_\mu. \]

Consider \( \xi^u = [P^u, \lambda_u, \mu_u] \) where \( \lambda_u(x,y) = \overline{u} \lambda(1^{-1}(x),1^{-1}(y))u \) and \( \mu_u(x) = \overline{u} \mu(1^{-1}(x))u. \) Clearly

\[ \xi^u = \begin{cases} 
    \xi_u(\xi) & \text{if } u\overline{u} = 1, \\
    c(\xi_u(\xi)) & \text{if } u\overline{u} = -1
\end{cases} \]

Hence \( \xi^u \in Q_\text{K}^r \) or \( Q_\text{K}^l. \)

Define \( j:P \to P^u \) by \( j(x) = 1(x) \times u^{-1} = 1(x \cdot u^{-1}). \) \( j \) is a \( A \)-module isomorphism and one easily checks

\[ \begin{align*}
    j \times j & \quad \text{and} \\
    P \times P & \quad \text{and} \\
    P^u \times P^u & \quad \text{and}
\end{align*} \]

\[ \begin{array}{c}
    P \xrightarrow{\lambda} A \\
    P \xrightarrow{\mu} A/\{x+(-1)^k x\}
\end{array} \]

commute. Hence, in \( Q_\text{K}(A,-) \), \( \xi \) and \( \xi^u \) are isomorphic.

If \( \xi \in Q_\text{K}(A,-) \) then the torsion of the isomorphism clearly lies in \( \{u\}. \)

Since \( c \circ c = \text{id} \), \( u_\Psi(\xi) = \begin{cases} 
    \xi & \text{if } u\overline{u} = 1, \\
    c(\xi) & \text{if } u\overline{u} = -1
\end{cases} \) where \( c \) equal means isomorphic. It is easy to check that \( j \) gives a natural isomorphism. Q.E.D.

The theorem plus the discussion in section 1 almost proves Theorems 1 and 2. We need only further note that, if \( \xi \) goes to an element of \( L_{2k}^r \) (or \( L_{2k}^l \)), then \( c(\xi) \) goes to the inverse element.

For Theorem 3, we have the following situation. We have rings with involution \( A \) and \( A_1, \ldots, A_n \); maps \( h_1:A_1 \to A \) of rings with involution; and units \( u \in A \), \( u_1 \in A_1 \) such that \( u\overline{u} = u_1 \overline{u}_1 = +1 \) and each

\[ \begin{array}{c}
    A_1 \xrightarrow{h_1} A \\
    r_{u_1} \circ h_1 \quad \text{and} \\
    r_u \circ h_1 \quad \text{and}
\end{array} \]

commutes.

Wall’s description ([5] page 72) of the groups \( L_{2k}^r(A;A_1,\ldots,A_n) \) (dropping mention of torsions gives \( L_{2k}^r \)) and our theorem easily show \( r_u \) induces \( \begin{cases} 
    \text{id} & \text{if } u\overline{u} = 1, \\
    \text{id} & \text{if } u\overline{u} = -1
\end{cases} \) on these relative Wall groups.

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Section 3: The Odd Case

Again A is a ring with involution - and u is a unit such that 
\( u^2 = 1 \). \( r_u \) induces a map on \( U(A) \) which covers the map induced on Wall groups. \( U(A) \) is the limit of the finite unitary groups \( U_{2n}(A) \).

\( r_u \) induces maps \( u_\#: U_{2n}(A) \rightarrow U_{2n}(A) \) which takes the matrix \( a_{ij} \) to \( (u^{-1}a_{ij}u) \). Let \( c_n(u) \) be the diagonal matrix

\[
\begin{pmatrix}
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

with \( u^{-1} \) in the first \( n \) positions and \( u \) in the last \( n \). Given

\( M \in U_{2n}(A) \), \( M' \) denotes the matrix with basis \( e_1, f_1 \) changed to \( e_1, -f_1 \) (see Wall [5] page 62).

Theorem: \( c_n(u) \in U_{2n}(A) \) and if \( M \in U_{2n}(A) \),

\[
u_\#(M) = \begin{cases} c_n(u)^*Mc_n(u)^{-1} & \text{if } u^{-1}u = 1 \\ c_n(u)^*M'c_n(u)^{-1} & \text{if } u^{-1}u = -1 \end{cases}
\]

Proof: That \( c_n(u) \in U_{2n}(A) \) is just a formal check. Our equation for \( u_\#(M) \) is obvious. Q.E.D.

Since Wall groups are abelian, and since \( M' \) is the inverse in the Wall group, we have proved Theorem 1 for \( u \in B \in K_1(A) \).

Now Farrell-Wagoner [3] showed \( L^A_{2k-1}(\mathbb{Z},-) \) and \( L^A_{2k}(\mathbb{Z},-) \) are naturally isomorphic. \( \lambda A = A/\mathfrak{M} A \), where \( \mathfrak{M} A \) is the ring of locally-finite matrices over \( A \) and \( \mathfrak{M} A \) is the ideal in \( A \) of matrices with only finitely many non-zero entries. Note \( A \in K_0(A) = K_1(\mathfrak{M} A) \) by Farrell-Wagoner [2].

Naturality shows \( r_u \) and \( r_\mu \) commute,

\[
\begin{array}{ccc}
L^A_{2k-1}(\mathbb{Z},-) & \xrightarrow{ru} & L^A_{2k-1}(\mathbb{Z},-) \\
\downarrow L^A_{2k}(\mathbb{Z},-) & & \downarrow L^A_{2k}(\mathbb{Z},-) \\
L^A_{2k}(\mathfrak{M} A) & \xrightarrow{r_\mu} & L^A_{2k}(\mathfrak{M} A)
\end{array}
\]

where \( \mu \) is the infinite diagonal matrix with all \( u \)'s. Our result for the even case now carries over to this case.

The groups \( \Gamma_{2k-1} \) are subgroups of odd Wall groups so we are done in this case also.
For $L_{2k-1}^h(A_1^1;\ldots,A_n^1)$ we use a relative version of the above
Farrell-Wagoner theorem to place ourselves in the case
$L_{2k}^S(\lambda A_1;\ldots,\lambda A_n^1)$, which we know.

The reader may wonder why we do not just use the well known
Shaneson-Wall splitting theorem instead of this strange theorem of
Farrell-Wagoner. The point is that the Shaneson-Wall formula is known
only for finitely generated, finitely presented integral group rings
and rings with $1/2$, whereas the Farrell-Wagoner formula is valid for
an arbitrary ring with involution. Non-finitely generated groups play
a role in surgery on paracompact manifolds, so we wish to avoid any
finiteness assumptions on $G$.

1. S. Cappell and J. Shaneson, The codimension two placement problem
   and homology equivalent manifolds, preprint, 1972.
2. T. Farrell and J. Wagoner, Infinite matrices in algebraic K-theory
4. S. Novikov, Algebraic construction and properties of Hermitian
   analogs of K-theory over rings with involution from the viewpoint
   of Hamiltonian formalism. Applications to differential topology