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SURGERY GROUPS AND INNER AUTOMORPHISMS

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The object of this paper is to investigate the map induced on surgery groups by an inner automorphism of the group. To describe our results, let $\omega: G \longrightarrow Z_2$ be a homomorphism. $L_n(G,\omega)$ is the n^{th} Wall group. This notation is ambiguous as there are Wall groups L_n^s for simple homotopy equivalence, L_n^h for homotopy equivalence, and there are other possibilities (see section 1 for a thorough discussion). L_n denotes any of these groups.

$\mathrm{Aut}(G,\omega)$ denotes the group of automorphisms of G which preserve ω. There is a homomorphism $G \longrightarrow \mathrm{Aut}(G,\omega)$ which takes an element in G to its inner automorphism. Wall groups are functors, so there is a homomorphism $\mathrm{Aut}(G,\omega) \longrightarrow \mathrm{Aut}(L_n(G,\omega))$. Finally there is a standard homomorphism from Z_2 to the automorphism group of any abelian group. If the group is written additively, $-1 \in Z_2$ just goes to multiplication by -1.

Theorem 1:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(G,\omega) \\ \downarrow{\scriptstyle\omega} & & \downarrow \\ Z_2 & \longrightarrow & \mathrm{Aut}(L_n(G,\omega)) \end{array}$$

commutes.

Corollary 1.1: If $G \xrightarrow{\omega} Z_2$ is trivial, then any inner automorphism induces the identity on $L_n(G,\omega)$.

Corollary 1.2: If $(\text{Center } G) \xrightarrow{\omega} Z_2$ is onto, then $L_n(G,\omega)$ is a Z_2-vector space, and any inner automorphism induces the identity.

Here are examples of non-trivial actions on $L_n(G,\omega)$. Let $\alpha: Z \longrightarrow \mathrm{Aut}(Z) \cong Z_2$ be onto, and let K be the semi-direct product of Z and Z by α. There is a homomorphism $\omega: K = Z \times_\alpha Z \longrightarrow Z \longrightarrow Z_2$ which is onto. By Wall [5], page 171, $L_1(K,\omega) \cong Z$ and $L_2(K,\omega) \cong Z \oplus Z_2$. Any element $k \in K$ such that $\omega(k) = -1$ gives an inner automorphism of K which does not induce the identity on L_1 and L_2. $K \times Z \times Z$ with the obvious ω has inner automorphisms which are the identity on neither L_1, L_2, L_3, nor L_4.

Cappell and Shaneson [1] have defined surgery groups $\Gamma_n(\mathcal{F})$, where $\mathcal{F}: Z[G] \longrightarrow \Lambda'$ is an epimorphism of rings with involution. Any $g \in G$ induces an automorphism of $\mathcal{F}$ by acting on $Z[G]$ via conjugation by g and on Λ' via conjugation by the image of g in Λ'. This gives a homomorphism $G \longrightarrow \mathrm{Aut}(\mathcal{F})$. The Γ_n are functors, so there is a homomorphism $\mathrm{Aut}(\mathcal{F}) \longrightarrow \mathrm{Aut}(\Gamma_n(\mathcal{F}))$. As with Wall groups, Γ_n denotes any of the possible torsions or projective classes which can be used to manufacture Cappell-Shaneson groups.

Theorem 2:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(\mathcal{F}) \\ \downarrow{\scriptstyle\omega} & & \downarrow \\ Z_2 & \longrightarrow & \mathrm{Aut}(\Gamma_n(\mathcal{F})) \end{array}$$

commutes.

There are also relative Wall and Γ groups associated to any groupoid of finite type. If $\mathcal{G}$ is the groupoid, there is given a homomorphism of groupoids $\omega: \mathcal{G} \longrightarrow Z_2$. $\mathrm{Aut}(\mathcal{G}, \omega)$ will denote the automorphism group of the groupoid whose elements preserve ω. Each element of $\prod_{G \in \mathcal{G}} G$ gives a collection of automorphisms, f_G, where f_G is just the inner automorphism on G given by the component of G in the product. $(\mathcal{G}) \subseteq \prod_{G \in \mathcal{G}} G$ is the subgroup such that $\{f_G\} \in \mathrm{Aut}(\mathcal{G})$. $(\mathcal{G})_\omega$ is the subgroup such that $\omega(g_{G_i}) = \omega(g_{G_j})$ where G_i, $G_j \in \mathcal{G}$ are arbitrary and g_G is the component of G in the product.

Conjecture:

$$\begin{array}{ccc} (\mathcal{G})_\omega & \longrightarrow & \mathrm{Aut}(\mathcal{G}, \omega) \\ \downarrow{\scriptstyle\omega} & & \downarrow \\ Z_2 & \longrightarrow & \mathrm{Aut}(L_n(\mathcal{G}, \omega)) \end{array}$$

commutes.

There is a similar conjecture for Γ groups.

Almost all that I can prove is

Theorem 3: The conjecture is true for the Wall groups L^s_{2k}, L^h_{2k}, and L^h_{2k+1} if $\mathcal{G}$ is a pair. I know nothing about relative Γ groups.

Remarks: The obvious versions of Corollaries 1.1 and 1.2 are valid for Theorems 2 and 3, and all are proved by simple diagram chasing.

Notice that the groups are not assumed to be finitely generated or finitely presented.

Section 1: Preliminaries.

Wall and Γ groups can actually be defined for any rings with involution, not just integral group rings. Even Wall and Γ groups are obtained by a Grothendieck construction from modules with quadratic form plus additional structure. If Λ is a ring with involution, and, if $A \subset \tilde{K}_0(\Lambda)$ is a subgroup invariant under the induced involution, then we can define Wall groups $L^A_{2k}(\Lambda,-)$ by insisting that our module be a projective module whose image in $\tilde{K}_0$ is contained in A. If $B \subset \tilde{K}_1(\Lambda)$ is invariant under the induced involution, and, if we insist that our modules be free and based, then we can define Wall groups $L^B_{2k}(\Lambda,-)$ by insisting that the torsion of the adjoint map lie in B.

Even Γ groups can also be defined. Given $\mathcal{F}:\Lambda \longrightarrow \Lambda'$ a local epimorphism (see [1]) we have quadratic modules over Λ with nice properties when tensored over Λ'. If $A \subset \tilde{K}_0(\Lambda')$ and $B \subset \tilde{K}_1(\Lambda')$ are invariant subgroups, we can define $\Gamma^A_{2k}(\mathcal{F})$ and $\Gamma^B_{2k}(\mathcal{F})$ by insisting in the first case that our modules when tensored are projective with class in A and in the second case that our modules are free and based over Λ and the adjoint map when tensored over Λ' has torsion in B (this uses lemma 1.2 of [1]).

Hopefully the above motivates considering $Q_k(\Lambda,-)$, the category whose objects are $\xi \in [P, \lambda, \mu]$ where P is a right Λ-module; $\lambda:P \times P \longrightarrow \Lambda$ is a map; $\mu:P \longrightarrow \Lambda/\{x+(-1)^k\bar{x}\}$ is a map; and the five relations of Wall [5] page 45 are satisfied (with $\chi_N(x) = 0$). Morphisms are just Λ-module homomorphisms which preserve this extra structure. $Q^t_k(\Lambda,-)$ is a similar category: the only difference is that now we require P to be free and based. The first two paragraphs of this section just say that Wall and Γ groups, with any torsions,etc. are constructed naturally from various subcategories of Q_k and Q_k^t. We remark that any map of rings with involution induces a functor on Q_k and on Q_k^t.

Odd Wall and Γ groups are more tricky. L_{2k+1}^B can be defined as a quotient of $U(\Lambda;B)$, where $U(\Lambda;B) \subseteq U(\Lambda)$ is the subgroup of the infinite unitary group over Λ whose elements are those matrices in $U(\Lambda)$ with torsions in B. L_{2k+1}^A can be defined directly (Novikov [4]) or by a theorem of Farrell-Wagoner which equates L_{2k+1} to L_{2k} of some other ring. We will use this latter method.

Γ_{2k+1}^B can be defined as those elements of $L_{2k+1}^B(\Lambda')$ which come from elements in $U(\Lambda)$ with a special property. One might define $\Gamma_{2k+1}^A = \{ x \in L_{2k+1}^A(\Lambda') \mid 2x$ is in the image of $\Gamma_{2k+1}^{\tilde{K}_1(\Lambda')}(\mathcal{F}) \longrightarrow L_{2k+1}^{K_1(\Lambda')}(\Lambda') \longrightarrow L_{2k+1}^A(\Lambda') \}$. I know of no use for groups like Γ^A, so the definition is not very important.

As is usual with surgery groups, the proof divides into an even and an odd case.

Section 2: The Even Case.

Let Λ be a ring with involution $-$, and suppose u is a unit of Λ such that $u\bar{u} = \pm 1$. It follows that $\bar{u}u = u\bar{u}$. There is an isomorphism of rings with involution $r_u: \Lambda \longrightarrow \Lambda$ given by $r_u(x) = u^{-1}xu$. Any such map induces functors $u_*: Q_k(\Lambda,-) \longrightarrow Q_k(\Lambda,-)$ and $u_*: Q_k^t(\Lambda,-) \longrightarrow Q_k^t(\Lambda,-)$.

On the category Q_k(or Q_k^t) there is a functor c such that $c([P, \lambda, \mu]) = [P, -\lambda, -\mu]$. Note $c \circ c$ is the identity.

Theorem: Let $\xi \in Q_k(\Lambda,-)$. Then $u_*(\xi)$ is naturally isomorphic to $\begin{cases} \xi & \text{if } u\bar{u} = 1 \\ c(\xi) & \text{if } u\bar{u} = -1 \end{cases}$. If $\xi \in Q_k^t(\Lambda,-)$ then the torsion of the equivalence is in $\{u\}$, where $\{u\}$ is the subgroup of $\tilde{K}_1(\Lambda)$ generated by the unit u.

proof: Under u_*, the module P goes to $P \otimes_\Lambda \Lambda = P^u$. where Λ is made into a Λ-module using r_u. Let $i: P \longrightarrow P^u$ be the natural map. Then i is r_u-linear, that is, $x \times a = i(i^{-1}(x) \cdot uau^{-1})$ where $\times$ is the product in P^u and $\cdot$ is the product in P. $u_*\lambda(x,y) = r_u(\lambda(i^{-1}(x), i^{-1}(y)))$ and

$u_*\mu(x) = r_u^k\mu(i^{-1}(x))$ where r_u^k is the map induced on $\Lambda/\{x+(-1)^k\bar{x}\}$ by r_u. If P is based, use i to base P^u.

Consider $\xi^u = [P^u, \lambda_u, \mu_u]$ where $\lambda_u(x,y) = \bar{u}\,\lambda(i^{-1}(x),i^{-1}(y))u$ and $\mu_u(x) = \bar{u}\,\mu(i^{-1}(x)u$. Clearly

$\xi^u = \begin{cases} u_*(\xi) & \text{if } u\bar{u} = 1 \\ c(u_*(\xi)) & \text{if } u\bar{u} = -1 \end{cases}$. Hence $\xi^u \in Q_k$ or Q_k^t.

Define $j: P \longrightarrow P^u$ by $j(x) = i(x) \times u^{-1} = i(x\cdot u^{-1})$. j is a Λ-module isomorphism and one easily checks

$$\begin{array}{ccc} P \times P & \xrightarrow{\lambda} & \Lambda \\ \downarrow j\times j & \nearrow \lambda_u & \\ P^u \times P^u & & \end{array} \quad \text{and} \quad \begin{array}{ccc} P & \xrightarrow{\mu} & \Lambda/\{x+(-1)^k\bar{x}\} \\ \downarrow j & \nearrow \mu_u & \\ P^u & & \end{array}$$

commute. Hence, in $Q_k(\Lambda,-)$, ξ and ξ^u are isomorphic.

If $\xi \in Q_k^t(\Lambda,-)$ then the torsion of the isomorphism clearly lies in $\{u\}$.

Since $c \circ c = id$, $u_*(\xi) = \begin{cases} \xi & \text{if } u\bar{u} = 1 \\ c(\xi) & \text{if } u\bar{u} = -1 \end{cases}$, where here equal means isomorphic. It is easy to check that j gives a natural isomorphism. Q.E.D.

The theorem plus the discussion in section 1 almost proves Theorems 1 and 2. We need only further note that, if ξ goes to an element of L_{2k} (or Γ_{2k}), then $c(\xi)$ goes to the inverse element.

For Theorem 3, we have the following situation. We have rings with involution Λ and $\Lambda_1,\ldots,\Lambda_n$; maps $h_i:\Lambda_i \longrightarrow \Lambda$ of rings with involution; and units $u \in \Lambda$, $u_i \in \Lambda_i$ such that $u\bar{u} = u_i\bar{u}_i = \pm 1$ and each

$$\begin{array}{ccc} \Lambda_i & \xrightarrow{h_i} & \Lambda \\ \downarrow r_{u_i} & & \downarrow r_u \\ \Lambda_i & \xrightarrow{h_i} & \Lambda \end{array}$$

commutes.

Wall's description ([5] page 72) of the groups $L_{2k}^s(\Lambda;\Lambda_1,\ldots,\Lambda_n)$ (dropping mention of torsions gives L_{2k}^h) and our theorem easily show r_u induces $\begin{cases} id & \text{if } u\bar{u} = 1 \\ -id & \text{if } u\bar{u} = -1 \end{cases}$ on these relative Wall groups.

Section 3: The Odd Case

Again Λ is a ring with involution - and u is a unit such that $u\bar{u} = \pm 1$. r_u induces a map on $U(\Lambda)$ which covers the map induced on Wall groups. $U(\Lambda)$ is the limit of the finite unitary groups $U_{2n}(\Lambda)$. r_u induces maps $u_*: U_{2n}(\Lambda) \longrightarrow U_{2n}(\Lambda)$ which takes the matrix (a_{ij}) to $(u^{-1}a_{ij}u)$. Let $c_n(u)$ be the diagonal matrix

$$\left(\begin{array}{ccc|ccc} u^{-1} & 0 & & & 0 & \\ 0 & \ddots & & & & \\ & & u^{-1} & & & \\ \hline & & & \bar{u} & 0 & \\ & 0 & & 0 & \ddots & \\ & & & & & \bar{u} \end{array}\right)_{2n \times 2n}$$

with u^{-1} in the first n positions and $\bar{u}$ in the last n. Given $M \in U_{2n}(\Lambda)$, M' denotes the matrix with basis e_i, f_i changed to e_i, $-f_i$ (see Wall [5] page 62).

Theorem: $c_n(u) \in U_{2n}(\Lambda)$ and if $M \in U_{2n}(\Lambda)$,

$$u_*(M) = \begin{cases} c_n(u)\cdot M\cdot c_n(u)^{-1} & \text{if } u\bar{u} = 1 \\ c_n(u)\cdot M'\cdot c_n(u)^{-1} & \text{if } u\bar{u} = -1 \end{cases}.$$

proof: That $c_n(u) \in U_{2n}(\Lambda)$ is just a formal check. Our equation for $u_*(M)$ is obvious. Q.E.D.

Since Wall groups are abelian, and since M' is the inverse in the Wall group, we have proved Theorem 1 for $\{u\} \subseteq B \subseteq \tilde{K}_1(\Lambda)$.

Now Farrell-Wagoner [3] showed $L^A_{2k-1}(\Lambda,-)$ and $L^A_{2k}(\lambda\Lambda,-)$ are naturally isomorphic. $\lambda\Lambda = \ell\Lambda/m\Lambda$, where $\ell\Lambda$ is the ring of locally-finite matrices over Λ and $m\Lambda$ is the ideal in Λ of matrices with only finitely many non-zero entries. Note $A \subseteq K_0(\Lambda) = \tilde{K}_1(\lambda\Lambda)$ by Farrell-Wagoner [2].

Naturality shows
$$\begin{array}{ccc} L^A_{2k-1}(\Lambda) & \xrightarrow{r_u} & L^A_{2k-1}(\Lambda) \\ \downarrow & & \downarrow \\ L^A_{2k}(\lambda\Lambda) & \xrightarrow{r_\mu} & L^A_{2k}(\lambda\Lambda) \end{array}$$
commutes, where μ is the infinite diagonal matrix with all u's. Our result for the even case now carries over to this case.

The groups Γ_{2k-1} are subgroups of odd Wall groups so we are done in this case also.

For $L^h_{2k-1}(\Lambda;\Lambda_1,\ldots,\Lambda_n)$ we use a relative version of the above Farrell-Wagoner theorem to place ourselves in the case $L^s_{2k}(\lambda\Lambda;\lambda\Lambda_1,\ldots,\lambda\Lambda_n)$, which we know.

The reader may wonder why we do not just use the well known Shaneson-Wall splitting theorem instead of this strange theorem of Farrell-Wagoner. The point is that the Shaneson-Wall formula is known only for finitely generated, finitely presented integral group rings and rings with 1/2, whereas the Farrell-Wagoner formula is valid for an arbitrary ring with involution. Non-finitely generated groups play a role in surgery on paracompact manifolds, so we wish to avoid any finiteness assumptions on G.

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