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$K(Z, 0)$ AND $K(Z_2, 0)$ AS THOM SPECTRA

BY

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Let X be a connected space. For a map $f\colon X \to BO$ or $f\colon X \to BF$, one can choose a filtration of f by restrictions $F_pX \to BO(n_p)$ or $F_pX \to BF(n_p)$, pull back the universal bundles or spherical fibrations, take Thom complexes, and so obtain a Thom spectrum Mf. These Thom spectra of maps were first introduced and studied by Barratt [unpublished] and Mahowald [10], [11]; their original work in this direction dates back to the early 1970's. A detailed analysis of this construction has recently been given by Lewis [2], [8]. Mf is independent of the choice of filtration and depends only on the homotopy class of f. For $f\colon X \to BO$, Mf is the same as $M(jf)$, $j\colon BO \to BF$. There is a Thom isomorphism under the evident orientability assumption.

If X is an H-space and f is an H-map, then Mf admits a product $Mf \wedge Mf \to Mf$ with two-sided unit (in the stable category). Here subtleties begin to enter. If X is homotopy associative, it need not follow that Mf is associative unless X and BO or BF admit associating homotopies compatible under f. However, the homology Thom isomorphism commutes with products and so the relevant homology algebras will be associative even if Mf is not. Lewis has determined the precise higher multiplicative structure present on Mf when X is an n-fold loop space and f is an n-fold loop map, but we shall not need anything so elaborate.

Mf is (-1)-connected, and $\pi_0 Mf$ is a cyclic group since Mf can be constructed to have a single zero cell. If f is non-orientable, so that $f^*(w_1) \neq 0$, then the zero cell extends over the Moore spectrum and $\pi_0 Mf = Z_2$. If f is both non-orientable and an H-map, then $\pi_* Mf$ is a Z_2 vector space and thus Mf is 2-local.

Our purpose in this note is to give a simple proof of the following striking theorem of Mark Mahowald [11].

THEOREM 1. *There is a map $f\colon \Omega^2S^3\langle 3\rangle \to BSF$ whose associated Thom spectrum is the Eilenberg-Mac Lane spectrum $K(Z, 0)$.*

Here $S^3\langle 3\rangle$ is the 3-connective cover of S^3. This is analogous to the following earlier result of Mahowald [10, 4.5].

THEOREM 2. *The Thom spectrum associated to the second loop map $\bar{\eta}\colon \Omega^2S^3 \to BO$ determined by the non-trivial map $\eta\colon S^1 \to BO$ is the Eilenberg-Mac Lane spectrum $K(Z_2, 0)$.*

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Madsen and Milgram sketched [9, p. 49] and Priddy completed [15] an easy proof of this result based on Kochman's calculation of the homology operations of BO. We shall give at least as easy a proof, using only Steenrod operations and not homology operations, of the following more general result.

THEOREM 3. (i) *The Thom spectrum associated to any H-map $\Omega^2 S^3 \to BF$ with non-zero first Stiefel-Whitney class is $K(Z_2, 0)$.*

(ii) *The Thom spectrum associated to any H-map $\Omega^2 S^3\langle 3\rangle \to BSF$ with non-zero second Stiefel-Whitney class and non-zero first Wu class at each odd prime is $K(Z, 0)$.*

We shall first prove Theorem 3 and then give its second part content by constructing an H-map of the prescribed sort, thus completing the proof of Theorem 1. We shall then comment on the bordism interpretation of these results and shall conclude with some observations, again due to Mahowald, about the Thom spectra obtained by restriction of $\bar{\eta}$ to the two natural filtrations of $\Omega^2 S^3$.

Let $Z_{(p)}$ denote the localization of Z at p. We shall prove the following cohomological assertion.

THEOREM 4. (i) *Let $f\colon \Omega^2 S^3 \to BF$ be an H-map with non-zero first Stiefel-Whitney class. Then the Thom class $\mu\colon Mf \to K(Z_2, 0)$ induces an isomorphism on mod 2 cohomology.*

(ii) *Let $f\colon \Omega^2 S^3\langle 3\rangle \to BSF$ be an H-map with non-zero second Stiefel-Whitney class. Then the Thom class $\mu\colon Mf \to K(Z_{(2)}, 0)$ induces an isomorphism on mod 2 cohomology.*

(iii) *Let $f\colon \Omega^2 S^3\langle 3\rangle \to BSF$ be an H-map with non-zero first mod p Wu class, $p > 2$. Then the Thom class $\mu\colon Mf \to K(Z_{(p)}. 0)$ induces an isomorphism on mod p cohomology.*

Theorem 3 is an immediate consequence. Indeed, in (i), both spectra are 2-local of finite type and μ is thus an equivalence. For the integral case, $\mu\colon Mf \to K(Z, 0)$ induces an isomorphism on integral homology and is thus an equivalence if and only if $\mu\colon Mf \to K(Z_{(p)}, 0)$ induces an isomorphism on mod p cohomology for each prime p.

The proof of Theorem 4 depends on the following observation, which is already enough to imply that MO and MF, MSO at $p = 2$, and MSF split as wedges of Eilenberg-Mac Lane spectra. An argument like this was first noticed by Peterson and Toda [14]. We write w_i for both the Stiefel-Whitney classes in mod 2 cohomology and the Wu classes in mod p cohomology for $p > 2$.

LEMMA 5. (i) *Let P_j^0 be the jth primitive basis element of the mod 2 Steenrod algebra; thus $P_1^0 = Sq^1$ and*

$$P_{k+1}^0 = [Sq^{2^k}, P_k^0] \quad \text{if } k \geq 1.$$

Then, in H^*MF for $k \geq 1$ and in H^*MSF for $k \geq 2$,

$$P_k^0 \mu = \mu \cup (w_{2^k - 1} + \textit{decomposables}),$$

where μ denotes the Thom class in mod 2 cohomology.

(ii) *Let Q_i, $i \geq 0$, and P_j^0, $j \geq 1$, be the standard primitive basis elements of the mod p Steenrod algebra for $p > 2$; thus $Q_0 = \beta$, $P_1^0 = P^1$,*

$$Q_{k+1} = [P^{p^k}, Q_k] \text{ for } k \geq 0 \quad \text{and} \quad P_{k+1}^0 = [P^{p^k}, P_k^0] \text{ for } k \geq 1.$$

Then, in H^*MSF for $k \geq 1$ (and setting $p(k) = 1 + p + \cdots + p^{k-1}$),

$$Q_k \mu = \mu \cup ((-1)^k \beta w_{p(k)} + \textit{decomposables})$$

and

$$P_k^0 \mu = \mu \cup ((-1)^{k+1} w_{p(k)} + \textit{decomposables}),$$

where μ denotes the Thom class in mod p cohomology.

Proof. With $P^i = Sq^i$ and Q_k and β ignored, the argument to follow applies verbatim in the case $p = 2$. The Wu relations give

(a) $P^r \beta^\varepsilon w_s \equiv (-1)^r (r, s(p-1) - r - 1 + \varepsilon) \beta^\varepsilon w_{r+s}$ mod decomposables.

In particular, a check of binomial coefficients yields

(b) $P^{p^j} \beta^\varepsilon w_{p(k+1) - p^j} \equiv 0$ if $1 - \varepsilon \leq j \leq k$ and $P^{p^j} w_{p^j + \cdots + p^k} \equiv w_{p^{j-1} + \cdots + p^k}$ if $1 \leq j \leq k$.

Using these facts, a simple induction on j gives

(c) $Q_j w_{p^j + \cdots + p^k} \equiv (-1)^j \beta w_{p(k+1)}$ and $P_j^0 w_{p^j + \cdots + p^k} \equiv (-1)^{j+1} w_{p(k+1)}$ for $1 \leq j \leq k$.

Since $P^i \mu = \mu \cup w_i$, the conclusion follows easily by use of induction on k, the Cartan formula, the first congruence of (b) with $j = k$, and the congruences of (c) with $j = k$.

The only further information we need to prove Theorem 4 is the structure of $H_* \Omega^2 S^3$ and $H_* \Omega^2 S^3 \langle 3 \rangle$ as Hopf algebras over the Steenrod algebra.

PROPOSITION 6. (i) *With mod 2 coefficients,*

$$H_* \Omega^2 S^3 = P\{x_n \mid n \geq 0\} \quad \text{and} \quad H_* \Omega^2 S^3 \langle 3 \rangle = P\{x_0^2, x_n \mid n \geq 1\} \subset H_* \Omega^2 S^3,$$

where x_0 is the fundamental class of $H_* S^1$ and $x_n = Q^{2^n} x_{n-1}$ for $n \geq 1$. Thus x_n is a primitive element of degree $2^{n+1} - 1$ and

$$Sq_*^{2k}(x_n^{2k}) = x_{n-1}^{2k+1} \quad \text{for } n \geq 1 \text{ and } k \geq 0.$$

(ii) *With mod p coefficients for $p > 2$,*

$$H_* \Omega^2 S^3 = E\{x_n \mid n \geq 0\} \otimes P\{\beta x_n \mid n \geq 1\}$$

and

$$H_*\Omega^2 S^3\langle 3\rangle = E\{x_n \mid n \geq 1\} \otimes P\{\beta x_n \mid n \geq 1\},$$

where x_0 *is the fundamental class of* $H_* S^1$ *and* $x_n = Q^{p^{n-1}} x_{n-1}$ *for* $n \geq 1$. *Thus* $\beta^\varepsilon x_n$ *is a primitive element of degree* $2p^n - 1 - \varepsilon$, $P_*^{p^k} x_n = 0$ *for* $n \geq 0$ *and* $k \geq 0$, *and*

$$P_*^{p^k}((\beta x_n)^{p^k}) = -(\beta x_{n-1})^{p^{k+1}} \quad \textit{for } n \geq 2 \textit{ and } k \geq 0.$$

Proof. See [5, I Section 4 and III Section 3]. The only point in mentioning the homology operations is that, via the Cartan formula and Nishida relations, they make the primitivity and the assertions about the Steenrod operations obvious.

Proof of Theorem 4. We have $\mu^*(a) = a\mu$ for $a \in A = H^*K(Z_2, 0)$ in case (i) or for $a \in A/A\beta = H^*K(Z_{(p)}, 0)$ in cases (ii) and (iii). In all cases, the domain and range of μ^* have the same finite dimension in each degree and μ^* is a morphism of coalgebras, hence it suffices to prove that μ^* is a monomorphism on primitive elements.

(i) By Lemma 5, it suffices to show that $f^*w_{2^k-1}$ is indecomposable in $H^*\Omega^2 S^3$ for all $k \geq 1$. Since x_n is primitive,

$$Sq^1 \cdots Sq^{2^{n-1}} w_{2^n} \equiv w_{2^{n+1}-1} \quad \text{mod decomposables},$$

$Sq_*^{2^{n-1}} \cdots Sq_*^1 x_n = x_0^{2^n}$, $f_* x_0 \neq 0$, and the 2^n-fold coproduct on w_{2^n} has the summand $w_1 \otimes \cdots \otimes w_1$, we have, for $n \geq 1$, that

$$\begin{aligned}\langle f^* w_{2^{n+1}-1}, x_n\rangle &= \langle f^* Sq^1 \cdots Sq^{2^{n-1}} w_{2^n}, x_n\rangle \\ &= \langle w_{2^n}, Sq_*^{2^{n-1}} \cdots Sq_*^1 f_* x_n\rangle \\ &= \langle w_{2^n}, (f_* x_0)^{2^n}\rangle \\ &= 1.\end{aligned}$$

(ii) The argument here is the same as for (i), except that $f_*(x_0^2) \neq 0$ and the summand $w_2 \otimes \cdots \otimes w_2$ of the 2^{n-1}-fold coproduct on w_{2^n} now ensure the non-triviality of the specified Kronecker bracket.

(iii) It suffices to show that $f^*\beta^\varepsilon w_{p(k)}$ is indecomposable for $\varepsilon = 0$ or 1 and $k \geq 1$. Since βx_n is primitive,

$$P^1 \cdots P^{p^{n-1}} w_{p^n} \equiv w_{p(n+1)} \quad \text{mod decomposables},$$

$P_*^{p^{n-1}} \cdots P_*^1 \beta x_{n+1} = (-1)^n (\beta x_1)^{p^n}$, $f_* \beta x_1 \neq 0$, and the p^n-fold coproduct on w_{p^n} has the summand $w_1 \otimes \cdots \otimes w_1$, we have, for $n \geq 0$, that

$$\begin{aligned}-\langle f^* \beta w_{p(n+1)}, x_{n+1}\rangle &= \langle f^* w_{p(n+1)}, \beta x_{n+1}\rangle \\ &= \langle f^* P^1 \cdots P^{p^{n-1}} w_{p^n}, \beta x_{n+1}\rangle \\ &= \langle w_{p^n}, P_*^{p^{n-1}} \cdots P_*^1 f_* \beta x_{n+1}\rangle \\ &= (-1)^n \langle w_{p^n}, (f_* \beta x_1)^{p^n}\rangle \\ &\neq 0.\end{aligned}$$

Of course, the second loop map $j \circ \bar{\eta}\colon \Omega^2 S^3 \to BO \to BF$ has non-trivial first Stiefel-Whitney class, hence we have already proven Theorem 2. Since $\Omega^2 S^3\langle 3\rangle$ and BSF are both torsion spaces of finite type, they are equivalent to the product (and to the wedge) of their localizations at the various primes. Thus any map $f\colon \Omega^2 S^3\langle 3\rangle \to BSF$ splits as the product over p of its localizations f_p. Further, f is an H-map if and only if each f_p is. Therefore, to construct a global map $f\colon \Omega^2 S^3\langle 3\rangle \to BSF$ with Thom spectrum $K(Z, 0)$, we need only construct a map $g_p\colon \Omega^2 S^3\langle 3\rangle \to BSF$ with suitable p-local properties for each prime p, as we can then take f_p to be the localization of g_p at p. Clearly the composite

$$\Omega^2 S^3\langle 3\rangle \longrightarrow \Omega^2 S^3 \xrightarrow{\bar{\eta}} BO \xrightarrow{j} BF$$

is a second loop map which lifts to a second loop map

$$g_2\colon \Omega^2 S^3\langle 3\rangle \to BSF$$

with non-trivial second Stiefel-Whitney class. We must still construct an H-map $g_p\colon \Omega^2 S^3\langle 3\rangle \to BSF$ with non-trivial first mod p Wu class for each prime $p > 2$.

The construction of g_p is based on our version [6] of the stable splitting, originally due to Snaith [18]:

$$\Omega^2 S^3 \simeq C_2 S^1 \overset{s}{\simeq} \bigvee_{q \geq 1} D_{2,q} S^1, \text{ where } D_{2,q} S^1 = C_{2,q}^+ \wedge_{\Sigma_q} S^q.$$

Here $D_{2,1} S^1 \simeq S^1$ and $D_{2,q} S^1$ is $(q-1)$-connected. In mod p homology, it is clear from Proposition 6 and the filtration of $C_2 S^1$ that $\tilde{H}_* D_{2,q} S^1 = 0$ for $1 < q < p$ and that $H_* D_{2,p} S^1 = H_* M$, where M denotes the Moore space $S^{2p-2} \cup_p e^{2p-1}$. Therefore $(D_{2,q} S^1)_p \simeq 0$ for $1 < q < p$ and $(D_{2,p} S^1)_p \simeq M$.

The splitting cited above is based on certain very explicit generalized James maps $j_q\colon C_2 X \to QD_{2,q} X$, where $QX = \varinjlim \Omega^n \Sigma^n X$. We shall analyze the multiplicative properties of these maps in [3]. In particular, we shall define certain products

$$QD_{2,r} X \times QD_{2,s} X \to QD_{2,r+s} X$$

such that, up to homotopy, we shall have a formula of the form

$$j_t(x + y) = \sum_{r+s=t} j_r(x) j_s(y),$$

where the sums are specified in terms of the standard H-space structures on $C_2 X$ and $QD_{2,t} X$. When X is S^1 (or any other odd dimensional sphere) and t is an odd prime p, the error terms vanish upon localization of the $D_{2,r} X$ and $D_{2,s} X$ at p, because either $1 < r < p$ or $1 < s < p$ in each such term. Thus the results of [3] will include the following.

LEMMA 7. *The composite*

$$\Omega^2S^3 \simeq C_2S^1 \xrightarrow{j_p} QD_{2,p}S^1 \xrightarrow{Q\lambda} Q(D_{2,p}S^1)_p \simeq QM$$

is an H-map, where λ is a localization of $D_{2,p}S^1$ at p.

The composite of $Q\lambda \circ j_p$ and the natural second loop map $\Omega^2S^3\langle 3\rangle \to \Omega^2S^3$ is an H-map k_p: $\Omega^2S^3\langle 3\rangle \to QM$. Since j_p restricts on $F_pC_2S^1$ to the composite of its projection onto $D_{2,p}S^1$ and the inclusion of the latter in $QD_{2,p}S^1$, k_p maps the span of $\{x_1, \beta x_1\}$ onto $\tilde{H}_*M \subset \tilde{H}_*QM$. Let α: $S^{2p-2} \to BSF$ be a map of mod p Hopf invariant one. (Thus α is of order p and has non-trivial Hurewicz image.) Then α extends over M, and there results a unique infinite loop map $\tilde{\alpha}$: $QM \to BSF$ which restricts on M to a given extension. Clearly $g_p = \tilde{\alpha} \circ k_p$: $\Omega^2S^3\langle 3\rangle \to BSF$ is an H-map with non-trivial first Wu class, and this completes the proof of Theorem 1.

We close with some remarks on the bordism interpretation of these results. Recall that Ω^2S^3 is equivalent to $\Omega_0^2S^2$ and that, by [5, pp. 59 and 226], there is a homology equivalence $\bar{\alpha}_2$: $\bar{C}_2S^0 \to \Omega_0^2S^2$, $\bar{C}_2S^0$ being equivalent to the classifying space BB_∞ of the infinite braid group B_∞. Clearly the restriction of $\bar{\eta}: \Omega^2S^3 \to BO$ to BB_∞ has the same associated Thom spectrum as does $\bar{\eta}$, namely $K(Z_2, 0)$. Making full use of May's infinite loop space machinery, Cohen [4] proved that this restriction is actually homotopic to the classifying map of the composite of the natural homomorphism $B_\infty \to \Sigma_\infty$ and the regular representation $\Sigma_\infty \to O$ and thus obtained the following interesting sharpening of Thom's theorem on the representability of mod 2 homology classes.

THEOREM 8. *Any mod 2 homology class of any space is the image of the fundamental class of a smooth compact manifold M such that the structural group of the stable normal bundle of M reduces to B_∞.*

Cohen [4] has begun the search for explicit examples of such manifolds, and Sanderson [17] has given a detailed analysis of what the reduction (or braid orientation) means geometrically. No such classical interpretation of the integral result is possible.

PROPOSITION 9. *Let G be a topogical group (possibly discrete) and let h: $G \to SF$ be a continuous homomorphism of monoids. Then $Q_1\mu = 0$ in the mod p cohomology of the Thom spectrum associated to Bh: $BG \to BSF$.*

Proof. For $g \in G$, $h(g)$ is a homeomorphism. Thus h factors through $STop$ In H^*MSTop. $Q_1\mu = 0$ for dimensional reasons [5, p. 169].

All we can conclude is the following. The notion of a normal space is defined in Quinn [16].

THEOREM 10. *Any integral homology class of any space is the image of the fundamental class of an oriented normal space N such that the classifying map of the stable normal spherical fibration of N factors through f: $\Omega^2S^3\langle 3\rangle \to BSF$.*

In both theorems, the representation is unique up to the appropriate bordism relation.

Finally, returning to the study of an H-map $f: \Omega^2 S^3 \to BF$, note that we may view $\Omega^2 S^3$ as either ΩMS^2 or $C_2 S^1$, where M is the James construction and C_2 is the little cubes approximation [12] used above. MS^2 and $C_2 S^1$ are filtered spaces, and there result two filtrations of $\Omega^2 S^3$. The following results (for $f = j\bar{\eta}$) are due to Mahowald [11, 6.2.11] and [10, 4.1].

PROPOSITION 11. *Let f_j be the restriction of f to $\Omega F_{2^{j+1}-1} MS^2$. Then, in mod 2 homology,*

$$H_* Mf_j = P\{\xi_i \mid 1 \le i \le j\} \subset P\{\xi_i \mid i \ge 1\} = A_* = H_* Mf.$$

Proof. $H_* MS^2 = P\{i_2\}$ and $H_* F_k MS^2$ is spanned by the i_2^q with $q \le k$. Since x_n in Proposition 6 suspends to $i_2^{2^n}$, a glance at the Serre spectral sequence shows that

$$H_* \Omega F_{2^{j+1}-1} MS^2 = P\{x_n \mid 0 \le n \le j-1\} \subset P\{x_n \mid n \ge 0\} = H_* \Omega^2 S^3.$$

Therefore $H_* Mf_j$ is a polynomial subalgebra of $H_* Mf$, necessarily the one specified in the statement.

PROPOSITION 12. *Let f_k be the restriction of f to $F_k C_2 S^1$. Then, in mod 2 cohomology,*

$$H^* Mf_k = M(k) \quad \text{where } M(k) = A/A\{\chi Sq^i \mid i > k\}.$$

Proof. $H_* F_k C_2 S^1$ is the subspace of $H_* C_2 S^1$ spanned by all monomials $\underset{q \ge 0}{\times} x_q^{n_q}$ such that $\sum_{q \ge 0} 2^q n_q \le k$. By [1, 1.3], $M(k)$ has as basis

$$\{\chi(Sq^I) \mid Sq^I \text{ is admissible, } I = (i_1, \ldots, i_r), \text{ and } i_1 \le k\}.$$

A standard counting argument (compare [13, p. 160]) shows that $H_* F_k C_2 S^1$ and $M(k)$ are Z_2 vector spaces of the same finite dimension in each degree. Since $H_* F_k C_2 S^1 \to H_* \Omega^2 S^2$ is a monomorphism, $Mf_k \to Mf$ induces an epimorphism $A = H^* Mf \to H^* Mf_k$. Thus it only remains to show that $\chi(Sq^i)\mu_k = 0$ for $i > k$, where μ_k is the Thom class of $H^* Mf_k$. This was much the easiest part of Mahowald's original proof [10, p. 252].

Ralph Cohen [7] has recently proven that Mf_k is equivalent to the Brown-Gitler spectrum $B(k)$ of [1].

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