\section*{Λ-SPLITTING 4-MANIFOLDS}

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\textbf{Given} a homotopy (or homology) equivalence, \( f : N \rightarrow X \), from a manifold to a Poincaré space, and some "decomposition" of \( X \), we may ask if \( f \) is homotopic to a map, \( g \), which restricts to an equivalence over each "piece" of the "decomposition". Finding \( g \) is called splitting \( f \). Splitting has been a fruitful technique in classifying manifolds of dimension at least five. Here we carry through a version of 4-dimensional splitting and give applications to 4-dimensional classification problems.

Let \((X_i, Y_i)\) and \((X_j, Y_j)\) be Poincaré pairs of formal dimension = 4, and with product neighborhoods \( Y_i \times [0, 1] \) of \( Y_i \). Let \( X_1 \) and \( X_2 \) be connected. Let \( Y_0 \) be a component of \( Y_i \) for \( i = 1 \) and \( 2 \) with \( \pi_i(Y_0) \rightarrow \pi_i(X_i) \) an isomorphism. Let \( \Lambda = Z[\pi_i(X)] \) be \( \Lambda \)-equivalence of pairs, i.e. inducing isomorphisms on homology with \( \Lambda \)-coefficients (\( \Lambda \) may be empty), where \((N, \partial)\) is a smooth manifold.

\textbf{Theorem 1.} The map \( f \) is homotopic (rel \( \partial N \)) to a map \( g \) with \( g|g^{-1}(X_i) \) and \( g|g^{-1}(Y_0) \) \( \Lambda \)-equivalences, \( i = 1 \) and \( 2 \).

\textbf{Proof.} Our proof is based on Browder's High-Dimensional Splitting Argument[1]. An elementary argument using \( \pi_i(X_i, Y_i) = 0 \) enables us to homotope \( f \) (we continue to call the map \( f \)) so that \( f^{-1}(X_i) \) and \( f^{-1}(Y_0) \) are connected.

Let \( N_i = f^{-1}(X_i) \), \( M = f^{-1}(Y_0) \). Choose handle decompositions \( \mathcal{H}(N_i, M) \) for \( N_i \) relative to \( M \). We may assume \( \mathcal{H}(N_i, M) \) contains no 0-handles. By a standard homotopy (see [13]) we may trade first the 1-handles of \( \mathcal{H}(N_i, M) \), then the 1-handles of \( \mathcal{H}(N_i, M) \). This again uses \( \pi_i(N_i, Y_0) = 0 \). This results in a new \( f \), \( N_i \), and \( M \) where \( N_i \) admits a handle decomposition \( \mathcal{H}(N_i, M) \) without 1-handles. Now consider:

\[\begin{array}{cccc}
\pi_3(f) & \rightarrow & \pi_3(N_i, M) & \rightarrow \pi_3(X_i, Y_0) \\
\downarrow & & \downarrow & \\
0 & \rightarrow & K_2(N_i, M; \Lambda) & \rightarrow H_2(N_i, M; \Lambda) \rightarrow H_2(X_i, Y_0; \Lambda) \rightarrow 0
\end{array}\]

The right-hand vertical is an isomorphism by the Hurewicz theorem and the centre vertical an epimorphism since \( \pi_3(M_i, N) = 0 \). Hence it follows easily that \( h \) is an epimorphism.

Let \( \{a_i\} \subset \pi_3(f) \) satisfy \( h(a_i) \) generates \( K_2(N_i, M; \Lambda) \). \( \{\delta a_i\} \) is easily seen to be represented by relatively imbedded 2-disks. (Because the cores of the 2-handles of \( \mathcal{H}(N_i, M) \) generate \( \pi_3(N_i, M) \) as \( \pi_3(M) \)-module. In fact \( \mathcal{H}(N_i, M) \) may be modified by handle passing so that \( \{\delta a_i\} \) is represented by cores of 2-disks).

According to Wall[5, p. 13], there is a unique regular homotopy class of relative immersions for which handle subtraction yields normal bordism. The following lemma shows that we do not need to worry about the relative regular homotopy class of our disks.

\textbf{Lemma.} Let \( (D^n, \partial) \rightarrow (K^n, \partial) \) be a relative map of the \( n \)-disk into a smooth \( 2n \)-manifold, \( n > 1 \). Then there is exactly one relative regular homotopy class of relative immersions homotopic to \( f \).

\textbf{Proof.} According to the Smale–Hirsch classification of immersions, the relative regular homotopy classes of relative immersions homotopic to \( f \) are in 1-1 correspondence with the stable homotopy classes of maps \( D^n \rightarrow K^n \) which induce homotopy equivalences of \( n \)-spheres.
homotopy classes of bundle maps:

\[ \tau(S^{n-1}) \xrightarrow{\text{injection}} (D^{2n-1} \times S^{n-1}) \to S^{n-1} \]

\[ \tau(D^n) \xrightarrow{\text{injection}} (D^{2n} \times D^n) \to D^n \]

and hence with the element of the homotopy group

\[ \pi_n \left( \frac{SO(2n)}{SO(n)} , \frac{SO(2n-1)}{SO(n)} \right) \approx \pi_n (SO(2n), SO(2n-1)) \]

\[ \approx \pi_n (S^{2n-1}) = 0 \]

Again by a standard homotopy, we may alter \( f \) to subtract a tubular neighborhood, \( U \), of the imbedded 2-disks representing \( \{ \partial a_i \} \) from \( N_i \), and add \( U \) to \( N_2 \). A calculation shows that \( K^* (N_i, M; \Lambda) = 0, * \neq 3 \). \( K^* (N_i, N_2; \Lambda) \neq 0, * = 3 \). It follows that \( K_3 (N_i, M; \Lambda) \) is the only non-zero kernel. By Theorem 2.3 [5], \( K_3 (N_i, M; \Lambda) \) is stably free. Note that \( N_i \) still has a handle decomposition rel \( M \) without 1-handles. Introduce some trivial (1-handle, 2-handle) pairs in \( \mathcal{M} (N_2, M) \) and trade the 1-handles over to \( \mathcal{M} (N_1, M) \). This makes \( K_3 (N_i, M; \Lambda) \) free. By the method above, represent a basis by imbedded disks and trade a tubular neighborhood by homotoping \( f \) to a map \( g \). A calculation [5, Chap. 4] shows that \( g : (N_2, M) \to (X_2, Y) \) is a \( \Lambda \)-equivalence of pairs. Since \( g \) induces a map between the Mayer–Vietoris sequences of

\[ \left( M \setminus N_i \setminus N_2 \setminus N \right) \]

and

\[ \left( Y \setminus X_1 \setminus X \setminus X \right), \]

inspection shows that \( g : N_1 \to X_1 \) is also a \( \Lambda \)-equivalence.

Note. Versions of Theorem 1 for simple-equivalences and for extending \( \Lambda \)-splittings on a boundary may be proved.

Putting our \( \Lambda \)-splitting theorem together with the "stable" surgery theorem of Cappell and Shaneson[2], we obtain a homology-surgery theorem for simply connected 4-manifolds minus a disk. Let \( R (\Sigma^3) \) denote the Rochlin invariant of a homology 3-Sphere \( \Sigma^3 \).

**Theorem 2.** Let \( f : K^* \to Z \) be a degree 1 normal map from a simply connected smooth 4-manifold to a Poincaré space (assume \( f \partial K \) is an integral homology equivalence if \( \partial K \neq \emptyset \)). Then \( f^* : K^* - D^* \to Z-D^* \) (if restricted to the complement of an open 4-disk) is normally bordant (rel \( \partial K \)) to an integral homology equivalence, \( g : P^* \to Z-D^* \) with \( \partial P^* = \partial K^* \cup \Sigma^3 \), where \( \Sigma^3 \) is an integral homology 3-sphere satisfying:

1. \( R (\Sigma^3) = [\sigma (Z) - \sigma (K)] / 8 \) (mod 2) and
2. \( \pi_1 (P) \) is in the normal closure of \( \text{inc} \circ \pi_1 (\Sigma^3) \).

**Proof.** Consider: \( K^* = K - D^* \cup Q \xrightarrow{(q \cup i)} Z - D^* \), where \( D^* \) is the interior of an imbedded 4-disk, \( Q \) is a framed bordism of \( S^3 \) to a homology 3-sphere \( \omega^3 \) with \( \sigma (Q) = \sigma (Z) - \sigma (K) \) (\( Q \) may be taken to be a plumbing construction on \( \bigoplus \pm E_3 \), and \( k \) is the canonical map \( Q \to S^3 \)). Then \( j \partial K^* \) is an integral homology equivalence. The surgery obstruction, \( \sigma (j) = 0 \in \Gamma (Z[0] - Z[0]) = L_d (0) = Z \). It follows from [2] that for some \( n, n_0 \),
\[ g = j \# \text{id} \] satisfy the conclusion of Theorem 1. Let \((N, M) = g^{-1}(X, Y)\). Since \(M\) also bounds \(N = g^{-1}(X)\), \(R(M) = 0\). Set \(P = N\) (an open tubular neighborhood of an imbedded arc connecting \(M\) and \(\omega^3\)). The normal bordism from \(K - D^4\) to \(P\) may be constructed as the union of three normal bordisms: \(K - D^4\) to \(K' - D^4\), \(K' - D^4\) to \(N\), (make \(H\) transverse to \((X, Y) \times I\)), and \(N_1\) to \(P\). Then \(\varphi^3 = M \# \omega^3\) and \(R(S^3) = R(M) + R(\omega^3) = R(M) = [(\sigma(Z) - \sigma(K))/8] \pm (2)\). Now \(X\) is simply connected and \(g: N = X\) so by Van Kampen's Theorem, \(0 = \pi_0(P) \ast \pi_0(N - P)\). Therefore the normal closure of \(\pi_1(S^3)\) in each side is the entire group.

\[ \square \]

We now give some applications to the homology-classification of 4-manifolds.

**Corollary 1A.** The kernel of Rochlin's homomorphism, \(R: \theta_0 \rightarrow Z_2\) (\(\theta_0 = \text{homology mod } \text{Cobordism classes of homology 3-spheres})\) is generated by homology 3-spheres \(S^3 = \partial M^4\), where \(M^4\) is spin and integral-homology-equivalent to \(S^2 \times S^2\) (and is a handle body on \(\partial M = \Sigma^3\) consisting of 2, 3, and 4-handles only).

**Corollary 1B.** \(\theta_0\) is generated by homology 3-spheres \(S^3 = \partial M^4\), where \(M^4\) is integral-homology-equivalent to \(S^2 \times S^2\) (and is a handle body on \(\partial M = \Sigma^3\) consisting of 2, 3, and 4-handles only).

**Proof of 1A.** Let \(R(S^3) = 0\). It is well known that \(\Sigma^3 = \partial M^4\), where \(M^4\) is framed and \(\sigma(M) = 0\). By [2], there is an integer \(n\) such that the surgery problem \(M^4 \# n(S^2 \times S^2) - D^4 \# n(S^2 \times S^2)\) can be solved rel \(\partial M\). Let the solution of \(f: N^4 \rightarrow D^4 \# n(S^2 \times S^2) (\partial N = \partial M = \Sigma^3)\). By Theorem 1, the copies of \(S^2 \times S^2\) may be split off inductively. The result is a bordism (homology equivalent to an \((n + 1)\)-punctured \(S^3\) from \(\omega^3\) to \(n\) homology 3-spheres (these will be taken as generators) each of which bounds a homology \(S^2 \times S^2 - D^4 \sim S^2 \times S^2\). The lemma follows.

**Proof of 1B.** The argument is similar to the one above. We generate \(\text{Ker}(R)\) by first doing "stable" surgery using \(\pm S^2 \times S^2\) in place of \(S^2 \times S^2\), and then split it. It then suffices to find a homology 3-sphere \(\Sigma^3\) with \(R(\Sigma^3) = 1\) and \(\Sigma^3 = \partial M^4\), \(M = S^2\). It is well known that the Poincaré-homology sphere, \(P\), \((R(P) = 1)\) is the boundary of the handle body consisting of the 4-ball with a 2-handle attached with framing +1 to the right-handed trefoil knot in \(\partial\) (4-ball). 1B follows.

We obtain some equivalences of certain optimistic conjectures.

**Conjecture** \(\partial_0^\ast\). *Every homology 3-sphere with Rochlin invariant zero bounds a contractible smooth manifold.*

**Conjecture** \(\partial_1^\ast\). *Every homology 3-sphere with Rochlin invariant zero bounds an acyclic smooth manifold.*

**Conjecture** \(S_0\). *If \(f: (M, \partial) \rightarrow (X, \partial)\) is a degree one normal map from a simply connected smooth manifold to a Poincaré space inducing a homology isomorphism on \(\partial\) and if \(\sigma(M) = \sigma(X)\) then \(f\) is normally bordant (rel \(\partial\)) to a homotopy equivalence.*

**Conjecture** \(S_1\). *With the preceding hypothesis conclude: \(f\) is normally bordant (rel \(\partial\)) to an integral homology equivalence.*

**Corollary 2.** \(\partial_0^\ast \Leftrightarrow S_0\) and \(\partial_1^\ast \Leftrightarrow S_1\).

**Proof.** The implications: \(S_0\) \(\Rightarrow \partial_0^\ast\) and \(\partial_1^\ast \Rightarrow S_1\) are well known. Assume \(\partial_1^\ast\). By Theorem 2, normally bordant \(f\) \(M - D^4\) to a homology equivalence \(g: N \rightarrow X - D^4\). Then \(f\) is normally bordant to a homology equivalence \(g \cup c: N \cup A \rightarrow X\) implying \(S_1\) \((c: (A, \Sigma^3) \rightarrow (D^4, \partial)\) is the collapsing map of an acyclic manifold). If \(\partial_0^\ast\) is assumed we may find an \(A\) so that \(\pi_1(N \cup A) = 0\) (by Theorem 2), yielding \(S_1\).

\[ \square \]

If \(M\) is an oriented 4-manifold let \(\hat{h}_M\) be its integral intersection form.
where $(\tilde{A}_n, \Sigma_k) \rightarrow (A_n - D^n, S^n)$, $(\tilde{A}_n, \Sigma_n) \rightarrow (A_n - D^n, S^n)$, and $(\tilde{A}_n, \partial) \rightarrow (A_n - 2D^n, S^n, 1\Sigma^n)$ are homology equivalences.

Proof. By [4], $M = \bigoplus_{i=0}^{n} A_n$. Now apply Theorem 1 inductively.

Example. If $\partial M$ is odd and indefinite, $\mathcal{H}_M = \bigoplus_{i=0}^{n} \mathcal{H}_{CP^n}$.

By the techniques of Theorem 1 and the lemmata version of the $(\pi - \pi)$-surgery theorem [5, Chap. 4] follows (easily):

**Theorem 3.** Suppose $f: (N, \partial) \rightarrow (X, Y)$ is a degree 1 normal map from a smooth manifold to a Poincaré pair (over $\Lambda = Z(\pi(X))$ coefficients) of formal dimension $= 4$. If $\pi_4(Y) \rightarrow \pi_4(X)$ is an isomorphism, then $f$ is normally cobordant to a $\lambda$-homology equivalence of pairs.

**References**


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