Computations of Gelfand-Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces

by

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Introduction

In this paper, we describe the relationship between the cohomology of certain function spaces and of Gelfand-Fuks cohomology to the cohomology of a certain construction $C(M, X)$ where M and X are compactly generated Hausdorff spaces and where X has a non-degenerate base-point $*$. In particular our work relates to the following problems:

(1) Let $\mathcal{L}^c_M$ be the Lie algebra of compactly supported C^∞ vector fields on a connected smooth manifold M and let $H^*\mathcal{L}^c_M$ be the continuous Lie algebra cohomology of M; Gelfand-Fuks, A. Haefliger, R. Bott, P. Trauber and others [3, 10, 11, 12, 13, 14, 22] have considered $H^*\mathcal{L}^c_M$ while Gelfand-Fuks gave a spectral sequence abutting to $H^*\mathcal{L}^c_M$ [10, 11, 13]. Those classes in total degree q of the E_1^{**}-term are just given by $H^q(C(M, X); \mathbb{R})$ for a certain easily described space X. Since the Gelfand-Fuks spectral sequence collapses if the rational Pontrjagin classes of M vanish [10, 11, 22], our computations of $H^*C(M, X)$ (in sections 1 and 2) gives the Gelfand-Fuks cohomology for a large class of manifolds. In general, we don't know $H^*C(M, X)$ in a closed form, but we give a spectral sequence abutting to it; the ingredients are (a) H^*M, (b) the dimension of M, and (c) H^*X. We

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remark that in case $M = \mathbb{R}^n$, these calculations specialize to $H^*(\Omega^n\Sigma^n X; \mathbb{R})$ (which is classical!)

(2) Let M be a smooth manifold and assume that M and X both have "nice" base-points. Let X^M be the space of based maps from M to X and give X^M the compactly generated topology obtained from the standard compact open topology on a function space. D. W. Anderson [1] and P. Trauber [22] have described a spectral sequence abutting to $H^*(X^M; R)$ where R is a field. We give a global description of the E_2^{**}-term for certain nice manifolds and where R is of characteristic zero. Our spectral sequence of section 2 here abuts to the Anderson-Trauber E_2^{**}-term in the cases where we don't give specific global answers. Our techniques also give information in characteristic $\neq 0$ and we include, in section 4, some conjectures about E_2^{**} and about collapse results. The basic ingredient is $H_*C(M, X)$.

(3) By specializing to certain simpler function spaces, we can give more complete results. In particular, let M be a smooth manifold, let τM denote its tangent bundle, and let E_M be the bundle obtained by forming the one-point compactification of each fibre in τM. Then D. McDuff considers the space of sections of E_M of degree k, say $\Gamma_k(M)$ [18]. We compute $H_*\Gamma_k(M)$ with coefficients taken in a field of characteristic zero in case $M = S^n$ or $M = V \times \mathbb{R}^n$ where V is a connected manifold. In general, we give a spectral sequence converging to $H_*\Gamma_k(M)$. The basic ingredient is $H_*C(M, S^0)$.

Remark: In characteristic zero (with field coefficients), it is likely that other methods involving minimal models would also be useful in the above problems. However, our approach is quite general and works in characteristic $\neq 0$ for various related problems. In addition, our methods relate back to an interesting and useful geometric construction, $C(M, X)$.

An outline of the paper is as follows: Section 1 gives the construction of $C(M, X)$ together with other pertinent geometric and algebraic facts. Section 2 describes the spectral sequence converging to $H^*C(M, X)$. Section 3 gives the promised relationships to Gelfand-Fuks cohomology while section 4 is related to the Anderson-Trauber spectral sequence and section 5 contains results on $H_*\Gamma_k(M)$. Section 6 contains the proofs of some technical facts; one should compare Theorem 1.1 given here with an analogous theorem of Barratt and Eccles [2, Theorem 2.1]. In addition, the reader is referred to [7, 8, 9, 17] for information on $C(\mathbb{R}^n, X)$ (which is $\Omega^n\Sigma^n X$ up to weak homotopy type for path-connected X).

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§1. C(M, X) and its homology

In this section, we give the necessary prerequisites for our calculations. We remark that all constructions are functorial on appropriate categories and do not elaborate on this point. Details are in [8, 9].

Throughout this paper all spaces are compactly generated and Hausdorff. X is assumed to be of finite type and to have a base-point $*$ such that $(X, *)$ is an NDR pair. M is a connected m-dimensional manifold of finite type. In addition all (co)homology groups are taken within field coefficients of characteristic zero unless otherwise stated.

Given a manifold M, define the configuration space $F(M, k)$ as the subspace of M^k given by $\{(m_1, \ldots, m_k) \mid m_i \neq m_j \text{ if } i \neq j\}$. Further define

$$C(M, X) = \coprod_{k \geq 0} F(M, k) \times_{\Sigma_k} X^k / \approx$$

where $\approx$ denotes the equivalence relation generated by

$$((y_1, \ldots, y_j), (x_1, \ldots, x_j)) \approx ((y_1, \ldots, \hat{y}_i, \ldots, y_j), (x_1, \ldots, \hat{x}_i, \ldots, x_j))$$

if $x_i = *$ ($\hat{a}$ means delete a), and $\coprod$ denotes disjoint union. Define a filtration and topology on $C(M, X)$ as follows. $\coprod_{j=0}^{r} F(M, j) \times X^j$ maps to $C(M, X)$ and $F_r C(M, X)$ denotes its image. $\coprod_{j=0}^{r} F(M, j) \times X^j$ is given the topology of the disjoint union and $F_r C(M, X)$ is given the quotient topology. $C(M, X)$ is given the topology of the union of the $F_r C(M, X)$. By convention, $F(M, 0) \times X^0 = *$, the base-point.

In addition, let $X^{[j]}$ denote the j-fold smash product of X and observe that Σ_j acts on $X^{[j]}$ by permuting coordinates. Define

$$D_j(M, X) = F(M, j) \times_{\Sigma_j} X^{[j]}/F(M, j) \times_{\Sigma_j} * .$$

Notice that C(M, X) and $D_j(M, X)$ are generalizations of May's construction C_nX [17] and that $C(\mathbb{R}^n, X)$ is weakly homotopy equivalent to $\Omega^n\Sigma^nX$ if X is path-connected [8].

It is shown in [8] that C(M, X) splits stably into a wedge $\bigvee_{j\geq 1} D_j(M, X)$ if X is path-connected. A more general homological result is true:

Theorem 1.1. If homology is taken with field coefficients of any characteristic, then $H_*C(M, X) \cong H_*\bigvee_{j\geq 0} D_j(M, X)$.

Remark 1.2. Theorem 1.1 is a generalization of a result due to Barratt-Eccles with $M = \varinjlim_n \mathbb{R}^n$ [2, Theorem 2.1] and the results of [7, III, §4] with $M = \mathbb{R}^n$.

We want to give a global description of $H_*C(M, X)$ at least over a field of characteristic zero. This seems difficult for arbitrary M (see section 2). However if $M = \mathbb{R}^n$ and X is path-connected, then $C(\mathbb{R}^n, X)$ is weakly homotopy equivalent to $\Omega^n\Sigma^nX$ [8]. Hence the precise calculations which we give here should be regarded as enriched versions of $H_*\Omega^n\Sigma^nX$ and which are analogues of the algebraic constructions of [7, III. 2]. Analogous geometric motivation is given by the fact that $C(M \times \mathbb{R}^n, X)$ is homotopy equivalent to an n-fold loop space if X is path-connected.

We state conditions on M which are (1) easily verifiable, and (2) give a global description of $H_*C(M, X)$. These conditions reflect the requisite formalism extracted from [7, III.12.1(2) and 12.4].

Here recall that $H^*F(\mathbb{R}^m, k)$ is presented as an algebra by elements A_{ij}, $k \geq i > j \geq 1$ subject to relations $A_{ir}A_{is} = A_{sr}(A_{is} - A_{ir})$ for $r \leq s$. With $A_{ji} = (-1)^m A_{ij}$ for $i > j$, the Σ_k-action on $H^*F(\mathbb{R}^m, k)$ is specified by requiring that $\sigma^* A_{ij} = A_{\sigma i, \sigma j}$ and that σ^* be a morphism of algebras for $\sigma \in \Sigma_k$. (See [7, III.7.4 and 7.7] with the notational change given by $\alpha_{i+1,j} = A_{ij}$). The degree of A_{ij} is $m-1$.

Let D be a graded algebra and define

$$B(m, k, D) = H^*F(\mathbb{R}^m, k) \otimes D^k / I$$

where I is the two sided ideal generated by

$$A_{ij}(1^{i-1} \otimes y \otimes 1^{k-i-1} - 1^{j-1} \otimes y \otimes 1^{k-j-1})$$

for $y \in D$. Observe that $B(m, k, D)$ is naturally an algebra and inherits a Σ_k-action from the natural diagonal Σ_k-action on $H^*F(\mathbb{R}^n, k) \otimes D^k$. $B(m, k, D)$ is filtered as follows. Define the weight of $A \otimes y_1 \otimes \dots \otimes y_k$ to be degree(A) and set

$$F_s B(m, k, D) = \{x \in B(m, k, D) \mid \text{weight}(x) \leq s\} .$$

Clearly $B(M, k, D)$ with the above filtration is a filtered algebra, the various filtrations are preserved by the action of Σ_k, and hence the associated graded $E_0^*B(m, k, D)$ is an algebra over Σ_k.

Remark: Since $H^*(F(\mathbb{R}^m, k); \mathbb{Z})$ is torsion free [7, III. 6], we may define the construction $B(m, k, D)$ and its natural filtration over a field of any characteristic.

We say that an m-dimensional manifold M satisfies $\underset{\sim}{\omega}$ provided

(1) $H^*F(M, k)$ is additively isomorphic to $B(m, k, H^*M)$ and

(2) $H^*F(M, k)$ is filtered as an algebra over Σ_k such that the associated graded algebra is isomorphic as an algebra over Σ_k to

$$E_0^*B(m, k, H^*M) \ .$$

Example 1: $V \times \mathbb{R}^n - \{Q\}$ satisfies $\underset{\sim}{\omega}$ if V is a connected manifold, $n \geq 1$, and Q is a discrete subset of $V \times \mathbb{R}^n$ [9].

Example 2: More generally a codimension zero subset of $M \times \mathbb{R}^n$ satisfies $\underset{\sim}{\omega}$ [9].

There is an algebraic construction which corresponds to $H_*C(M, X)$ in case M satisfies $\underset{\sim}{\omega}$. We give this construction before stating the corresponding theorem.

Let V_i, $i = 1, 2$, be non-negatively graded vector spaces and $\sigma^n V_i$ be the n-fold suspension of V_i for $n \in \mathbb{Z}$. Let $L(\sigma^n V_2)$ be the free graded

Lie algebra on $\sigma^n V_2$ for $n \geq 0$. Define

$$AL_n(V_1, V_2)$$

as the free commutative algebra on

$$V_1 \otimes \sigma^{-n} L(\sigma^n V_2) \quad .$$

Recall that the square of an odd degree element is required to be zero in a free commutative algebra in characteristic $\nmid 2$.

We define a weight function w on $V_1 \otimes \sigma^{-n} L(\sigma^n V_2)$: First recall that a basis for $L(\sigma^n V_2)$ is given by the "collection process" of P. Hall in terms of the so-called basic commutators in addition to the elements

$$[\sigma^n x, \sigma^n x]$$

where degree$(\sigma^n x)$ is odd for x a basis element for V_2 [15]. In the evident way we think of such basis elements as given in terms of $\sigma^n x_1, \ldots, \sigma^n x_k$, $x_i \in V_2$ and we (cavalierly) write such a basis element as $[x_1, \ldots, x_k]$ where the interior brackets are arranged in some order (which we deliberately ignore here). Then a typical basis element of $V_1 \otimes \sigma^{-n} L(\sigma^n V_2)$ is given by

$$t = v_1 \otimes [x_1, \ldots, x_r]$$

for v_1 a basis element of V_1. Define

$$w(t) = (n+1)(r-1) \quad .$$

Define $w(t+t') = \max\{w(t), w(t')\}$.

Observe that w extends in the usual way to a weight function on $AL_n(V_1, V_2)$ by requiring that $w(x_1 \cdot x_2) = w(x_1) + w(x_2)$ for $x_i \in V_1 \otimes \sigma^{-n}L(\sigma^n V_2)$.

Filter $AL_n(V_1, V_2)$ by setting

$$F_s AL_n(V_1, V_2) = \{x \in AL_n(V_1, V_2) \,|\, w(x) \leq s\} \ .$$

Notice that

$$F_o AL_n(V_1, V_2) \cong A(V_1 \otimes V_2)$$

where $A(V_1 \otimes V_2)$ is the free commutative algebra on $V_1 \otimes V_2$.

Example 4: If $V_2 = H_*(\text{point})$, then $AL_n(V_1, V_2)$ reduces to two cases depending on the parity on n. Let ε be a basis element for V_2. Then it is easy to check that (a) $L\sigma^{2n}V_2$ consists of one element $\sigma^{2n}\varepsilon$ and (b) $L\sigma^{2n+1}V_2$ consists of two elements $\sigma^{2n+1}\varepsilon$ and $[\sigma^{2n+1}\varepsilon, \sigma^{2n+1}\varepsilon]$. Hence

$$AL_n(V_1, V_2) = \begin{cases} \text{free commutative algebra on } V_1 \otimes V_2 & \text{if } n = 2k \text{ and} \\ \text{free commutative algebra on } V_1 \otimes H_* S^n & \text{if } n = 2k+1 \ . \end{cases}$$

We remark that if $n = 2k$, a typical element in $AL_n(V_1, V_2)$ is $(a_1 \dots a_\ell)\varepsilon^\ell$ for $a_i \in V_1$. We may identify in another way those elements of $AL_n(V_1, V_2)$ spanned by $(a_1 \dots a_\ell)\varepsilon^\ell$ for fixed ℓ, say T_ℓ. In particular, assume that V_1 in degree zero is the ground field R and let JV_1 be the cokernel of the natural map $V_1 \longrightarrow R$ which is an isomorphism in degree zero. Let $A(JV_1)$ be the free commutative algebra on JV_1 which is filtered by requiring $F_r A(JV_1)$ to be spanned by products of no more than r indecomposables. Then it is easy to check that $F_\ell A(JV_1)$ is isomorphic to T_ℓ. Similar remarks apply to the case $n = 2k+1$.

Theorem 1.2. If M is an m-manifold which satisfies $\underline{\omega}$, then there is a homomorphism

$$\phi : AL_{m-1}(H_*M, \tilde{H}_*X) \longrightarrow H_*C(M, X)$$

which is an isomorphism over a field of characteristic zero.

The proofs of Theorem 1.2 and the next theorem are given in [9].

The only compact manifold, M, for which we have a reasonable description of $H_*C(M, X)$ is $M = S^n$. Here, let $i : \mathbb{R}^n \longrightarrow S^n$ be a fixed open embedding.

Theorem 1.3. Let X be path-connected. Then $\tilde{H}^*C(S^n, X) \cong E[\iota] \otimes B_n$ as a vector space where ι is of degree n if n is odd and $C(i, 1)^* : H^*C(S^n, X) \longrightarrow H^*C(\mathbb{R}^n, X)$ is an injection on B_n.

Remark 1.4: B_n is a submodule of $H^*C(\mathbb{R}^n, X)$ although it is not a subalgebra. In particular 1 is not in B_n.

Remark 1.5: The coproduct in $H_*C(M, X)$ can be given in terms of the generators of Theorem 1.2. Hence we describe the cohomology algebra of C(M, X); details are given in [9].

§2. Spectral sequences converging to $H^*F(M, k)$ and $H^*C(M, X)$

In this section we give a spectral sequence abutting to $H^*(F(M, k); \mathbb{F})$ where $\mathbb{F}$ is a field of any characteristic; appropriate modifications serve to give a spectral sequence abutting to $H^*C(M, X)$ with field coefficients of characteristic zero. We continue the convention of taking field coefficients of characteristic zero unless otherwise stated. However, if a statement is true over an arbitrary field, in this section we specifically write coefficients taken in $\mathbb{F}$ where $\mathbb{F}$ is assumed to be any field.

Our spectral sequence is the spectral sequence associated to an increasing filtration on a cochain complex. For technical reasons, it is more convenient for us to drop the usual conventions on differential bidegrees and filtration degrees; in particular we agree that q denotes filtration degree in $E_*^{p,q}$.

Recall the algebra $B(m, k, D)$, its natural filtration, and its associated graded algebra of section 1. To avoid duplicity of notation, we write

$$Gr_q B(m, k, D)^{p+q}$$

for those classes concentrated in degree $p+q$ of $F_q B(m, k, D)/F_{q-1}B(m, k, D)$.

Theorem 2.1. Let M^m be a connected manifold which is oriented with $\mathbb{F}$ coefficients and where $H_*(M; \mathbb{F})$ of finite type. Then there exists a spectral sequence of algebras over Σ_k with

$$E_1^{p,q} = Gr_q B(m, k, H^*(M; \mathbb{F}))^{p+q}$$

abutting to $H^{p+q}(F(M, k); \mathbb{F})$ as an algebra over Σ_k.

Remarks 2.2: (a) d_r has bidegree $(r, 1-r)$.

(b) $E_1^{p,q} = 0$ unless $q = (m-1)\ell$, $0 \leq \ell \leq k-1$. Hence $d_r = 0$ unless $r \equiv 1(m-1)$.

(c) $E_1^{p,q} = 0$ for $q > (m-1)(k-1)$. Hence $E^{**}_{(m-1)(k-1)+2} = E^{**}_\infty$.

(d) $E_1^{p,o} \cong H^p(M^k; \mathbb{F})$ and the edge homomorphism

$$E_1^{p,o} \longrightarrow E_\infty^{p,o} \subset H^p(F(M, k); \mathbb{F})$$

is the map $H^p(M^k; \mathbb{F}) \longrightarrow H^p(F(M, k); \mathbb{F})$ induced by the natural inclusion $F(M, k) \subset M^k$.

(e) $E_1^{o,q} = H^q(F(\mathbb{R}^m, k); \mathbb{F})$ and the edge homomorphism

$$H^q(F(M, k); \mathbb{F}) \longrightarrow E_\infty^{o,q} \subset E_1^{o,q}$$

is the map induced by any open embedding $\mathbb{R}^m \subset M$.

We also describe the first non-zero differential, d_m. Let $T(\tau_M)$ be the Thom space of the tangent bundle of M. Since M is oriented there is a Thom class in $H^m(T(\tau_M); \mathbb{F})$. $H^m(T(\tau_M); \mathbb{F}) \cong H^m(M \times M, F(M, 2); \mathbb{F})$ and there is the natural map $H^m(M \times M, F(M, 2); \mathbb{F}) \longrightarrow H^m(M \times M; \mathbb{F})$. Let Δ (or Δ_M) denote the image of the Thom class in $H^m(M \times M; \mathbb{F})$.

As an algebra, $E_1^{p,q}$ is generated by $H^*(M; \mathbb{F})$ and the A_{ij}. Since the elements of $H^*(M; \mathbb{F})$ are all infinite cycles, we need only define

$d_m A_{ij} \in H^m(M^k; \mathbb{F})$. Let $f_{ij} : M^k \longrightarrow M \times M$ be defined by $f_{ij}(m_1, \ldots, m_k) = (m_i, m_j)$. Set $\Delta_{ij} = f_{ij}^* \Delta$ and we have: $d_m A_{ij} = \Delta_{ij}$.

Spanier [21] p. 347 describes Δ for M compact. Let $\{b_i\}$ be a basis for $H^*(M; \mathbb{F})$ and let $\{b_i^*\}$ be the dual basis under the non-singular pairing $H^*(M; \mathbb{F}) \otimes H^{m-*}(M; \mathbb{F}) \longrightarrow \mathbb{F}$ given by cup product evaluated on the fundamental class. Then $\Delta = \sum_i (-1)^{|b_i|} b_i \times b_i^* \in H^m(M \times M; \mathbb{F})$.

Remarks 2.3: (1) If $i : N \subset M$ is a codimension zero embedding

$$i^* \Delta_M = \Delta_N \ .$$

(2) $\Delta_M = 0$ if $M = V \times \mathbb{R}$ and hence if $M \subset V \times \mathbb{R}$ is any open subset.

Now from [9] we have that the spectral sequence collapses if and only if $\Delta = 0$. Our condition $\underset{\sim}{\mathcal{W}}$ is clearly equivalent to the spectral sequence collapsing, and so we get

Theorem 2.4. M satisfies $\underset{\sim}{\mathcal{W}}$ if and only if $\Delta_M = 0$.

We remark that in [9], an analogous spectral sequence is given with coefficients in a ring and for those M for which $H_*(M)$ is not necessarily of finite type. The E_1 term is slightly harder to write down and so we defer the exposition of this spectral sequence to [9]. We also have an analogous spectral sequence in case M is not orientable.

To obtain a spectral sequence abutting to $H^*C(M, X)$, we use the spectral sequence of Theorem 2.1 to compute $H^*D_j(M, X)$ and then appeal to Theorem 1.1. The spectral sequence given here works only in characteristic zero.

Recall that the Leray-Serre spectral sequence for a finite-sheeted covering space collapses with characteristic zero coefficients. We require a lemma which is checked in section 6. Let Σ_k act on $H^*F(M, k)$ via the natural action on $F(M, k)$, let Σ_k act on $(\widetilde{H}^*X)^k$ by permuting factors (with standard sign conventions), and let Σ_k act diagonally on the tensor product of these last two modules.

<u>Lemma</u> 6.2. $H^*D_k(M, X)$ is isomorphic to the vector space of elements in $H^*F(M, k) \otimes (\widetilde{H}^*X)^k$ invariant under the Σ_k-action.

To describe the spectral sequence abutting to $H^*D_k(M, X)$, we first filter

$$B(m, k, H^*M) \otimes (\widetilde{H}^*X)^k$$

by the natural weight filtration obtained from that of $B(m, k, H^*M)$. We then obtain a spectral sequence with

$$E_1^{p,q} = Gr_q(B(m, k; H^*(M)) \otimes (\widetilde{H}^*X)^k)^{p+q}$$

converging to $H^*(F(M, k)) \otimes (\widetilde{H}^*X)^k$. This spectral sequence is a spectral sequence of Σ_k-modules. Since the characteristic of our field G is zero, all our modules are projective $G[\Sigma_k]$-modules, and we have a spectral sequence

$$E_1^{**}(D_k(M, X)) = Gr_*(B(m, k; H^*(M)) \otimes (\tilde{H}^*X)^k)^{\Sigma_k}$$

converging to $H^{p+q}(D_k(M, X))$ where L^{Σ_k} denotes the Σ_k-invariants in a Σ_k-module L.

This spectral sequence has additional internal structure. Recall the spectral sequence $E_r^{**}(F(M, k))$ of Theorem 2.1. Define

$$E_r^{p, q, s}(D_k(M, X)) = (E_r^{p+mk, q}(F(M, k)) \otimes \tilde{H}^{s-mk} X^{[k]})^{\Sigma_k}$$

where $(L)^{\Sigma_k}$ are the invariants in a Σ_k-module L.

Remark: Consider $E_r^{**}(D_k(M, X))$ of Theorem 2.4. Notice that

$$E_r^{p, q}(D_k(M, X)) = \sum_s E_r^{p-s, q, s}(D_k(M, X))$$

and our spectral sequence consequently decomposes as the direct sum of spectral sequences one for each s.

Since $H^*C(M, X) \cong H^* \bigvee_{j \geq 0} D(M, X)$ by Theorem 1.1, we obtain a spectral sequence (tri-graded!) with

$$E_r^{p, q, s}C(M, X) = \sum_k E_r^{p, q, s}(D_k(M, X)) .$$

Summing this up, we have

Theorem 2.5. Let M be an oriented connected m-dimensional manifold of finite type. Let X be of finite type. There exists a spectral sequence (over a field of characteristic zero!) with

$$E_1^{p,q} = \mathrm{Gr}_q AL_{m-1}(H^* M, \widetilde{H}^* X)^{p+q}$$

abutting to $H^{p+q}C(M, X)$. Moreover

$$E_r^{p,q} = \sum_s E_r^{p-s,q,s} C(M, X) \ .$$

Remark 2.6: $d_r = 0$ unless $r = (m-1)\ell+1$. We have given an implicit description of d_m. Because of our method of summing the spectral sequences for the $D_k(M, X)$, we no longer have $E_r^{p,q} = 0$ for large q. The gains from this method of amalgamation are: first, we can compute $E_1^{p,q}$ in closed form; second, $\sum_{r+s=p} E_\infty^{r,s,q}$ is $E_2^{p,q}$ of the Gelfand-Fuks and the Anderson-Trauber spectral sequences.

As a caveat to the reader we remark that our spectral sequence is a second quadrant spectral sequence: i.e., $E_1^{p,q} = 0$ unless $p \leq 0$ and $q \geq 0$.

Remark 2.7: If X is path connected then the total degree lines are bounded and the spectral sequence in 2.5 actually converges. This case covers most of our applications. Since the spectral sequence is just a sum of convergent spectral sequences it does converge in some weak sense even if X is not path-connected.

In addition, information on products in this spectral sequence and in $H^* C(M, X)$ is given in [9].

§3. Gelfand-Fuks cohomology and its relation to C(M, X)

Let M be a connected smooth (paracompact) manifold without boundary. Consider the smooth vector fields on M with compact support; these vector fields have the natural structure of a topological Lie algebra denoted $\mathcal{L}^c_M$ [10].

Gelfand and Fuks [10, 11, 12] amongst others [3, 13, 14] considered the continuous Lie algebra cohomology of $\mathcal{L}^c_M$, $H^*\mathcal{L}^c_M$ (with real coefficients), and gave a spectral sequence abutting to $H^*\mathcal{L}^c_M$. In this section, we show that for certain spaces X depending on the dimension of M, $H^q(C(M, X); \mathbb{R})$ is isomorphic as a vector space to the elements concentrated in total degree q of the E_2-term of the Gelfand-Fuks spectral sequence abutting to $H^*\mathcal{L}^c_M$.

The use of these observations is the following: For many manifolds M, we know $H^*C(M, X)$ and hence we know the E_2^{**} term of the Gelfand-Fuks spectral sequence. If the rational Pontrjagin classes of M vanish, then the spectral sequence collapses [10, 22]. Hence in these cases, we have the equation $H^*\mathcal{L}^M_c \cong H^*(C(M, X); \mathbb{R})$. For example, in case $M = \mathbb{R}^n$, then $H^*\mathcal{L}^{R^n}_c \cong H^*(\Omega^n\Sigma^n X; \mathbb{R})$ which is known classically [5].

As particular examples, we use the results of section 1 to compute $H^*\mathcal{L}^c_V$ where V satisfies $\underline{\omega}$ and where the rational Pontrjagin classes of V vanish. In addition, we describe $H^*\mathcal{L}^c_{S^n} = H^*\mathcal{L}_{S^n}$. The results here should be compared to those of A. Haefliger [14], who obtains analogous results in case $V = M \times \mathbb{R}^n$ and $V = S^n$.

We remark that the spectral sequence of section 2 abuts to the

E_2-term of the Gelfand-Fuks spectral sequence because our spectral sequence converges to $H^*C(M, X)$; in addition, our spectral sequence is given solely in terms of the dimension and cohomology of M. Furthermore, we shall see in Theorem 3.2 that $E_2^{p,q}$ of the Gelfand-Fuks spectral sequence is given by the $\sum_s E^{p-s,s,q}$-term of Theorem 2.5.

We describe the Gelfand-Fuks spectral sequence; for the remainder of this section all (co)homology groups are taken with real coefficients.

Our notations and definitions can be found in work of Gelfand-Fuks and in work of A. Haefliger [10, 11, 12, 13]. Let $H^*\mathcal{L}_o(\mathbb{R}^n)$ be the cohomology of the Lie algebra of formal vector fields on $\mathbb{R}^n$, M^k is the k-fold product of M, and M^k_{k-1} is the subspace of M^k given by k-tuples of points in which at least two coordinates coincide. Let $H^\infty_*(M^k, M^k_{k-1})$ denote the homology with infinite chains of the pair (M^k, M^k_{k-1}); observe that the natural action of Σ_k on M^k given by permutation of coordinates passes to M^k_{k-1} and hence Σ_k acts naturally on $H^\infty_*(M^k, M^k_{k-1})$. In addition, Σ_k acts on

$$\sum_{\substack{q_1+\dots+q_k=q \\ q_i>0}} H^{q_1} \otimes \dots \otimes H^{q_k}$$

by interchanging factors together with the standard sign convention with $H^{q_i} = H^{q_i}\mathcal{L}_o(\mathbb{R}^n)$. We give

$$\sum_{\substack{q_1+\dots+q_k=q \\ q_i>0}} H^\infty_p(M^k, M^k_{k-1}) \otimes H^{q_1} \otimes \dots \otimes H^{q_k}$$

the diagonal Σ_k-action where Σ_k acts on each factor as given above.

Theorem 3.1 (Gelfand-Fuks) [10, 11]. There exists a spectral sequence abutting to $H^*\mathcal{L}^c_M$ with $E_2^{-p,q}$ given as the vector space of elements invariant under the Σ_k-action given above on

$$\sum_{\substack{q_1+\ldots+q_k=q \\ q_i>0}} H_p^\infty(M^k, M^k_{k-1}) \otimes H^{q_1} \otimes \ldots \otimes H^{q_k}$$

summed over all $k \geq 0$.

We remark that $H^q\mathcal{L}_o(\mathbb{R}^n) = 0$ for $0 < q \leq 2n$ and that all products are trivial [12]. Hence $H^*\mathcal{L}_o(\mathbb{R}^n)$ is isomorphic as an algebra to $H^*\Sigma^n X$ for some path-connected space X (which, for example, may be chosen as a wedge of spheres). We show

Theorem 3.2. Let M be an m-dimensional manifold and X as above. Then the summand of E_2^{**} concentrated in degree q in Theorem 3.1 is isomorphic to $H^qC(M, X)$. Moreover the $E_2^{p,q}$-term of the Gelfand-Fuks spectral sequence is isomorphic to the $\sum_s E_\infty^{p-s,s,q}$-term of the spectral sequence of Theorem 2.5.

Corollary 3.3. Let M be an m-dimensional manifold of finite type which (1) has vanishing rational Pontrjagin classes and which (2) satisfies $\underset{\sim}{\mathcal{W}}$. Then

$$H^*\mathcal{L}^c_M \cong A_{m-1}(H_*M, \widetilde{H}_*X) \quad .$$

Example: Since $M = V \times \mathbb{R}^j$ satisfies $\underline{\mathcal{W}}$, we obtain the Gelfand-Fuks cohomology of $V \times \mathbb{R}^j$ provided the rational Pontrjagin classes vanish. The reader should compare these results with those of A. Haefliger [14, §3.3].

Example: If U is an open connected subspace of $M = V \times \mathbb{R}^j$ with H_*U of finite type and vanishing rational Pontrjagin classes, then

$$H^*\mathcal{L}^c_U \cong AL_{m-1}(H_*U, \tilde{H}_*X) .$$

This improves a result of A. Haefliger [14, p. 519 Remarques].

Corollary 3.4. $\tilde{H}^*\mathcal{L}_{S^n}$ is additively $E[x] \otimes B_n$ with $B_n \subseteq H^*\mathcal{L}^c_{\mathbb{R}^n}$ and $E[x]$ is an exterior algebra on a class of degree m if m is odd and a class of degree $2m-1$ if m is even.

Note: B_n is not a subalgebra of $H^*\mathcal{L}^c_{\mathbb{R}^n}$ and the description of B_n seems sufficiently messy to avoid giving it a good global description.

These results follow directly from our computations of $H_*C(M, X)$ together with Theorems 3.1 and 3.2 and the requisite observations on products. We now prove the key ingredient, Theorem 3.2.

To start, we state a lemma most conveniently proven in section 6. Observe that Σ_k acts diagonally in a natural way on $H^*F(M, k) \otimes (\tilde{H}^*X)^k$. (See section 2 for details.) We assume that X is of finite type (and of course coefficients are taken in a field of characteristic zero).

Lemma 6.2. $H^*D(M, k)$ is isomorphic to the vector space elements in $H^*F(M, k) \otimes (\widetilde{H}^*X)^k$ invariant under the natural action of Σ_k.

Proof of Theorem 3.2: We first apply the Lefschetz duality isomorphism [4]

$$L : H_p^{\infty}(M^k, M_{k-1}^k) \longrightarrow H^{nk-p}(M^k - M_{k-1}^k)$$

since (M^k, M_{k-1}^k) is orientable where dimension$(M) = n$.

In addition, consider the isomorphism

$$H^{q_i}\mathscr{L}_o(\mathbb{R}^n) \cong H^{q_i}\Sigma^n X \xrightarrow{\sigma^{-n}} H^{q_i - n} X .$$

For any fixed integer k, using the above we define a map

$$\theta : \sum_{\substack{q_1 + \ldots + q_k = q \\ q_i > 0}} H_p^{\infty}(M^k, M_{k-1}^k) \otimes H^{q_1} \otimes \ldots \otimes H^{q_k}$$

$$\longrightarrow \sum_{\substack{q_1 + \ldots + q_k = q \\ q_i > 0}} H^{nk-p}F(M, k) \otimes H^{q_1 - n} X \otimes \ldots \otimes H^{q_k - n} X$$

by the formula

$$\theta(a \otimes y_1 \otimes \ldots \otimes y_k) = (-1)^{\lambda} L(a) \otimes \sigma^{-n} y_1 \otimes \ldots \otimes \sigma^{-n} y_k$$

where $\lambda = n \sum_{i=1}^{k-1} (k-i)|y_i|$ ($|x|$ is the degree of x).

(Remark: the sign $(-1)^{\lambda}$ comes from standard sign conventions when one commutes a graded homomorphism past a variable; here the homomorphisms are σ^{-n} and the variables are the y_j.)

Notice that θ is certainly an isomorphism of vector spaces. In addition, we claim that θ is Σ_k-equivariant where Σ_k acts diagonally on the right-hand vector space in the natural way. We check this in case $k = 2$ and leave the details in case $k > 2$ to the reader:

Let $a \otimes x \otimes y \in H_p^{\infty}(M^2, M_1^2) \otimes H^{q_1} \otimes H^{q_2}$. Then

(i) $\theta(a \otimes x \otimes y) = (-1)^{n|x|} L(a) \otimes \sigma^{-n}x \otimes \sigma^{-n}y.$

Let τ be the non-trivial element in Σ_2. Then

(ii) $\tau\theta(a \otimes x \otimes y) = (-1)^{n|x|+(n+|x|)(n+|y|)} \tau L(a) \otimes \sigma^{-n}y \otimes \sigma^{-n}x$ and

(iii) $\theta \circ \tau(a \otimes x \otimes y) = (-1)^{n|y|+|x|(y)} L(\tau a) \otimes \sigma^{-n}y \otimes \sigma^{-n}x.$

To compare $\tau L(a)$ and $L(\tau a)$, we recall that the Lefschetz duality isomorphism is given by $L(a) = u/a$ where u is the orientation class and that by [21, pp. 287, 297]

$$[(\tau \times \tau)^* u]/a = \tau^*(u/\tau_* a) \quad .$$

Since $(\tau \times \tau)^* u = (-1)^n u$ and τ is of order 2, we have the formula

(iv) $\tau L(a) = (-1)^n L(\tau a).$

Comparing formulas (ii)-(iv) gives $\tau\theta = \theta\tau$ and so θ is an isomorphism of Σ_k-modules.

Consequently the elements in

$$\sum_{\substack{q_1+\dots+q_k=q \\ q_i>0}} H_p^{\infty}(M^k, M^k_{k-1}) \otimes H^{q_1} \otimes \dots \otimes H^{q_k}$$

invariant under Σ_k are isomorphic as vector spaces to those elements in $H^* D_k(M, X)$ of total degree $nk-p + \sum_1^k (q_i - n) = q-p$ by Lemma 6.2. Summing over $k \geq 0$ and observing that $H^* C(M, X) \cong \sum_{k \geq 0} H^* D_k(M, X)$ if X is of finite type (by Theorem 1.1) finishes the proof of the additive structure.

A comparison of the definition of the $E_2^{p,q}$ term of the spectral sequence of Theorem 3.1 with the definition of the $\sum_s E_\infty^{p-s,s,q}$-term of Theorem 2.5 finishes the proof.

Proofs of Corollaries 3.3-3.4:

By Theorem 3.2, we need only compute $H_* C(M, X)$ to compute the E_2^{**}-term of the Gelfand-Fuks spectral sequence. This is done in section 1 for those M which satisfy $\mathcal{W}$ or $M = S^n$.

Remark 6.3. Gelfand-Fuks proved that their spectral sequence abutting to $H^* \mathcal{L}_c^M$ collapses if the rational Pontrjagin classes of M vanish [11]. There is an error in the proof. (See the proof below statement 2.3 on p. 115 of the English translation of [11]. The translation is published by Consultants Bureau, New York.) The statement in error is the following: Consider the natural map

$$\pi_* : \sum_{\substack{q_1+\cdots+q_k=q \\ q_i>0}} H^\infty_* M^k \otimes H^{q_1} \otimes \cdots \otimes H^{q_k}$$

$$\longrightarrow \sum_{\substack{q_1+\cdots+q_k=q \\ q_i>0}} H^\infty_*(M^k, M^k_{k-1}) \otimes H^{q_1} \otimes \cdots \otimes H^{q_k}$$

induced by the map of pairs $M^k \longrightarrow (M^k, M^k_{k-1})$. They assert that π_* is onto. That this assertion is false is easily checked. (For example, let $M = S^3$ and $k = 2$.)

A proof of the collapse result has been given by P. Trauber [22].

§4. On the Anderson-Trauber spectral sequence

In this section assume that M^m is a smooth manifold without boundary, M has a base-point $*$, and X is of finite type. Let X^M be the space of based maps from M to X with compact support. X^M is given the compactly generated topology inherited from the standard compact open topology on X^M. D. W. Anderson [1] and P. Trauber [22] have given a spectral sequence abutting to H^*X^M with coefficients taken in any field. We assume that all (co)homology groups in this section are taken with coefficients in this fixed field.

Set

$$D^k(M) = \left\{ (m_1, \ldots, m_k) \in M^k \,\middle|\, \begin{array}{l} x_i = * \text{ for some } i \text{ or} \\ x_i = x_j \text{ some } i \neq j \end{array} \right\} .$$

<u>Theorem</u> 4.1 (Anderson-Trauber). Let dimension $(M) \leq$ connectivity (X). Then there is a spectral sequence abutting to $H^*(X^M)$ with $E_1^{p,q}$ equal to

$$\sum_{k \geq 0} H_*(C_*^\infty(M^k, D^k(M)) \otimes_{\Sigma_k} (\tilde{H}^* X)^k)$$

where Σ_k acts in the natural way.

The reader should compare this E_1^{**}-term with the E_2^{**} of the Gelfand-Fuks spectral sequence in section 3. In fact, using arguments similar to those given in section 3, it can be shown that as a vector space

$$\sum_{s-r=q-p} E_2^{-r,s} = \sum_{k \geq 0} H^{q-p} D_k(M-*, Y) = H^{q-p} C(M-*, Y)$$

where Y is a space such that $\widetilde{H}_*\Sigma^m Y \cong \widetilde{H}_* X$. Details of these last assertions will probably appear in work of D. W. Anderson and/or P. Trauber. In fact, using these identifications together with Snaith's stable splitting of $\Omega^n\Sigma^n X$ [8 or 18], Anderson observed that this spectral sequence must collapse for $\Omega^n\Sigma^n X$ for path-connected X. Also the spectral sequence of Theorem 2.5 gives the $E_2^{p,q}$-term given in Theorem 4.1. Hence we are two spectral sequences away from computing the cohomology of a function space; also recall that the E_1-term reported in Theorem 2.5 is just given solely in terms of the cohomology and dimensions of the relevant spaces.

Conjecture 1: $H_*(C(\mathbb{R}^n \times M-*, X); \mathbb{F})$ is a functor of the dimension of M, the homology of M and the homology of X for any field $\mathbb{F}$ if $n > 1$.

Conjecture 2: $C(M, X)$ is weakly homotopy equivalent to some function space related to $\Omega^n\Sigma^n X$ if X is path-connected. Note that $C(\mathbb{R}^n, X)$ is weakly equivalent to $\Omega^n\Sigma^n X$ in that case.

Theorem 4.2. Let $M-*$ be an m-manifold which satisfies $\mathcal{W}$ and consider the spectral sequence of Theorem 4.1 abutting to X^M (as described in this section). If X is of finite type, then additively

$$E_2^{**} \cong AL_{m-1}(H_*M-*, \widetilde{H}_*X) .$$

Proof: Since the vector space of classes in total degree q in E_2 of this spectral sequence is isomorphic to $H^q C(M-*, X)$, the result follows from

Theorem 1.2 of section 1.

Remark 4.3: In general, one again uses the spectral sequence of Theorem 2.5 to compute the E_2^{**}-term of the Anderson-Trauber spectral sequence.

Remark 4.4: The spectral sequence considered in this section arises from a filtration given by P. Trauber of a bicomplex due to D. W. Anderson.

§5. The homology of $\Gamma_k(M)$

In this section, we relate our computations to the work of D. McDuff [18] which we review in part. Assume that M is a smooth m-dimensional manifold without boundary and let E_M be the space obtained from the tangent bundle of M by forming the one-point compactification of each fibre in the tangent bundle. E_M is a fibre bundle over M with fibre S^m. Let $\Gamma(M)$ be the space of cross-sections of E_M with compact support. $\Gamma(M)$ has $\mathbb{Z}$-components; let $\Gamma_k(M)$ be the cross-sections of degree k. We recall McDuff's results:

Theorem 5.1 [McDuff]. Let M be a closed compact manifold. Then there are maps $F(M, k)/\Sigma_k \longrightarrow \Gamma_k(M)$ which, for each n, induce isomorphisms $H_n F(M, k)/\Sigma_k \longrightarrow H_n \Gamma_k(M)$ when k is sufficiently large.

Theorem 5.2 [McDuff]. Let M be an open, paracompact manifold. Then there are maps $F(M, k)/\Sigma_k \longrightarrow \Gamma_k(M)$ which induce an isomorphism

$$\lim_{k\to\infty} H_* F(M, k)/\Sigma_k \cong \lim_{k\to\infty} H_* \Gamma_k(M) \quad .$$

The sense in which the $F(M, k)/\Sigma_k$ form a directed system are given in McDuff's paper [18]. Also, it is observed there that if M is open, the homotopy type of $\Gamma_k(M)$ is independent of k.

We remark that there is a comparison between $C(M, X)$ and $\Gamma(M) = \coprod_{k\in\mathbb{Z}} \Gamma_k(M)$. In particular

$$C(M, S^o) = \coprod_{k \geq 0} F(M, k)/\Sigma_k \ .$$

Hence McDuff's theorems can be thought of as a map $F(M, k)/\Sigma_k \longrightarrow \Gamma_k(M)$ giving a natural map

$$C(M, S^o) \longrightarrow \Gamma(M)$$

which is "trying" to be a group completion in homology. (By a theorem originally proven in [6] and [18], this last statement is true for $M = \mathbb{R}^n$.) If M satisfies our axioms for $\underline{\mathcal{W}}$, then this is the essential content of our next theorem although we do not know how to prove an analogue for arbitrary M.

Theorem 5.3. Let M be an m-dimensional manifold of finite type which satisfies $\underline{\mathcal{W}}$. With coefficients in any field R of characteristic zero,

$$H_*\Gamma(M) = R[\mathbb{Z}] \otimes \overline{A}_{m-1}(M)$$

where $R[\mathbb{Z}]$ is the group ring of $\mathbb{Z}$ and

$$\overline{A}_{m-1}(M) = \begin{cases} \text{free commutative algebra on } \tilde{H}_*M \text{ if } m \text{ is odd, and} \\ \text{free commutative algebra on } \tilde{H}_*M \otimes H_*S^{m-1} \text{ if } m \text{ is even.} \end{cases}$$

In case $M = S^m$, we have

Proposition 5.4. With coefficients taken in a field of characteristic zero,

$$H_*F(S^m,\ k)/\Sigma_k \cong \begin{cases} H_*S^m & \text{if } m \text{ is odd and} \\ H_*S^{2m-1} & \text{if } m \text{ is even .} \end{cases}$$

Hence $H_*\Gamma_k S^m$ is the homology of a sphere for $*$ depending on k (as in Theorem 5.1).

Proof of Theorem 5.3:

Recall from Theorem 1.2 that

$$H_*C(M,\ S^o) = AL_{m-1}(H_*M,\ \widetilde{H}_*S^o)$$

if M satisfies $\underline{\mathcal{W}}$. It is easy to check that the directed system in Theorem 5.2

$$\phi_* : H_*F(M,\ k)/\Sigma_k \longrightarrow H_*F(M,\ k+1)/\Sigma_{k+1}$$

is given by $\phi_*(x) = x \circ \varepsilon$ where ε is the non-zero class in $\widetilde{H}_*S^o$. Comparing ϕ_* with example 4 of section 1, we see that

$$\lim_{\overrightarrow{k}} H_*F(M,\ k)/\Sigma_k$$

is additively isomorphic to

(a) the free commutative algebra on $\widetilde{H}_*M$ if $m = 2k+1$, and

(b) the free commutative algebra on $\widetilde{H}_*M \otimes H_*S^{m-1}$ if $m = 2k$.

Since all components of $\Gamma_k(M)$ are homotopy equivalent [18], Theorem 5.3 follows by summing over components.

The reader should compare the computation of ϕ_* with the easily understood case of $M = \mathbb{R}^n$. In [9], ϕ_* is studied in more detail.

Proof of Proposition 5.4:

To compute $H_*F(S^n, k)/\Sigma_k$, we must only compute the elements in $H^*F(S^n, k)$ invariant under the Σ_k-action. We recall that $H^*F(S^n, k)$ is presented in the following way [9]

n odd:

$H^*F(S^n, k) \cong H^*S^n \otimes H^*F(\mathbb{R}^n, k-1)$ as Σ_k-algebra. The action of Σ_k on H^*S^n is trivial and the natural map $H^*F(S^n, k) \longrightarrow H^*F(\mathbb{R}^n, k-1)$ [given by $(x_1, \ldots, x_{k-1}) \longmapsto (x_1, \ldots, x_{k-1}, \infty)$] is Σ_{k-1}-equivariant. (We do not require the full action of Σ_k here to compute fixed points.)

n even:

$H^*F(S^n, k) \cong H^*S^{2n-1} \otimes A_n$ as Σ_k-algebra where A_n is a subalgebra of $H^*F(\mathbb{R}^n, k)$ invariant under the action of Σ_k. Furthermore, a basis for the elements in A_n concentrated in degrees $n-1$ is given by $A_{21}-A_{ij}$ with $i > 2$ and $k \geq i > j \geq 1$ [9]. The Σ_k action is that given on $H^*F(\mathbb{R}^n, k)$ in section 1.

The computation of $H_*F(S^n, k)/\Sigma_k$ is broken up into two cases.

n odd: We first show that the elements in $H^*F(\mathbb{R}^n, k-1)$ fixed by Σ_k are trivial. Since the natural map $H^*F(\mathbb{R}^n, k-1) \longrightarrow H^*F(S^n, k)$ is Σ_{k-1}-equivariant, it suffices to show that $H^*F(\mathbb{R}^n, k-1)/\Sigma_{k-1} = \{0\}$ in order to show that $H^*F(S^n, k)/\Sigma_k \cong H^*S^n$. But, by [7, III.3.3] $\widetilde{H}_*F(\mathbb{R}^n, k-1)/\Sigma_{k-1}$ is trivial if $\widetilde{H}_*\Omega^n_{(k-1)}S^n$ is trivial. Since n is odd, this result is obvious. (See remark 3.15 of [7, III].) Since H^*S^n in $H^*F(S^n, k)$ is invariant under the action of Σ_k, the results for n odd follow.

n even: The method of proof here is similar to that given above. We know that $H_*F(\mathbb{R}^n, k)/\Sigma_k = H_*S^{n-1}$ if n is even. (See remark 3.15 of [7, III] or for a direct calculation, see [7, III §9].) Consequently, the only elements in A_n fixed by Σ_k must lie in dim n-1. A basis for the elements in degree n-1 of A_n is given by

$$A_{21}-A_{ij}, \; i > 2, \; k \geq i > j \geq 1 \; .$$

Recall that the element in $H^*F(\mathbb{R}^n, k)$ fixed by Σ_k is given by $\sum_{k\geq i>j\geq 1} A_{ij}$ [7, p. 279]. It is trivial to check that $\sum_{k\geq i>j\geq 1} A_{ij}$ is not in the image of $H^*F(S^n, k) \longrightarrow H^*F(\mathbb{R}^n, K)$ and that consequently, the only fixed points in A_n are trivial. The result follows.

§6. Proofs of some earlier assertions

We prove Theorem 1.1 and Lemma 6.2 in this section. Recall

Theorem 1.1. If homology is taken with field coefficients of any characteristic, then $H_*C(M, X) \cong H_* \bigvee_{j\geq 0} D_j(M, X)$.

Notice that Theorem 1.1 follows directly from the geometric splitting in [8, Theorem 1.1] if X is path-connected.

Proof of 1.1:

To prove 1.1, notice that there is the standard spectral sequence converging to $H_*C(M, X)$ obtained from the filtration of $C(M, X)$ given in section 1. The E^1_{**} term of this spectral sequence is clearly isomorphic to $H_*\bigvee D_j(M, X)$ since by [8, Lemma 4.4]

$$F_{j-1}C(M, X) \xrightarrow{\iota_j} F_jC(M, X) \xrightarrow{\pi_j} D_j(M, X)$$

is a cofibre sequence. To show that the spectral sequence collapses (from which Theorem 1.1 follows) it suffices to show that ι_j is an injection in homology. Since the natural quotient map $\rho : F_j(M, j) \times_{\iota_j} X^j \longrightarrow D_j(M, X)$ factors through $F_jC(M, X)$, it suffices to prove

Lemma 6.1. ρ_* is an epimorphism where homology is taken with any field coefficients.

Proof: Let A be the subspace of X^j given by

$\{(x_1, \ldots, x_j) | \text{some } x_i = *\}$. Then by [17, Appendix], (X^j, A) is an equivariant NDR pair. Hence there is a cofibre sequence

$$F(M, j) \times_{\Sigma_j} A \xrightarrow{i} F(M, j) \times_{\Sigma_j} X^j \xrightarrow{\rho} D_j(M, X) \quad .$$

To prove 6.1, we show that i_* is monic.

Notice that a basis for H_*X^j is given by $S = \{y_1 \otimes \ldots \otimes y_j | y_i$ runs over basis elements for $H_*X\}$. Let e_0 be the class of the base-point. Then we may partition S as

$$S = \overline{T} \cup \overline{U}$$

where $y_1 \otimes \ldots \otimes y_j \in \overline{T}$ if some $y_i = e_0$ and $y_1 \otimes \ldots \otimes y_j \in \overline{U}$ if $y_i \neq e_0$ for all y_i.

Let T (resp. U) be the vector subspace of H_*X^j spanned by the elements of $\overline{T}$ (resp. $\overline{U}$). Also observe that $H_*C_*F(M, k) \otimes_{\Sigma_k} C_*X^k \cong H_*(C_*F(M, k) \otimes_{\Sigma_k} (H_*X)^k)$ [16, Lemma 1.1].

Notice that T and U are invariant under the action of Σ_j and that $H_*A \cong T$ as Σ_j-module. Clearly $H_*X^j \cong T \oplus U$ as Σ_j modules. Hence we have a splitting

$$H_*(C_*F(M, j) \otimes_{\Sigma_j} H_*X^j) = H_*(C_*F(M, j) \otimes_{\Sigma_j} (T \oplus U)) \quad .$$

Consequently

$$H_*(F(M, j) \times_{\Sigma_j} X^j) \cong (H_*(C_*F(M, j) \otimes_{\Sigma_j} T)) \oplus (H_*(C_*F(M, j) \otimes_{\Sigma_j} U)) \quad .$$

But the natural map $H_*(A) \longrightarrow T$ is an isomorphism of Σ_j-modules and so i_* is an injection.

The proof of Lemma 6.2 is similar to that of Lemma 6.1. For the evident reasons (see the proof), we assume that homology is taken with field coefficients of characteristic zero.

Lemma 6.2. $H^*D_k(M, X)$ is isomorphic to the vector space of elements in $H^*F(M, k) \otimes (\widetilde{H}^*X)^k$ invariant under the natural action of Σ_k.

Proof: First recall that since X is of finite type, homology and cohomology are dual. By the proof of 6.1,

$$H_*D_j(M, X) = H_*(C_*F(M, j) \otimes_{\Sigma_j} U) \quad .$$

But in characteristic zero, it is clear that $H_*(C_*F(M, j) \otimes_{\Sigma_j} U)$ is just the vector space of invariants in $H_*F(M, j) \otimes U$ under the action of Σ_j. But $U \cong (\widetilde{H}_*X)^j$ as a Σ_j-module. The result follows.

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