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On the Representation Theory Associated to the Cohomology of Configuration Spaces

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Dedicated to Mark Mahowald on his 60th birthday.

ABSTRACT. The space of ordered n-tuples in $\mathbf{R}^m$ has an action of the symmetric group on n letters on it. In sections 1 thru 3, we describe the induced action on homology and cohomology with any coefficients via direct sum and induction from certain representations. The rest of the paper is devoted to analyzing these.

The cohomology groups of the configuration space of ordered k-tuples of distinct points in Euclidean space give natural modules over the group ring of the symmetric group. On the other hand, these modules are basic building blocks for certain free graded Lie algebras, [1,2] and Theorem 1.7 here. Thus we consider the representation theory of these modules in some detail. These results overlap with some recent results of Erich Ossa [unpublished] and results of G. Lehrer and L. Solomon, [4,5]. In particular, the papers [4,5] give information when cohomology groups are taken with complex coefficients. The referee informs us that a lot of the methods of Lehrer and Solomon work over the integers or the cyclotomic integers. We wish to thank the referee and Gus Lehrer for pointing out some of the overlap. Namely, in characteristic zero, Theorems 3.9, 5.1 and 5.3 are contained in the papers [4,5] while some of the information concerning graphs in the proof of Theorem 4.8 of the present paper is contained in the paper [5].

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1. Preliminary calculations.

For any space, M, let $F(M,r)$ denote the set of r distinct points in M. The symmetric group, Σ_r, acts on $F(M,r)$ by permuting the coordinates: specifically $\sigma(x_1,\cdots,x_r) = (x_{\sigma(1)},\cdots,x_{\sigma(r)})$, which gives a right action. We know at least two ways to study $F(M,r)$ when M is a manifold. The first goes back to Fadell and Neuwirth, [3]. There is a map, $\pi: F(M,r) \to F(M,s)$ defined by $\pi(x_1,\cdots,x_r) = (x_1,\cdots,x_s)$, whenever $r \geq s$.

THEOREM 1.1. *For any manifold, M, without boundary the map*

$$\pi: F(M,r) \to F(M,s)$$

is a fibre bundle. If $(y_1,\cdots,y_s) \in F(M,s)$, then the fibre over $(y_1,\cdots,y_s)$ is $F(M-\{y_1,\cdots,y_s\}, r-s)$.

The second method to study $F(M,r)$ depends on some embeddings: $\iota: F(M,r) \subset F(M,r-1) \times M$ defined by $\iota(x_1,\cdots,x_r) = (x_1,\cdots,x_r)$ and $\Delta_i: F(M,r-1) \subset F(M,r-1) \times M$ defined by $\Delta_i(x_1,\cdots x_{r-1}) = (x_1,\cdots,x_{r-1},x_i)$, $1 \leq i \leq r-1$. Note $\iota(F(M,r)) \amalg_{1\leq i\leq r-1} \Delta_i(F(M,r-1)) = F(M,r-1) \times M$ and that Δ_i is the restriction of an embedding $M^{r-1} \to M^r$. If M is a manifold, the normal bundle of Δ_i, denoted τ_i, is the pull-back of the tangent bundle of M over the map $p_i: F(M,r-1) \to M$ defined by $p_i(x_1,\cdots,x_{r-1}) = x_i$. We can choose the various τ_i so that their total spaces are disjoint.

THEOREM 1.2. *Up to homotopy, there is a cofibration sequence,*

$$F(M,r) \xrightarrow{\iota} F(M,r-1) \times M \to \bigvee_{1\leq i\leq r-1} T(\tau_i)$$

where $T(\tau_i)$ denotes the Thom space of the bundle, τ_i.

It was demonstrated in [1] that the Fadell–Neuwirth fibrations for $M = \mathbf{R}^m - \{y_1,\cdots,y_t\}$ are orientible and the resulting spectral sequences collapse. As in [1], one can use this to inductively compute $H^*(F(\mathbf{R}^m,r);\mathbf{Z})$. The map ι has a section if $M = \mathbf{R}^m$ and we can also use this approach to inductively compute $H^*(F(\mathbf{R}^m,r);\mathbf{Z})$. With some additional work, one gets the full equivariant algebra structure. Notationally, if $\sigma \in \Sigma_r$, we will use also σ to denote the map induced on $H^*(F(\mathbf{R}^m,r);\mathbf{Z})$. This gives a left action.

Let $S^{m-1} \subset \mathbf{R}^m$ as the set of unit vectors in the standard metric. The antipodal map on S^{m-1} is just the restriction of multiplication by -1 on $\mathbf{R}^m$. The map $i: S^{m-1} \to F(\mathbf{R}^m,2)$ defined by $i(x) = (x,-x)$ is an equivariant homotopy equivalence. The maps $f: F(\mathbf{R}^m,2) \to S^{m-1}$, defined by $f(x_1,x_2) = \dfrac{x_1-x_2}{||x_1-x_2||}$; and $H: F(\mathbf{R}^m,2) \times [0,1] \to F(\mathbf{R}^m,2)$, defined by $H(x_1,x_2,t) = \left(tx_1 + (1-t)\dfrac{x_1-x_2}{||x_1-x_2||}, tx_2 + (1-t)\dfrac{x_2-x_1}{||x_1-x_2||}\right)$ give the equivariant homotopy equivalence. Orient $\mathbf{R}^m$, which is equivalent to choosing a Thom class for its tangent bundle. The cofibration sequence from 1.2, for $r = 2$, induces a long exact sequence in cohomology: define $A_{21} \in H^{m-1}(F(\mathbf{R}^m, 2); \mathbf{Z})$ to be the element whose coboundary is the Thom class. The equivariant homotopy equivalence shows that $A_{21}^2 = 0$ and that, if $\tau \in \Sigma_2$ is the non-trivial element, $\tau(A_{21}) = (-1)^m A_{21}$.

Let $\pi_{ij}: F(\mathbf{R}^m, r) \to F(\mathbf{R}^m, 2)$ be defined by $\pi_{ij}(x_1, \cdots, x_r) = (x_i, x_j)$. Define $A_{ij} = \pi_{ij}^*(A_{21})$. It follows that $A_{ij} = (-1)^m A_{ji}$ and $A_{ij}^2 = 0$. Furthermore, for $\sigma \in \Sigma_r$, $\sigma(A_{ij}) = A_{\sigma(i)\,\sigma(j)}$, since $\pi_{ij} \circ \sigma = \pi_{\sigma(i)\,\sigma(j)}$.

We say that $I = ((i_1, j_1), \cdots, (i_s, j_s))$ is of *class* r provided $i_\ell \neq j_\ell$ and $1 \leq i_\ell, j_\ell \leq r$ for $1 \leq \ell \leq s$. For any I of class r, let $A_I \in H^{(m-1)s}(F(\mathbf{R}^m, r); \mathbf{Z})$ denote the product $A_{i_1 j_1} \cdots A_{i_s j_s}$. We say the sequence I is *ordered* if $i_\ell > j_\ell$ for all ℓ; $i_1 \leq \cdots \leq i_s$; and if $i_t = i_{t+1}$, $j_t \leq j_{t+1}$. Given any I of class r, there is an associated ordered sequence, $o(I) = ((a_1, b_1), \cdots, (a_s, b_s))$ and a permutation, $\sigma \in \Sigma_s$, so that $(a_\ell, b_\ell) = (i_{\sigma(\ell)}, j_{\sigma(\ell)})$ if $i_{\sigma(\ell)} > j_{\sigma(\ell)}$ or else $(a_\ell, b_\ell) = (j_{\sigma(\ell)}, i_{\sigma(\ell)})$ if $i_{\sigma(\ell)} < j_{\sigma(\ell)}$. It follows that $A_I = \pm A_{o(I)}$. We say that I is *weakly ordered* provided $i_\ell > j_\ell$ for all $1 \leq \ell \leq s$. We call the set $\{i_1, \cdots, i_s\}$ the *set of first coordinates* of I: we call the tuple $(i_1, \cdots, i_s)$ the *tuple of first coordinates* of I: we call the set $\{j_1, \cdots, j_s\}$ the *set of second coordinates* of I: we call the tuple $(j_1, \cdots, j_s)$ the *tuple of second coordinates* of I.

Given an I of class r and a $\sigma \in \Sigma_r$, let

$$\sigma(I) = \Big(\big(\sigma(i_1), \sigma(j_1)\big), \cdots, \big(\sigma(i_s), \sigma(j_s)\big)\Big)$$

and let $I(\sigma)$ denote the associated ordered sequence. It follows that $\sigma(A_I) = A_{\sigma(I)} = \pm A_{I(\sigma)}$.

Let $\delta: H^{\ell-1}(\vee_{1 \leq i \leq r-1} T(\tau_i); \mathbf{Z}) \to H^\ell(F(\mathbf{R}^m, r); \mathbf{Z})$ denote the coboundary in the cofibration sequence, with δ_s denoting the restriction to $T(\tau_s)$. The cofibration sequence in cohomology is a long exact sequence of $H^*(F(\mathbf{R}^m, r-1); \mathbf{Z})$ modules. The diagram

$$\begin{array}{ccccc}
F(M, r) & \xrightarrow{\iota} & F(M, r-1) \times M & \to & \vee_{1 \leq i \leq r-1} T(\tau_i) \\
\downarrow{\scriptstyle \pi_{rs}} & & \downarrow{\scriptstyle \pi_{rs}} & & \downarrow{\scriptstyle t_s} \\
F(M, 2) & \xrightarrow{\iota} & M \times M & \to & T(\tau)
\end{array}$$

commutes, where $t_s: \vee_{1 \leq i \leq r-1} T(\tau_i) \to T(\tau)$ sends all the $T(\tau_i)$ to the base point except for $T(\tau_s)$ where it is the map of Thom spaces induced by the map p_i. Let U_s denote the Thom class in $T(\tau_s)$. The diagram shows that $\delta_t(U_s) = 0$ for $t \neq s$ and $\delta_s(U_s) = A_{rs}$. Let $A_\emptyset$ denote the unit in cohomology. For $M = \mathbf{R}^m$, ι has a section, so we get

LEMMA 1.3. *$H^*(F(\mathbf{R}^m, r); \mathbf{Z})$ is a free $H^*(F(\mathbf{R}^m, r-1); \mathbf{Z})$ module on generators $A_\emptyset$ and A_{rs} for $1 \leq s \leq r-1$.*

REMARK. We see that the single suspension of $F(\mathbf{R}^m, r)$ is homotopy equivalent to a wedge of spheres.

As a special case of 1.3, we see that $H^{2(m-1)}\big(F(\mathbf{R}^m, 3); \mathbf{Z}\big)$, is a free $\mathbf{Z}$ module spanned by $A_{21}A_{31}$ and $A_{21}A_{32}$. Hence $A_{31}A_{32}$ is a linear combination of $A_{21}A_{31}$ and $A_{21}A_{32}$.

PROPOSITION 1.4.

$$A_{31}A_{32} = A_{21}A_{32} - A_{21}A_{31} \ .$$

PROOF: Write $A_{31}A_{32}$ as a linear combination of $A_{21}A_{31}$ and $A_{21}A_{32}$. The given formula is the only Σ_3 equivariant formula. A routine calculation for each parity of m establishes the result. ∎

COROLLARY 1.5 (THE 3–TERM RELATION). *If $i < j < k$, then*

$$A_{ki}A_{kj} = A_{ji}A_{kj} - A_{ji}A_{ki}$$

For an ordered $I = \big((i_1, j_1), \cdots, (i_s, j_s)\big)$ of class r, suppose we have an $i_t = i_{t+1}$. If $j_t < j_{t+1}$, then, using the 3–term relation, we get two new sequences: $I\langle 0\rangle$ is the ordered sequence associated to the weakly ordered sequence $\big((i_1, j_1), \cdots, (j_{t+1}, j_t), (i_t, j_{t+1}), \cdots, (i_s, j_s)\big)$; $I\langle 1\rangle$ is the ordered sequence associated to the weakly ordered sequence $\big((i_1, j_1), \cdots, (j_{t+1}, j_t), (i_t, j_t), \cdots, (i_s, j_s)\big)$. It follows that $A_I = A_{I\langle 0\rangle} - A_{I\langle 1\rangle}$. The choice of t may not be unique, but we call any formula like $A_I = A_{I\langle 0\rangle} - A_{I\langle 1\rangle}$ an *elementary 3–term expansion*. If $j_t = j_{t+1}$, then $A_I = 0$.

REMARKS. Notice that the set of second coordinates for I and $I\langle 0\rangle$ are identical: the set of second coordinates for $I\langle 1\rangle$ is contained in the set of second coordinates for I. The tuple of first coordinates for $I\langle 0\rangle$ and $I\langle 1\rangle$ are the same. The tuple of second coordinates for $I\langle 0\rangle$ and $I\langle 1\rangle$ differ only in the t–th position.

Order tuples so that $(a_1, \cdots, a_s) < (b_1, \cdots, b_s)$ provided there exists an ℓ, $1 \le \ell \le s$ such that $a_\ell < b_\ell$ and $a_t = b_t$ for all t, $\ell < t \le s$. The tuple of first coordinates for I is bigger than the tuple of first coordinates for $I\langle 0\rangle$. The tuple of second coordinates for $I\langle 0\rangle$ is bigger than the tuple of second coordinates for $I\langle 1\rangle$.

The ordering shows the process of taking elementary 3–term expansions must eventually terminate, either in 0 or in a sum of terms, $A_{I\langle t\rangle}$, where $I\langle t\rangle = \big((i_1^t, j_1^t), \cdots, (i_s^t, j_s^t)\big)$ with $i_1^t < \cdots < i_s^t$. This suggests calling any ordered $I = \big((i_1, j_1), \cdots, (i_s, j_s)\big)$ with $i_1 < \cdots < i_s$ *admissible*. We call the resulting A_I an admissible monomial. We will also call the empty set admissible of class r for any r and we call $A_\emptyset$ an admissible monomial.

If I is admissible of class $r - 1$, then any $A_I A_{rs}$ with $1 \le s \le r - 1$ is an admissible monomial of class r. It follows from Lemma 1.3 and these remarks that

THEOREM 1.6. *$H^*(F(\mathbf{R}^m, r); \mathbf{Z})$ is the free $\mathbf{Z}$ module based on the set of admissible monomials of class r.*

Consider a graded vector space over a field, V, with V concentrated in degrees greater than 0. Then consider the free graded Lie algebra generated by V, $L[V]$. Write $L[V] = \oplus L_r[V]$ where $L_r[V]$ is the module of Lie tensors of weight exactly r. Further let $\sigma^n V$ be the graded vector space obtained from V by raising all degrees by n.

THEOREM 1.7. *If $m \geq 2$ and $r \geq 1$ then, as vector spaces, $L_r[\sigma^{m-1}V]$ is isomorphic to $\sigma^{1-m}\left(H^{(m-1)(r-1)}(F(\mathbf{R}^m, r); \mathbf{Z}) \otimes_{\Sigma_r} V^{\otimes r}\right)$.*

This theorem shows the close connection between the structure of free Lie algebras and the representations of the symmetric groups considered here. For a proof see [2, p. 112-115] or [1].

2. Terminology for symmetric groups.

(2.1) For any finite set, X, let $|X|$ denote the cardinality of X. We say $|X|$ is the *norm* of X. We let RX denote the free R module with basis the elements of X. Let X and Y be two finite subsets of some ordered set, $\mathcal{O}$; say $X = \{x_1 < \cdots < x_r\}$ and $Y = \{y_1 < \cdots < y_s\}$. We say $X < Y$ (equivalently $Y > X$) iff either $|X| < |Y|$ or else $|X| = |Y| = r$ and $x_\ell < y_\ell$ for some ℓ, $1 \leq \ell \leq r$ with $x_j = y_j$ for all j, $\ell < j \leq r$. Note that for any two finite subsets sets of $\mathcal{O}$, say X and Y, exactly one of $X < Y$, $X = Y$ or $X > Y$ holds. Hence the set of finite subsets of $\mathcal{O}$ is ordered: we refer to this ordering as the *induced ordering*.

(2.2) If S is a set of positive integers, a *partition*, $\mathcal{P}$, of S is a collection of disjoint subsets of S whose union is S. We say that a set, T, is a *component* of $\mathcal{P}$ provided T is one of the subsets of S occurring in $\mathcal{P}$. We use $\pi_0(\mathcal{P})$ to denote the set of components of $\mathcal{P}$.

(2.3) Let Σ_S denote the symmetric group of bijections of S. Let $\Sigma_r = \Sigma_{\{1,\cdots,r\}}$. Given two finite sets, S_1 and S_2, of the same norm, there is a unique order preserving bijection between them. This bijection induces an isomorphism $\Sigma_{S_1} \cong \Sigma_{S_2}$.

(2.4) Let H be a group and define the wreath product, $\Sigma_S \wr H$ to be the semidirect product $H^S \rtimes \Sigma_S$, where H^S in the direct sum of $|S|$ copies of H indexed by the elements of S, and Σ_S acts by permuting the factors.

(2.5) Let Part(S) denote the set of partitions of S. There is an induced action of Σ_S on Part(S). Let $\Sigma\langle\mathcal{P}\rangle$ denote the subgroup of Σ_S which leaves $\mathcal{P}$ fixed : we call it the *partition subgroup* corresponding to $\mathcal{P}$.

(2.6) A partition of a positive integer, r, is a collection of positive integers whose sum is r. If μ is a partition of r, we use $\#\mu$ to denote the number of terms in μ. We use $\#\mu\langle i\rangle$ to denote the number of times i occurs in μ. Given a partition, $\mathcal{P}$, of S, the norm *induces* a partition of $|S|$, denoted $\mu_{\mathcal{P}}$.

(2.7) For each i, $1 \leq i \leq |S|$, let $\mathcal{P}\langle i\rangle \subset S$ denote the union over all components of $\mathcal{P}$ of norm i. Let $e_i(\mathcal{P})$ denote the number of components of $\mathcal{P}$ with norm i.

LEMMA 2.8.
$$\Sigma\langle\mathcal{P}\rangle \subset \Sigma_{\mathcal{P}\langle 1\rangle} \times \cdots \times \Sigma_{\mathcal{P}\langle |S|\rangle}$$
and if we write $\Sigma\langle\mathcal{P}\rangle\langle i\rangle$ *for* $\Sigma\langle\mathcal{P}\rangle \cap \Sigma_{\mathcal{P}\langle i\rangle}$ *then*
$$\Sigma\langle\mathcal{P}\rangle = \Sigma\langle\mathcal{P}\rangle\langle 1\rangle \times \cdots \times \Sigma\langle\mathcal{P}\rangle\langle |S|\rangle .$$

(2.9) Let $\Sigma[\mathrm{o}\mathcal{P}]$ denote the subgroup of $\Sigma\langle\mathcal{P}\rangle$ so that $\sigma \in \Sigma[\mathrm{o}\mathcal{P}]$ iff σ restricted to each component of $\mathcal{P}$ is order preserving. Note $\Sigma[\mathrm{o}\mathcal{P}]$ acts on $\pi_0(\mathcal{P})$.

(2.10) The *Young subgroup* corresponding to $\mathcal{P}$, $Y\langle\mathcal{P}\rangle$ is the subgroup of Σ_S which maps each component of $\mathcal{P}$ into itself: $Y\langle\mathcal{P}\rangle = \times\Sigma_T$ where the product is over the components of $\mathcal{P}$. Note that the Young subgroup is normal in the partition subgroup and we have
$$\Sigma\langle\mathcal{P}\rangle = Y\langle\mathcal{P}\rangle \rtimes \Sigma[\mathrm{o}\mathcal{P}] .$$

LEMMA 2.11.
$$\Sigma[\mathrm{o}\mathcal{P}] = \times\Sigma_{e_i(\mathcal{P})}$$
where the product is over all $e_i(\mathcal{P}) \neq 0$. *Furthermore,*
$$\Sigma\langle\mathcal{P}\rangle\langle i\rangle = \Sigma_{e_i(\mathcal{P})} \wr \Sigma_i .$$
Hence the partition subgroup is a product of wreath products.

(2.12) Let $\mathrm{Part}(|S|)$ denote the set of all partitions of the positive integer, $|S|$. Taking the induced partition defines a map $p\colon \mathrm{Part}(S) \to \mathrm{Part}(|S|)$. Since $\mathrm{Part}(S)$ has an ordering induced by the one on S, there is a section map, $\mathrm{Part}(|S|) \to \mathrm{Part}(S)$, which sends a partition, μ, of $|S|$ to the minimal partition of S which induces it, say $\mathcal{L}_\mu$.

PROPOSITION 2.13. *Let* $\mathcal{L} = \{T_1, \cdots, T_k\}$ *and* $\mathcal{P} = \{U_1, \cdots, U_q\}$ *be two partitions of* S *with the same induced partition of* $|S|$. *There exists a unique element* $\nu \in \Sigma_S$ *with the following three properties:*

a. $\nu(\mathcal{L}) = \mathcal{P}$;

b. if $T_j < T_\ell$ *then* $\nu(T_j) < \nu(T_\ell)$;

c. ν *is order preserving when restricted to each* T_ℓ, $1 \leq \ell \leq k$.

This shows that the map p identifies $\mathrm{Part}(|S|)$ with the orbit space $\mathrm{Part}(S)/\Sigma_S$ and that, within each coset of $\Sigma_S/\Sigma\langle\mathcal{L}\rangle$, the ν defined above is a preferred representative.

(2.14) We say that two partitions are *similar* if they lie in the same orbit of Σ_S on $\mathrm{Part}(S)$.

RECALL OF INDUCTION. Let V be a G module; let $H \subset G$ be a subgroup; and let $W \subset V$ be an H submodule. Write $W|^G$ for $W \otimes_{\mathbf{Z}H} \mathbf{Z}G$. There is a natural map $W|^G \to V$ whose image is the G module generated by W. A choice of coset representatives for G/H yields an isomorphism $W|^G \to \oplus_{G/H} W$ as H modules. We have $H^*(G; W|^G) \cong H^*(H; W)$ with similar formulae for homology and Tate cohomology.

A CONSTRUCTION IN REPRESENTATION THEORY. Let H be a group; let M be a graded H module; and let q be a positive integer. Form the wreath product (2.4), $\Sigma_q \wr H$. Form the q–fold graded tensor product of M over $\mathbf{Z}$, $M^{\otimes q}$. To each element $h \in H$ there is a homomorphism, $M \to M$: taking the tensor product of homomorphisms gives an action of $(H \times \cdots \times H)$ on $M^{\otimes q}$. The symmetric group, Σ_q, acts on $M^{\otimes q}$ by permuting the factors with sign. Specifically, let (ij) denote the transposition which exchanges i and j: then $(ij)(x_1 \otimes \cdots \otimes x_q) = (-1)^{|x_i|\cdot|x_j|} x_{(ij)(1)} \otimes \cdots \otimes x_{(ij)(q)}$, where $|x_\ell|$ denotes the dimension of x_ℓ. These two actions extend to an action of $\Sigma_q \wr H$ on $M^{\otimes q}$. Denote this graded $\Sigma_q \wr H$ module by $\Sigma_q \wr M^{\otimes}$.

3. First description of $H^*(F(\mathbf{R}^m, r); \mathbf{Z})$.

To any monomial, $A_I \in H^*(F(\mathbf{R}^m, r); \mathbf{Z})$, define a graph, $\Gamma(I)$ as follows. Let $A_I = A_{i_1 j_1} \cdots A_{i_\ell j_\ell}$. The vertices are the set $\{1, \cdots, r\}$ and there is an edge between i_t and j_t for all t, $1 \le t \le \ell$. To each path component, x, of $\Gamma(I)$, we get a subset, $T_x \subset \{1, \cdots, r\}$, which consists of all the vertices in x. The sets T_x are disjoint and their union is $\{1, \cdots, r\}$: i.e. we have a partition of $\{1, \cdots, r\}$. We denote this partition by $\mathcal{P}(I)$ and call it the *partition associated to* A_I. Conversely, to any partition, $\mathcal{P}$, of $\{1, \cdots, r\}$, there exist admissible monomials, A_I, whose associated partition is $\mathcal{P}$.

For any partition, $\mathcal{P}$, of $\{1, \cdots, r\}$, let $T(\mathcal{P}, m)$ denote the submodule of $H^*(F(\mathbf{R}^m, r); \mathbf{Z})$ spanned by the admissible monomials, A_I, whose associated partition is $\mathcal{P}$.

LEMMA 3.1. *If $\sigma \in \Sigma_r$ and if $\mathcal{P}$ is a partition of $\{1, \cdots, r\}$, the map induced by σ on $H^*(F(\mathbf{R}^m, r); \mathbf{Z})$ sends $T(\mathcal{P}, m) \to T(\sigma(\mathcal{P}), m)$.*

PROOF: Consider the monomial $\sigma(A_I)$. There is an associated ordered monomial, A_J with $\sigma(A_I) = \pm A_J$: we also have $\Gamma(J) = \Gamma(\sigma(I))$, and the associated graphs are the same. The monomial A_J is not usually admissible, but we can apply the 3–term relation repeatedly to write it as a linear combination of admissible monomials.

Let A_J be an ordered monomial and suppose that we can do an elementary 3–term expansion: suppose $A_J = A_{J\langle 0\rangle} - A_{J\langle 1\rangle}$. Let G denote the simplicial complex which is $\Gamma(J)$ plus one more edge between j_t and j_{t+1} and one 2–simplex whose three vertices are i_t, j_t and j_{t+1}: i.e. the complex, G, is obtained from $\Gamma(J)$ by an elementary expansion. Hence the inclusion $\Gamma(J) \subset G$ is a homotopy equivalence. But G is also obtained from $\Gamma(J\langle 0\rangle)$ (and from $\Gamma(J\langle 1\rangle)$) by an elementary expansion. Hence the associated partitions for J, $J\langle 0\rangle$ and $J\langle 1\rangle$ are all the same. ∎

PROPOSITION 3.2. *Let* $I = ((i_1, j_1), \cdots, (i_k, j_k))$ *be any sequence of pairs with* $i_\ell \neq j_\ell$ *for all* ℓ. *Then* $A_I = 0$ *iff* $H_1(\Gamma(I)) \neq 0$.

PROOF: It is no loss of generality to assume that I is ordered, since ordering it does not change the graph and only changes the monomial by a sign. As we apply the 3–term relation, we get terms whose graphs have the same homology as $\Gamma(I)$ (see the last proof); also, exactly one of the terms in the expansion has the same set of second coordinates as the original I. The expansion using the 3–term relation eventually terminates: either because we get two copies of an A_{ij} or because the monomials are all admissible. If we get two copies of an A_{ij}, then $H_1(\Gamma(I)) \neq 0$. If the monomials are admissible, $H_1(\Gamma(I)) = 0$. Hence, A_I expands to a sum of admissible monomials iff $H_1(\Gamma(I)) = 0$: otherwise it expands to a sum of terms each of which is 0. There is exactly one term in the expansion with the same set of second coordinates as I since the other sets of second coordinates are properly contained in the set for I. Hence, if A_I expands to a sum of admissible monomials, then $A_I \neq 0$. ∎

It follows from Lemma 3.1, that $\Sigma\langle\mathcal{P}\rangle$ acts on $T(\mathcal{P}, m)$. Moreover, $T(\mathcal{P}, m)$ is a submodule which is concentrated in dimension $(m-1)(r - \#\mu)$ where μ is the partition of r associated to $\mathcal{P}$.

LEMMA 3.3.

$$T(\mathcal{L}, m)\big|^{\Sigma_r} = \bigoplus_{\mathcal{P}} T(\mathcal{P}, m)$$

where the sum is over all partitions, $\mathcal{P}$, *similar to* $\mathcal{L}$. *Both sides are* Σ_r *modules and the isomorphism is an isomorphism of* Σ_r *modules.*

PROOF: If A_I is an admissible monomial whose associated partition is $\mathcal{L}$, and if ν is the map defined in Proposition 2.13, then $A_{\nu(I)}$ is an admissible monomial whose associated partition is $\mathcal{P}$. Hence we see that the natural map $T(\mathcal{L}, m)\big|^{\Sigma_r} \to H^*\big(F(\mathbf{R}^m, r); \mathbf{Z}\big)$ is onto $\oplus_{\mathcal{P}} T(\mathcal{P}, m)$. The ranks of the different $T(\mathcal{P}, m)$ are the same and the number of them is equal to the number of cosets in $\Sigma_r/\Sigma\langle\mathcal{L}\rangle$. Hence a rank count shows that our map is injective. ∎

THEOREM 3.4. *As a* Σ_r *module, we have*

$$H^*\big(F(\mathbf{R}^m, r); \mathbf{Z}\big) \cong \bigoplus_{\mu} T(\mathcal{L}_\mu, m)\big|^{\Sigma_r}$$

where the sum is over all partitions, μ, *of* r.

PROOF: By Theorem 1.6, $H^*\big(F(\mathbf{R}^m, r); \mathbf{Z}\big) = \oplus_{\mathcal{P}} T(\mathcal{P}, m)$ where the sum is over all partitions of $\{1, \cdots, r\}$. By Lemma 3.3, the sum on the right is the same sum. ∎

COROLLARY 3.5. *Let p be a prime and let V be a $\mathbf{Z}/p\mathbf{Z}[\Sigma_p]$ module. Then, for Tate cohomology,*

$$\check{H}^*\left(\Sigma_p; H^i(F(\mathbf{R}^m,p);\mathbf{Z}/p\mathbf{Z})\otimes V\right)=0 \qquad \text{for} \qquad 0<i<(m-1)(p-1)\ .$$

PROOF: The coefficient group is 0 unless $i=(m-1)e$: in this case

$$H^i(F(\mathbf{R}^m,p);\mathbf{Z}/p\mathbf{Z})=\bigoplus_{\mu} T(\mathcal{L}_\mu,m)\Big|^{\Sigma_p}$$

where the sum is over all partitions μ of p with $p-\#\mu=e$. In our case, $1<\#\mu<p$, so $\Sigma\langle\mathcal{L}_\mu\rangle$ has order prime to p and it follows that $T(\mathcal{L}_\mu,m)$ is a projective $\mathbf{Z}/p\mathbf{Z}[\Sigma\langle\mathcal{L}_\mu\rangle]$ module: hence $T(\mathcal{L}_\mu,m)\big|^{\Sigma_p}$ is a projective $\mathbf{Z}/p\mathbf{Z}[\Sigma_p]$ module. It follows that $T(\mathcal{L}_\mu,m)\big|^{\Sigma_p}\otimes V$ is also a projective $\mathbf{Z}/p\mathbf{Z}[\Sigma_p]$ module and the result follows. ∎

COROLLARY 3.6. *Let p be a prime and let V be a $\mathbf{Z}/p\mathbf{Z}[\Sigma_p]$ module. Then*

$$\check{H}^*\left(\Sigma_p; H^{(m-1)(p-1)}(F(\mathbf{R}^m,p);\mathbf{Z}/p\mathbf{Z})\otimes V\right)\cong \check{H}^{*+\epsilon}(\Sigma_p;V)$$

where $\epsilon=p$ if m is even and $\epsilon=1$ if m is odd.

PROOF: Consider the spectral sequence for $H^*(C_*(F(\mathbf{R}^m,p));V)$ where C_* denotes the singular chains. There is a differential

$$d\colon H^*\left(\Sigma_p; H^{(m-1)(p-1)}(F(\mathbf{R}^m,p);\mathbf{Z}/p\mathbf{Z})\otimes V\right)\to H^{*+(m-1)(p-1)+1}(\Sigma_p;V)\ .$$

Since $C_*(F(\mathbf{R}^m,p))$ is a finite dimensional complex, it follows from Corollary 3.5 that, for all large enough $*$, d must be an isomorphism. Since the Tate cohomology for Σ_p is periodic with period $2(p-1)$ the result follows. ∎

We conclude this section with a description of $T(\mathcal{P},m)$. Let $\mathbf{T}(S,m)$ denote $T(\mathcal{P},m)$ where $\mathcal{P}$ is the partition one of whose components is S and the rest are single element sets. Let $\mathbf{T}(r,m)$ be $\mathbf{T}(\{1,\cdots,r\},m)$. If S_1 and S_2 have the same norm, then the order preserving map between S_1 and S_2 induces a Σ_s isomorphism between $\mathbf{T}(S_1,m)$ and $\mathbf{T}(S_2,m)$ (where $s=|S_1|=|S_2|$ and we use (2.3) to identify Σ_{S_1},Σ_{S_2} and Σ_s). If S has only a single element, let $\mathbf{T}(S,m)$ be isomorphic to $\mathbf{Z}$ generated by $A_\emptyset$.

LEMMA 3.7. *$\mathbf{T}(S,m)$ is isomorphic to $H^{(m-1)(|S|-1)}(F(\mathbf{R}^m,|S|);\mathbf{Z})$ as Σ_S modules.*

LEMMA 3.8. *Let $T_1,\cdots,T_k$ denote the components of $\mathcal{P}$. The product in $H^*(F(\mathbf{R}^m,r);\mathbf{Z})$ induces an embedding,*

$$\mathbf{T}(T_1,m)\otimes\cdots\otimes\mathbf{T}(T_k,m)\subset H^{(m-1)e}(F(\mathbf{R}^m,r);\mathbf{Z})\ .$$

The image of this embedding is $T(\mathcal{P},m)$: $\Sigma\langle\mathcal{P}\rangle$ acts on $\mathbf{T}(T_1,m)\otimes\cdots\otimes\mathbf{T}(T_k,m)$ and the isomorphism between $\mathbf{T}(T_1,m)\otimes\cdots\otimes\mathbf{T}(T_k,m)$ and $T(\mathcal{P},m)$ is $\Sigma\langle\mathcal{P}\rangle$ equivariant. Use (2.8) and (2.11) to write $\Sigma\langle\mathcal{P}\rangle=\times_{i=1}^{k}\Sigma_{e_i(\mathcal{P})}\wr\Sigma_i$. We get an *equivariant isomorphism between* $\mathbf{T}(T_1, m) \otimes \cdots \otimes \mathbf{T}(T_k, m)$ *and* $\otimes_{i=1}^{k} \Sigma_{e_i(\mathcal{P})} \wr \mathbf{T}(i, m)^{\otimes}$.

Putting 3.8 and 3.4 together, we get

THEOREM 3.9. *As a* Σ_r *module,*

$$H^*\big(F(\mathbf{R}^m, r); \mathbf{Z}\big) \cong \bigoplus_{\mu} \Big(\bigotimes_{i=1}^{r} \Sigma_{\#\mu\langle i\rangle} \wr \mathbf{T}(i, m)^{\otimes} \Big) \Big|^{\Sigma_r}$$

where the sum is over all partitions, μ, *of* r.

Hence we have described $H^*\big(F(\mathbf{R}^m, r); \mathbf{Z}\big)$ in terms of standard representation theoretic constructions and the various $H^{(m-1)(r-1)}(F(\mathbf{R}^m, r); \mathbf{Z}) = \mathbf{T}(r, m)$.

4. Preliminary properties of $\mathbf{T}(r, m)$ for $m > 1$.

We introduce some more notation. Any admissible monomial in $\mathbf{T}(r, m)$ is an A_I for $I = \big((2,1), (3, j_3), \cdots, (r-1, j_{r-1}), (r, j_r)\big)$, with $1 \le j_i < i$. This makes the first coordinates redundant so we write A_I as $[1, j_3, \cdots, j_{r-1}, j_r]_m$. As a $\mathbf{Z}$ module, $\mathbf{T}(r, m)$ is free with a basis consisting of all the admissible monomials.

Big $\mathbf{T}(r, m)$ can be constructed from smaller ones as follows.

DEFINITION 4.1. For $m > 1$ and $1 \le s \le r$, let $V(r, s, m)$ denote the free abelian group spanned by the symbols A_I for $I = \big((s+1, j_{s+1}), \cdots, (r, j_r)\big)$ with $1 \le j_i < i$ for every i, $s+1 \le i \le r$. Note $V(r, s, m)$ is a $\mathbf{Z}$ submodule of $H^*\big(F(\mathbf{R}^m, t); \mathbf{Z}\big)$ for any $t \ge r$. Hence both $V(r, s, m)$ and $\mathbf{T}(s, m)$ are submodules of $H^*\big(F(\mathbf{R}^m, r); \mathbf{Z}\big)$: the product gives a map $i_{s,r,m}\colon \mathbf{T}(s, m) \otimes V(r, s, m) \to \mathbf{T}(r, m)$.

LEMMA 4.2. *For* $m > 1$, *the map* $i_{r,s,m}$ *is a* $\Sigma_s \times \Sigma_{\{s+1, \cdots, r\}}$ *equivariant isomorphism.*

PROOF: Consider the collection of $J = \big((i_{s+1}, j_{s+1}), \cdots, (i_r, j_r)\big)$ which satisfy

each $i_t > s$ and each j_t satisfies $1 \le j_t < i_t$;

$i_{s+1} \le \cdots \le i_r$;

if $i_t = i_{t+1}$ then $j_t < j_{t+1}$ and $j_{t+1} > s$.

The admissible monomials satisfying these conditions span $V(r, s, m)$. If we perform an elementary 3–term expansion on A_J, we see that the two terms continue to satisfy these conditions. Let $\sigma \in \Sigma_s \times \Sigma_{\{s+1, \cdots, r\}}$ and let A_I be an admissible monomial in $V(r, s, m)$. Then $I(\sigma)$ satisfies the above conditions. ∎

DEFINITION 4.3. We can define a $\Sigma_s \times \Sigma_{\{s+1, \cdots, r\}}$ equivariant filtration on $V(r, s, m)$ by letting $\mathrm{F}_t V(r, s, m)$ be the submodule spanned by the A_I with $I = \big((s+1, j_{s+1}), \cdots, (r, j_r)\big)$ and with at least t of the j_ℓ being less than or equal to s. Since $V(r, 1, m) = \mathbf{T}(r, m)$, we have defined a $\Sigma_{\{2, \cdots, r\}}$ filtration on

$\mathbf{T}(r,m)$: $F_t\mathbf{T}(r,m)$ is spanned by the $[1,j_3,\cdots,j_{r-1},j_r]_m$ where $j_\ell = 1$ for at least t different ℓ.

The next result is an observation.

LEMMA 4.4. *Let ι: $\mathbf{T}(r,m) \to \mathbf{T}(r,m+2)$ be defined by*

$$\iota([1,j_2,\cdots,j_{r-1},j_r]_m) = [1,j_2,\cdots,j_{r-1},j_r]_{m+2}$$

for $m > 1$. Then ι is a Σ_r equivariant isomorphism.

For our next result, let $\mathbf{Z}^-$ denote the group $\mathbf{Z}$ made into a Σ_r module via the sign representation.

LEMMA 4.5. *For $m > 1$, let ι^-: $\mathbf{T}(r,m) \to \mathbf{Z}^- \otimes \mathbf{T}(r,m+1)$ be defined by $\iota^-([1,j_2,\cdots,j_{r-1},j_r]_m) = 1 \otimes [1,j_2,\cdots,j_{r-1},j_r]_{m+1}$. Then ι^- is a Σ_r equivariant isomorphism.*

PROOF: Check that $\mathbf{T}(2,2\ell) = \mathbf{Z}$ and $\mathbf{T}(2,2\ell+1) = \mathbf{Z}^-$, so the result holds for $r = 2$.

We induct on r using the maps $i_{r,s,m}$ defined above. Using $s = r-1$, we see that ι^- is a $\Sigma_{\{1,\cdots,r-1\}}$ isomorphism. Using $s = r-2$, we can check that ι^- is a $\Sigma_{\{1,\cdots,r-2\}} \times \Sigma_{\{r-1,r\}}$ isomorphism. It follows that ι^- is a Σ_r isomorphism. ∎

It follows from the last two lemmas that we really only need to study the case $\mathbf{T}(r,3)$ and we mostly restrict ourselves to this case in the sequel.

Let $[1,2,\cdots,r-1]$ be denoted by A_r^{max}.

THEOREM 4.6. *For $m > 1$, the map*

$$\Psi\colon \mathbf{Z}\Sigma_{\{1,\cdots,r-1\}}A_r^{max} \to \mathbf{T}(r,m)$$

is a $\Sigma_{\{1,\cdots,r-1\}}$ equivariant isomorphism.

PROOF: Since both sides are free $\mathbf{Z}$ modules of the same rank, it suffices to prove that Ψ is onto. The result can be checked for $r = 2$. The isomorphism $i_{r,r-1,m}$ shows that $T(r,m) = \oplus_{i=1}^{r-1} T(r-1,m)A_{r\,i}$. Let $\sigma \in \Sigma_{\{1,\cdots,r-1\}}$ and note that if $xA_{r\,r-1} \in \mathbf{T}(r-1,m)A_{r\,r-1}$, then $\sigma(xA_{r\,r-1}) = \sigma(x)A_{r\,\sigma(r-1)}$ where $\sigma(x)$ is the result of letting $\sigma \in \Sigma_{r-1}$ act on $x \in T(r-1,m)$. By induction we see that $\Psi(\mathbf{Z}\Sigma_{\{1,\cdots,r-2\}}A_r^{max})$ is onto $\mathbf{T}(r-1,m)A_{r\,r-1}$ and the result follows. ∎

COROLLARY 4.7. *Let $S \subset \{1,\cdots,r\}$ denote any subset with $|S| = r-1$. Then, for $m > 1$, $\mathbf{T}(r,m)$ is a free $\mathbf{Z}\Sigma_S$ module on one generator.*

The $\Sigma_{\{2,\cdots,r\}}$ filtration defined on $\mathbf{T}(r,m)$ in (4.3) can be used to give an alternate proof of Corollary 3.6.

THEOREM 4.8. *Let $m > 1$ be odd. As $\Sigma_{\{2,\cdots,r\}}$ modules,*

$$F_t\mathbf{T}(r,m)/F_{t+1}\mathbf{T}(r,m) \cong H^{(m-1)(r-t-1)}\big(F(\mathbf{R}^m, r-1); \mathbf{Z}\big)$$

where we make $H^*\big(F(\mathbf{R}^m, r-1); \mathbf{Z}\big)$ into a $\Sigma_{\{2,\cdots,r\}}$ module using (2.3).

PROOF: Let Γ be a graph with vertices between 2 and r and with no 1 cycles. Define a map $\bar{\rho}(A_\Gamma) \in \mathbf{T}(r,m)$ by setting $\bar{\rho}(A_\Gamma) = A_\Gamma A_M$ where $M = \big((m_1,1)\cdots,(m_t,1)\big)$; where Γ has t path components; and where $m_1 < \cdots < m_t$ run over the minimal elements in each of these path components. Let $\alpha: \{1,\cdots,r-1\} \to \{2,\cdots r\}$ be the unique order preserving homomorphism. Define $\hat{\rho}: H^*\big(F(\mathbf{R}^m, r-1); \mathbf{Z}\big) \to \mathbf{T}(r,m)$ by $\hat{\rho}(A_\Gamma) = \bar{\rho}(A_{\alpha(\Gamma)})$. If we can do a 3–term expansion in A_Γ, suppose we get $A_\Gamma = A_{\Gamma+} - A_{\Gamma-}$. It follows that $\hat{\rho}(A_\Gamma) = \hat{\rho}(A_{\Gamma+}) - \hat{\rho}(A_{\Gamma-})$.

The map $\hat{\rho}$ has an inverse given by just dropping all the terms in the product for A_Γ which have a 1 in their second coordinate and then applying α^{-1}.

If we restrict $\hat{\rho}$ to a particular dimension, it lands in a particular filtration: specifically $\hat{\rho}: H^{(m-1)(r-t-1)}\big(F(\mathbf{R}^m, r-1); \mathbf{Z}\big) \to F_t\mathbf{T}(r,m)$. We use $\hat{\rho}_t$ to denote the map into the associated graded, $\hat{\rho}_t: H^{(m-1)(r-t-1)}\big(F(\mathbf{R}^m, r-1); \mathbf{Z}\big) \to F_t\mathbf{T}(r,m)/F_{t+1}\mathbf{T}(r,m)$.

Since $\hat{\rho}$ is an isomorphism, so is $\hat{\rho}_t$. Moreover, we shall check that $\hat{\rho}_t$ is equivariant. Let $\sigma \in \Sigma_{\{1,\cdots,r-1\}}$ and let $\tau \in \Sigma_{\{2,\cdots,r\}}$ be the result of applying α to σ. Let $\sigma(A_\Gamma) = (-1)^e A_{\sigma(\Gamma)}$ and note $\tau(A_{\alpha(\Gamma)}) = (-1)^e A_{\tau(\alpha(\Gamma))}$. Since $\alpha(\sigma(\Gamma)) = \tau(\alpha(\Gamma))$, $\hat{\rho}_t(A_{\sigma(\Gamma)}) = \bar{\rho}(A_{\alpha(\sigma(\Gamma))}) = \bar{\rho}(A_{\tau(\alpha(\Gamma))}) = A_{\tau(\alpha(\Gamma))}A_M$. Let $\hat{\rho}_t(A_{\alpha(\Gamma)}) = A_{\alpha(\Gamma)}A_K$: $\tau\Big(\hat{\rho}_t(A_\Gamma)\Big) = \tau\Big(\bar{\rho}(A_{\alpha(\Gamma)})\Big) = \tau(A_{\alpha(\Gamma)})A_{\tau(K)} = (-1)^e A_{\tau(\alpha(\Gamma))}A_{\tau(K)}$. In general, $\tau(K) \neq M$ because it might happen that $\tau(k_i)$ is not be the minimal element in its path component. If $\tau(k_i)$ is not the minimal element in its path component, then the 3–term relation can be applied with $(\tau(k_i), 1)$ as one of the terms. Modulo F_{t+1}, there is only one term in the 3–term relation: we can continue to expand until we get to the minimal element. Hence $A_{\tau(\alpha(\Gamma))}A_{\tau(K)} = A_{\tau(\alpha(\Gamma))}A_M$ in F_t/F_{t+1}, so $\tau\Big(\hat{\rho}_t(A_\Gamma)\Big) = \hat{\rho}_t(\sigma(A_\Gamma))$ in F_t/F_{t+1}. ∎

COROLLARY 4.9. *For $m > 1$ and odd, $H^*\big(F(\mathbf{R}^m, r); \mathbf{C}\big)$ is the regular representation.*

We need a generalization of (4.8) for later. Let $\mathcal{P}$ be a partition of $\{s+1,\cdots,r\}$. Since $\Sigma[o\mathcal{P}]$ acts on $\pi_0(\mathcal{P})$ and Σ_s acts on $\{1,\cdots,s\}$, $\Sigma_s \times \Sigma[o\mathcal{P}]$ acts on $X_{s,\mathcal{P}}$, the set of maps from $\pi_0(\mathcal{P})$ to $\{1,\cdots s\}$. The epimorphism $\Sigma\langle\mathcal{P}\rangle \to \Sigma[o\mathcal{P}]$ extends the above action to an action of $\Sigma_s \times \Sigma\langle\mathcal{P}\rangle$ on $X_{s,\mathcal{P}}$. Hence $\Sigma_s \times \Sigma\langle\mathcal{P}\rangle$ acts on $T(\mathcal{P},m) \otimes \mathbf{Z}X_{s,\mathcal{P}}$

PROPOSITION 4.10. *When m is odd we have, as $\Sigma_s \times \Sigma_{\{s+1,\cdots,r\}}$ modules,*

$$F_tV(r,s,m)/F_{t+1}V(r,s,m) \cong \bigoplus_{\mathcal{P}} T(\mathcal{P},m) \otimes \mathbf{Z}X_{s,\mathcal{P}}\Big|^{\Sigma_s\times\Sigma_{\{s+1,\cdots,r\}}}$$

where the sum is over a set of partitions $\mathcal{P}$ of $\{s+1,\cdots,r\}$ where we take one partition in each similarity class with $|\mathcal{P}|=t$.

PROOF: The proof is similar to that of Theorem 4.8. Define $\bar{\rho}(A_\Gamma \otimes f) = A_\Gamma A_M \in V(r,s,m)$ where $M = \big((m_1, f([m_1])),\cdots,(m_t, f([m_t]))\big)$ with $m_1 < \cdots < m_t$ being the minimal elements in the components of Γ; with $[m_i]$ denoting the component of Γ containing m_i; and with f being a function from the components of Γ to $\{1,\cdots,s\}$. As before $\bar{\rho}$ maps to the associated graded and we check that it is equivariant. ∎

5. $H^*\big(F(\mathbf{R}^m,r);\mathbf{Q}\big)$ as a Σ_r module.

Since Σ_r acts freely on $F(\mathbf{R}^m,r)$, the Lefschetz number of any $\sigma\in\Sigma_r$, $\sigma\neq \mathrm{id}$ is 0. (Even though $F(\mathbf{R}^m,r)$ is not compact, it has a compact, equivariant deformation retract, so the Lefschetz fixed point theorem applies.) If m is odd, the Lefschetz number of σ is also the character: it follows that $H^*\big(F(\mathbf{R}^m,r);\mathbf{Q}\big)$ is just the regular representation (which also follows from Theorem 4.8 and Corollary 4.7). If m is even, $H^*\big(F(\mathbf{R}^m,r);\mathbf{Q}\big) = A_e(r,m)\oplus A_o(r,m)$, where $A_e(r,m)$ (resp. $A_o(r,m)$) denotes the submodule of elements concentrated in even (resp. odd) dimensions. The Lefschetz fixed point argument shows that $A_e(r,m)\otimes\mathbf{Q}$ and $A_o(r,m)\otimes\mathbf{Q}$ are the same representation: let $\chi^r_{1/2}$ denote the character.

THEOREM 5.1. *The representations $A_e(r,m)\otimes\mathbf{Q}$ and $A_o(r,m)\otimes\mathbf{Q}$ are $1_{(12)}\big|^{\Sigma_r}$, where $1_{(12)}$ is the trivial representation on the subgroup generated by the transposition* (12). *The character is given by*

$$\chi^r_{1/2}(\sigma) = \begin{cases} r!/2 & \text{if } \sigma = \mathrm{id} \\ (r-2)! & \text{if } \sigma \text{ is a single transposition} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF: Let $c\colon \mathbf{R}^m\to\mathbf{R}^m$ denote the homeomorphism which is multiplication by -1 on the first coordinate and the identity on the other coordinates: let $F(c,r)$ denote the induced homeomorphism on $F(\mathbf{R}^m,r)$. The map induced by $F(c,r)$ on $H^*\big(F(\mathbf{R}^m,r);\mathbf{Q}\big)$ is the identity in even dimensions and multiplication by -1 in odd dimensions: hence the Lefschetz number of $F(c,r)\circ\sigma$ is just $2\chi^r_{1/2}(\sigma)$. The map $F(c,r)\circ\sigma$ has no fixed points unless $\sigma^2=\mathrm{id}$.

If $r=2$, the result can be checked, so we may assume by induction that the formula holds for $\chi^{r-1}_{1/2}$. Recall $H^*\big(F(\mathbf{R}^m,r);\mathbf{Q}\big) = H^*\big(F(\mathbf{R}^m,r-1);\mathbf{Q}\big)\otimes (\mathbf{Q}^+\oplus V(r,r-1,m)\otimes\mathbf{Q})$ as Σ_{r-1} modules (where $\mathbf{Q}^+$ is a trivial Σ_{r-1} module): $\chi^r_{1/2}(\sigma)$ can now be calculated for any σ with a fixed point and our formula holds.

This completes the proof except for the case that $r=2\ell$ and $\sigma_\ell = (12)\cdots(r-1\,r)$. For this case, note $H^*\big(F(\mathbf{R}^m,r);\mathbf{Q}\big) = H^*\big(F(\mathbf{R}^m,r-2);\mathbf{Q}\big)\otimes(W\oplus$

$V(r, r-2, m) \otimes \mathbf{Q})$, where W is the $\Sigma_{r-2} \times \Sigma_{\{r-1,r\}}$ module with $\mathbf{Q}$ basis $\{1\} \cup \{A_{r-1\,1}, \cdots, A_{r-1\,r-2}\} \cup \{A_{r1}, \cdots, A_{r\,r-1}\}$.

Let $\sigma_\ell = \sigma_{\ell-1} \times (r-1\,r)$ and note $\chi^r_{1/2}(\sigma_\ell) = \chi^{r-2}_{1/2}(\sigma_{\ell-1}) \cdot \chi(\sigma_\ell)$, where $\chi(\sigma_\ell)$ is the trace of σ_ℓ acting on $W \oplus V(r, r-2, m) \otimes \mathbf{Q}$. If $r = 4$, one can calculate $\chi^4_{1/2}(\sigma_2) = 0$ and then the full result follows by induction.

The alternate description follows from the character formula. ∎

From Corollary 4.9 and Theorem 5.1 we see

COROLLARY 5.2. *If m is odd, there is one copy of the trivial representation in $H^*(F(\mathbf{R}^m, r); \mathbf{Q})$ and it occurs in dimension 0 If m is even, there are two copies of the trivial representation, one in dimension 0 and the other in dimension $(m-1)$ generated by $\sum_{1 \le j < i \le r} A_{ij}$.*

THEOREM 5.3. *Let $\xi_r : \mathbf{Z}/r\mathbf{Z} \to \mathbf{C}$ be the faithful representation defined by $1 \mapsto e^{\frac{2\pi i}{r}}$. The representation $\mathbf{T}(r, 2k+1) \otimes \mathbf{C}$, $k > 0$, is given by $\xi_r \Big|^{\Sigma_r}$, where $\mathbf{Z}/r\mathbf{Z} \subset \Sigma_r$ is the r cycle. The character, χ, is given by*

$$\chi(\sigma) = \begin{cases} 0 & \text{unless } \sigma \in \Sigma_r \text{ is the product of } \ell \text{ disjoint } L \text{ cycles} \\ (\ell-1)!\,(L)^{\ell-1}\mu(L) & \text{in this case} \end{cases}$$

PROOF: To begin, we compute the character of $\xi_r \Big|^{\Sigma_r}$. First, $\chi(\sigma) = 0$ unless σ is conjugate to an element in $\mathbf{Z}/r\mathbf{Z}$. This happens iff σ is a product of ℓ disjoint L cycles, where $r = \ell L$. In this case,

$$\chi(\sigma) = \frac{1}{r}\left|Z_{\Sigma_r}(\sigma)\right| \sum_{\phi \in Aut(\mathbf{Z}/L\mathbf{Z})} \xi_r(\phi(\ell))$$

where $Z_{\Sigma_r}(\sigma)$ denotes the centralizer of σ. It is a classical result that

$$\sum_{\phi \in Aut(\mathbf{Z}/L\mathbf{Z})} \xi_r(\phi(\ell)) = \mu(L)$$

and one can compute $\left|Z_{\Sigma_r}(\sigma)\right| = \ell!\,(L)^\ell$. Hence $\chi(\sigma) = (\ell-1)!\,(L)^{\ell-1}\mu(L)$.

The result on $\left|Z_{\Sigma_r}(\sigma)\right|$ may be seen as follows. It suffices to do the case

$$\sigma = (1 \cdots L) \cdots (aL+1 \ \cdots (a+1)L) \cdots ((\ell-1)L+1 \ \cdots r)\ .$$

There is a normal subgroup $N \subset Z_{\Sigma_r}(\sigma)$ which leaves the cycles in the same order. The quotient, $Z_{\Sigma_r}(\sigma)/N$ is Σ_ℓ. We claim that N consists of all elements of the form

$$(1 \cdots L)^{e_1} \cdots (aL+1 \ \cdots (a+1)L)^{e_{a+1}} \cdots ((\ell-1)L+1 \ \cdots r)^{e_\ell}$$

so there are L^ℓ of these. Clearly all these elements are in N; also,

$$N \subset \Sigma_{\{1,\cdots,L\}} \times \cdots \times \Sigma_{\{(\ell-1)L+1,\cdots,r\}}$$

so the result follows from the fact that the centralizer of the t cycle in Σ_t is just the subgroup generated by that t cycle.

Now, we start to compute the character, $\hat{\chi}_r$, of Σ_r acting on $T(r, 2k+1)$. First of all, if $\sigma' \in \Sigma_{r-1}$, $\hat{\chi}_r(\sigma') = 0$. Since $\hat{\chi}_r$ is conjugation invariant, we see that $\hat{\chi}_r(\sigma') = 0$ whenever σ' has a fixed point. Hence we can assume that σ' is a product of disjoint cycles with no element of $\{1, \cdots, r\}$ missing. We may further assume that $\sigma' = \sigma \times (s+1 \cdots r)$ and that the cycle $(s+1 \cdots r)$ is as short as any cycle in σ. Let us further assume that $\sigma \neq \text{id}$ and that $(s+1 \cdots r) \neq \text{id}$ (so $s+1 < r$).

Next we compute $\hat{\chi}_r(\sigma \times (s+1 \cdots r))$. We get $\hat{\chi}_r(\sigma \times (s+1 \cdots r)) = \hat{\chi}_s(\sigma) \cdot \chi_V(\sigma \times (s+1 \cdots r))$, where $\chi_V(\sigma \times (s+1 \cdots r))$ is the trace of the transformation $\sigma \times (s+1 \cdots r)$ acting on $V(r, s, m)$. We need only do the case in which $\sigma \in \Sigma_s$ satisfies $\hat{\chi}_s(\sigma) \neq 0$: by induction, we may assume that σ is the product of disjoint cycles, all of the same size, say L. Note $L \geq r-s$ by our previous assumptions.

We have $\chi_V(\sigma \times (s+1 \cdots r)) = \sum_{i=1}^{r-s-1} \chi_{V_i}(\sigma \times (s+1 \cdots r))$, where $\chi_{V_i}(\sigma \times (s+1 \cdots r))$ is the trace of $\sigma \times (s+1 \cdots r)$ acting on $V_i = F_i V(r, s, m)/F_{i+1} V(r, s, m)$.

By Proposition 4.10, $\chi_{V_i}(\sigma \times (s+1 \cdots r)) = \sum \chi_{\mathcal{P}}(\sigma \times (s+1 \cdots r))$, where $\chi_{\mathcal{P}}$ is the character of the representation $T(\mathcal{P}, m) \otimes \mathbf{Z}X_{s,\mathcal{P}}\big|^{\Sigma_s \times \Sigma_{\{s+1,\cdots,r\}}}$; $\chi_{\mathcal{P}}(\sigma \times (s+1 \cdots r)) = 0$ unless $\sigma \times (s+1 \cdots r)$ is conjugate to an element in $\Sigma_s \times \Sigma\langle\mathcal{P}\rangle$. Without loss of generality, we can assume that $\sigma \times (s+1 \cdots r) \in \Sigma_s \times \Sigma\langle\mathcal{P}\rangle$ and then $\chi_{\mathcal{P}}(\sigma \times (s+1 \cdots r)) = \chi_{T(\mathcal{P},m)}(\sigma) \cdot \chi_X(\sigma \times (s+1 \cdots r))$, where χ_X is the character of the representation $\mathbf{C}X_{s,\mathcal{P}}$.

Since this is a permutation representation, $\chi_X(\sigma \times (s+1 \cdots r)) = e$ where e is the number of elements in $X_{s,\mathcal{P}}$ left fixed by $\sigma \times (s+1 \cdots r)$. Since σ is a product of disjoint L cycles, $e = 0$ unless L divides the order of each orbit of $\Sigma[o\mathcal{P}]$ acting on $\pi_0(\mathcal{P})$. There are at most $r-s$ elements in $\pi_0(\mathcal{P})$, and since $r - s \leq L$, we get $L = r-s$; $\Sigma[o\mathcal{P}] = \Sigma_{\{s+1,\cdots,r\}}$; and each component of $\mathcal{P}$ has exactly one element.

In this case, $T(\mathcal{P}, m) = \mathbf{Z}$ (acted on trivially) and there are $\ell \cdot L$ fixed elements in $X_{s,\mathcal{P}}$. By induction, it follows that $\hat{\chi}_r(\sigma \times (s+1 \cdots r)) = (\ell-1)!(L)^{\ell-1}\mu(L) = \xi_r\big|^{\Sigma_r}(\sigma \times (s+1 \cdots r))$ and hence that $\hat{\chi}_r(\sigma') = \xi_r\big|^{\Sigma_r}(\sigma')$ for all $\sigma' \in \Sigma_r$ except possibly for the r cycle. The inner product formula for characters now shows that these two characters also agree on the r cycle iff they contain the same number of copies of the trivial representation. By Corollary 5.2, $\mathbf{T}(r, m)$ contains no copies of the trivial representation. The representation $\xi_r\big|^{\Sigma_r}$ contains no copies of the trivial representation by Frobenius reciprocity. ∎

We can improve Theorem 5.3 somewhat.

PROPOSITION 5.4. *Let $N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})$ denote the normalizer of the r cycle in*

Σ_r. *Then there exists a projective* $\mathbf{Z}[\frac{1}{r}]\left[N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})\right]$ *module,* P*, such that*

$$P\Big|^{\Sigma_r} = \mathbf{T}(r,m)\otimes\mathbf{Z}[\tfrac{1}{r}]$$

PROOF: It is known that the normalizer of the r cycle in Σ_r, $N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})$, is isomorphic to the semi–direct product of $\mathbf{Z}/r\mathbf{Z}$ and $\mathrm{Aut}(\mathbf{Z}/r\mathbf{Z})$.

There is an epimorphism, $\mathbf{Z}\Sigma_r \to \mathbf{T}(r,m)$ defined by letting $x \in \mathbf{Z}\Sigma_r$ act on A^{max}. Let $e \in \mathbf{Q}\mathbf{Z}/r\mathbf{Z}$ be the central idempotent corresponding to the faithful irreducible rational representation of $\mathbf{Z}/r\mathbf{Z}$. There is a formula for e which shows that $e \in \mathbf{Z}[\frac{1}{r}]\mathbf{Z}/r\mathbf{Z}$ and that e is invariant under conjugation by any element in $\mathrm{Aut}(\mathbf{Z}/r\mathbf{Z})$. We may consider e as an element of $\mathbf{Z}[\frac{1}{r}]\Sigma_r$, and then note that $e\mathbf{Z}[\frac{1}{r}]\Sigma_r$ maps onto $e\mathbf{T}(r,m)\otimes\mathbf{Z}[\frac{1}{r}]$.

It follows that $e\mathbf{T}(r,m)\otimes\mathbf{Z}[\frac{1}{r}]$ is an $N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})$ summand of $\mathbf{T}(r,m)\otimes\mathbf{Z}[\frac{1}{r}]$ and $e\mathbf{Z}[\frac{1}{r}]\Sigma_r$ is an $N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})$ summand of $\mathbf{Z}[\frac{1}{r}]\Sigma_r$. One can calculate directly that $e\mathbf{Z}[\frac{1}{r}]\Sigma_r$ induced up to Σ_r is all of $\mathbf{Z}[\frac{1}{r}]\Sigma_r$. It follows that $e\mathbf{T}(r,m)\otimes\mathbf{Z}[\frac{1}{r}]$ induced up to Σ_r is all of $\mathbf{T}(r,m)\otimes\mathbf{Z}[\frac{1}{r}]$. We let $P = e\mathbf{T}(r,m)\otimes\mathbf{Z}[\frac{1}{r}]$.

It further follows that the map $\mathbf{Z}[\frac{1}{r}]\left[N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})\right] \to P$ induces an isomorphism from $e\mathbf{Z}[\frac{1}{r}]\left[N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})\right]$ to P. Hence P is an $N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})$ summand of $\mathbf{Z}[\frac{1}{r}]\left[N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})\right]$ and hence projective. ∎

REMARK. The group $N_{\Sigma_r}(\mathbf{Z}/r\mathbf{Z})$ acts on the ring of algebraic integers, $\mathbf{Z}[\zeta_r]$, where ζ_r is a primitive root of unity of order r. A natural candidate for $T(r,m)$ is to induce this representation up to Σ_r.

6. Results on H^0 and H_0.

Our goal in this section is to compute H^0 and H_0 for the modules $H^{(m-1)e}\big(F(\mathbf{R}^m,r)\big)$. Let R denote a commutative ring. For any abelian group, A, let ${}_mA$ denote the subgroup of elements annihilated by m: for any symmetric group, let A^+ denote A with the trivial action; let A^- denote A with the sign representation. Some notation will make the subsequent discussion easier: let $\mathcal{H}^{\pm}(s;V) = H^0\big(\Sigma_s;(V^{\otimes s})^{\pm}\big)$.

PROPOSITION 6.1.

$$H^0\big(\Sigma_r;H^{(m-1)(r-t)}(F(\mathbf{R}^m,r);R^{\pm}\big) \cong \oplus_{\mu}\big(\bigotimes_{\#\mu\langle\ell\rangle>0}\mathcal{H}^{\lambda}(\#\mu\langle\ell\rangle;H^0(\Sigma_\ell;\mathbf{T}(\ell,m)\otimes R^{\pm})\big)$$

where μ *runs over all partitions of* r *with* $\#\mu = t$ *and where* λ *is* $+$ *except when* $\#\mu\langle\ell\rangle > 1$*;* ℓ *is odd; and either*

we have R^+ *coefficients on the left and* m *is even or*

we have R^- *coefficients on the left and* m *is odd.*

PROOF: It follows from Theorem 3.9 that we need to analyze $H^0\big(\Sigma\langle\mathcal{P}\rangle;T(\mathcal{P},m)\otimes R^{\pm}\big)$. Recall $\Sigma\langle\mathcal{P}\rangle \cong \times\,\Sigma_{\#\mathcal{P}\langle\ell\rangle}\wr\Sigma_\ell$ where the sum is over all ℓ for which $\#\mathcal{P}\langle\ell\rangle > 0$. Hence we need to compute $H^0\big(\Sigma_{\#\mathcal{P}\langle\ell\rangle} \wr \Sigma_\ell; \Sigma_{\#\mathcal{P}\langle\ell\rangle} \wr \mathbf{T}(\ell, m) \otimes R^{\pm}\big)$.

It follows (e.g. from the spectral sequence) that

$$H^0\big(\Sigma_{\#\mathcal{P}\langle\ell\rangle} \wr \Sigma_\ell; \Sigma_{\#\mathcal{P}\langle\ell\rangle} \wr \mathbf{T}(\ell, m) \otimes R^{\pm}\big) = \begin{cases} \mathcal{H}^+(\#\mathcal{P}\langle\ell\rangle; H^0(\Sigma_\ell; \mathbf{T}(\ell, m) \otimes R^{\pm})) \\ \mathcal{H}^-(\#\mathcal{P}\langle\ell\rangle; H^0(\Sigma_\ell; \mathbf{T}(\ell, m) \otimes R^{\pm})) \end{cases}$$

where the top line occurs in case (a) and the bottom line occurs in case (b). It is these two cases which account for the cases in the result.

To begin, consider the sign representation on Σ_r restricted to $\Sigma_{\#\mathcal{P}\langle\ell\rangle} \wr \Sigma_\ell$. We get the sign representation on each copy of Σ_ℓ: we get the sign representation on $\Sigma_{\#\mathcal{P}\langle\ell\rangle}$ if ℓ is odd; we get the trivial representation on $\Sigma_{\#\mathcal{P}\langle\ell\rangle}$ if ℓ is even. A further wrinkle occurs when $\#\mathcal{P}\langle\ell\rangle = 1$ since in this case the sign representation on $\Sigma_{\#\mathcal{P}\langle\ell\rangle}$ is trivial.

Case (b) occurs in two situations. Suppose first that we have R^+ coefficients on the left. Then, we can only be in case (b) iff $\mathbf{T}(\ell, m)$ has odd dimension: this can happen iff m is even and ℓ is odd. Now suppose that we have R^- coefficients on the left. The $\Sigma_{\#\mathcal{P}\langle\ell\rangle}$ action is the sign representation if ℓ is odd; trivial if ℓ is even. If ℓ is odd and m is even, we are in case (a) because we get the tensor product of two copies of the sign representation. If ℓ is even, we are always in case (a): if ℓ and m are both odd, we are in case (b). ∎

To completely evaluate the right hand side of the formula in Proposition 6.1, it suffices to do the case $m = 3$, since from Lemma 4.4 and Lemma 4.5 we can then get the rest. Our first result is an observation.

PROPOSITION 6.2. $\mathbf{T}(1, 3) \cong \mathbf{Z}^+$; $\mathbf{T}(2, 3) \cong \mathbf{Z}^-$.

PROPOSITION 6.3. *Let* $r > 2$. *Then*

$$H^0(\Sigma_r; \mathbf{T}(r, 3) \otimes A^+) = 0 : \; H^0(\Sigma_2; \mathbf{T}(2, 3) \otimes A^+) = [1, 2] \otimes {}_2A \; .$$

PROOF: The result for $r = 2$ can be verified. The remaining conclusion is equivalent to the statement that $\mathbf{T}(r, 3) \otimes A^+$ has no fixed elements for $r > 2$. Recall that $\Sigma_{\{2,\cdots,r\}}$ acts on $\mathbf{T}(r, 3) \otimes A^+$ and it follows from Corollary 5.2 that the fixed submodule of $\mathbf{T}(r, 3) \otimes A^+$ is one dimensional. By inspection, the element $[1, 1, \cdots, 1]$ is fixed. Therefore, any Σ_r fixed submodule of $\mathbf{T}(r, 3) \otimes A^+$ must be contained in the submodule generated by $[1, 1, \cdots, 1]$. If we now apply the transposition (12) to $[1, 1, \cdots, 1]$ we get $-[1, 2, 2, \cdots, 2]$ so there is no fixed submodule. ∎

PROPOSITION 6.4. *Let* $r > 3$. *Then* $H^0(\Sigma_r; \mathbf{T}(r, 3) \otimes A^-) = 0$; $H^0(\Sigma_3; \mathbf{T}(3, 3) \otimes A^-) = ([1, 2] + [1, 1]) \otimes {}_3A$; $H^0(\Sigma_2; \mathbf{T}(2, 3) \otimes A^-) = [1, 2] \otimes A$.

PROOF: The result for $r = 2$ is immediate, so assume $r > 2$. Again recall that $\Sigma_{\{2,\cdots,r\}}$ acts on $\mathbf{T}(r, 3)$, so it follows from Theorem 4.6 that the fixed submodule of $\mathbf{T}(r, 3) \otimes \mathbf{Z}^-$ is a one dimensional $\mathbf{Z}$ summand: let x denote a generator. Any fixed subgroup of $\mathbf{T}(r,3)\otimes A^-$ must lie in the A summand generated by x. If the fixed subgroup is non-zero for any A^-, then $(12)x$ must be a multiple of x, say $(12)x = mx$. If this happens then $x\otimes a \in \mathbf{T}(r,3)\otimes A^-$ is fixed iff $(m+1)a = 0 \in A$. The result for $r=3$ can be verified by checking that $x = [1,1]-2[1,2]$ with $m=2$ in this case. In case $r>3$ we will show that there is no such m.

To do this, we need to identify the element x more precisely. Recall the $\Sigma_{\{2,\cdots,r\}}$ filtration, $F_s\mathbf{T}(r,3)$. If $x\in F_s\mathbf{T}(r,3)$, let $[x]_s$ denote the image of x in $H^{2(r-s-1)}\big(F(\mathbf{R}^3,r-1);\mathbf{Z}\big)$: for $\sigma\in\Sigma_{r-1}$, $\sigma[x]_s = (-1)^{|\sigma|}[x]_s$. If $[x]_s\neq 0$, then $H^{2(r-s-1)}\big(F(\mathbf{R}^3,r-1);\mathbf{Z}^-\big)\neq 0$, and hence $H^{2(r-s-1)}\big(F(\mathbf{R}^3,r-1);\mathbf{Q}^-\big)\neq 0$. It follows from Theorem 4.6 that this can happen for only one value of s.

Let $r = 2\ell$ or $r = 2\ell+1$. It follows by induction on r and (6.1) that $H^{2(r-\ell-1)}\big(F(\mathbf{R}^3,r-1);\mathbf{Q}^-\big)\neq 0$: the corresponding partition of $r-1$ consists of $\ell-1$ 2's and one 1 or ℓ 2's.

It follows by induction on s that $[x]_s = 0$ for $s>\ell$. Mod 2, x is fixed under the $\Sigma_{\{2,\cdots,r\}}$ action, so $x = a_1[1,1,\cdots,1] + 2\sum a_{(j_3,\cdots,j_r)}[1,j_3,\cdots,j_r]$ where a_1 is odd and at least one $j_t\neq 1$ in every term in the sum.

It is easy to compute the action of the transposition (12) on any sum like that for x: $(12)x = -a_1[1,2,\cdots,2] - 2\sum a_{(j_3,\cdots,j_r)}[1,(12)(j_3),\cdots,(12)(j_r)]$ and none of the $[1,(12)(j_3),\cdots,(12)(j_r)]$ are equal to $[1,2,\cdots,2]$ since at least one $j_t\neq 1$. It follows that $[(12)x]_1\neq 0$. Since $\ell>1$, it follows that $(12)x$ can not be a multiple of x. ∎

LEMMA 6.5.

$$H_0(\Sigma_r;\mathbf{T}(r,3)^{\pm})$$

is finite cyclic if $r>2$*:* $H_0(\Sigma_2;\mathbf{T}(2,3)^{\pm})\cong\mathbf{Z}$.

PROOF: By Theorem 4.6, there exists unique $\Sigma_{\{2,\cdots,r\}}$ equivariant map, $\mathbf{T}(r,3)\to\mathbf{Z}^+$ and a unique $\Sigma_{\{2,\cdots,r\}}$ equivariant map, $\mathbf{T}(r,3)\to\mathbf{Z}^-$. This shows that the groups are cyclic. The result for $r=2$ may be verified. It may further be verified that, for $r>2$, $H_0(\Sigma_r;\mathbf{T}(r,3)^{\pm}\otimes\mathbf{Q}) = 0$. Hence $H_0(\Sigma_r;\mathbf{T}(r,3)^{\pm})$ is finite. Since $\mathbf{T}(r,3)\otimes\mathbf{Z}[\frac{1}{r}]$ is projective, it follows that $H_0(\Sigma_r;\mathbf{T}(r,3)^{\pm})$ is a quotient of $\mathbf{Z}/r\mathbf{Z}$.

Use induction to check that for $s+1\le r$,

$$\big((23\cdots r)+\cdots+(2\cdots s+1)\big)A_S^{max} = - A_{31}A_{43}\cdots A_{s\,s-1}A_{s+1\,s}A_{s+1\,2}A_{s+2\,s+1}\cdots A_{r\,r-1}$$

$$(23\cdots s)A_S^{max} = -A_{31}A_{43}\cdots A_{s\,s-1}A_{s\,2}A_{s+1\,2}A_{s+2\,s+1}\cdots A_{r\,r-1}\ .$$

For $s=1$ we get

$$\big((23\cdots r)+\cdots+(23)\big)A_S^{max} = -A_{21}A_{31}A_{43}\cdots A_{r\,r-1}\ .$$

Hence

$$((23\cdots r)+\cdots+(23)+e)A_S^{max}=A_{21}A_{31}A_{43}\cdots A_{r\,r-1}=-(12)A_S^{max}\ ,$$

so

$$((23\cdots r)+\cdots+(23)+(12)+e)A_S^{max}=0\ .$$

If r is odd, we see that $H_0(\Sigma_r;\mathbf{T}(r,3)^-)=0$ and we recover that $H_0(\Sigma_r;\mathbf{T}(r,3)^+)$ is a quotient of $\mathbf{Z}/r\mathbf{Z}$. If $r=2\ell$,

$$((1r)(2,r-1)\cdots(\ell\,\ell+1))[1,2,\cdots,r-1]=-[1,2,\cdots,r-1]$$

so, $H_0(\Sigma_r;\mathbf{T}(r,3)^+)$ is a quotient of $\mathbf{Z}/2\mathbf{Z}$ when r is even. Moreover, $H_0(\Sigma_r;\mathbf{T}(r,3)^-)$ is a quotient of $\mathbf{Z}/2\mathbf{Z}$ when $r\equiv 0 \pmod 4$. ∎

The cohomology periodicity argument shows that $H_0(\Sigma_r;\mathbf{T}(r,3)^+)\cong\mathbf{Z}/r\mathbf{Z}$ if r is a prime.

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