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RIGIDITY OF FIBRATIONS OVER NONPOSITIVELY CURVED MANIFOLDS†

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0. INTRODUCTION

THE THEORY of manifold approximate fibrations is the correct bundle theory for topological manifolds and singular spaces. This theory plays the same role in the topological category as fiber bundle theory plays in the differentiable category and as block bundle theory plays in the piecewise linear category. For example, neighborhoods in topologically stratified spaces can be characterized and classified within the theory of manifold approximate fibrations (see [11, 3]).

This paper is concerned with manifold approximate fibrations over closed manifolds of nonpositive curvature. The main result is that this curvature assumption on the base space implies that such manifold approximate fibrations are rigid in the sense that if two manifold approximate fibrations over a manifold of nonpositive curvature are homotopy-theoretically equivalent, then they are equivalent in a much stronger, geometric way appropriate in this theory (they are controlled homeomorphic).

Although we work within the setting of manifold approximate fibrations, our results also hold, and are new, for the more classical theories of fibrations and fiber bundle projections between manifolds.

These results are established by working with the full moduli space of all manifold approximate fibrations over a closed manifold B of nonpositive curvature. We study a forgetful map from this moduli space which assigns to a manifold approximate fibration over B, the underlying map of a closed manifold to B. We call this map the forget control map for manifold approximate fibrations and show that it is homotopy-split injective.

This paper is the second in a series dealing with controlled topology over manifolds of nonpositive curvature. In the first paper [8] we establish that the theory of controlled homeomorphisms coincides with the theory of bounded homeomorphisms over Hadamard manifolds. This point of view is extended in the present paper by showing that the theory of manifold approximate fibrations coincides with the theory of manifold bounded fibrations over Hadamard manifolds. We combine this finding with our classifying differential [6] to prove the homotopy injectivity of the forget control map mentioned above.

In the future papers in this series [9, 10], we will show that the splitting of the forget control map for manifold approximate fibrations is compatible with various other constructions, such as taking the underlying Hurewicz fibration or taking the surgery-theoretic normal invariant. This will allow us to prove homotopy split injectivity for the forgetful

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map on controlled homotopy-topological structures and prove generalized Novikov conjectures (on the split injectivity of certain assembly maps).

1. THE MAIN RESULTS

Let B denote a metric space with a fixed metric. Recall the following definition.

Definition 1.1. If $c > 0$, then a map $p: M \to B$ is a *c-fibration* if for every commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & M \\ \times 0 \downarrow & & \downarrow p \\ X \times [0,1] & \xrightarrow{F} & B \end{array}$$

there is a map $\tilde{F}: X \times [0, 1] \to M$ such that $\tilde{F} | X \times 0 = f$ and $d(p\tilde{F}, F) < c$.

If $p: M \to B$ is a c-fibration for some $c > 0$, then we say p is a *bounded fibration*.

We assume that the reader is familiar with the basic properties of *approximate fibrations* as discussed in [6, Appendix]. In particular, if M and B are ANRs (as they always will be in this paper), then a map $p: M \to B$ is a c-fibration for every $c > 0$ if and only if p is an approximate fibration.

A *manifold approximate fibration* is a proper approximate fibration between manifolds without boundary. Important special cases to keep in mind are fibrations (i.e. maps with the homotopy lifting property for all spaces) and projection maps of locally trivial fiber bundles, as long as these maps are proper and have manifolds without boundary as domain and range.

The appropriate notion of equivalence for manifold approximate fibrations is *controlled homeomorphism* (manifold approximate fibrations where classified up to controlled homeomorphism in [6, 7]). If $p: M \to B$ and $q: N \to B$ are two maps, then a *controlled homeomorphism from p to q* is a homeomorphism $h: M \times [0, 1) \to N \times [0, 1)$ such that h is fiber preserving over $[0, 1)$, and the compositions

$$M \times [0,1) \xrightarrow{h} N \times [0,1) \xrightarrow{q \times \mathrm{id}} B \times [0,1)$$

and

$$N \times [0,1) \xrightarrow{h^{-1}} M \times [0,1) \xrightarrow{p \times \mathrm{id}} B \times [0,1)$$

continuously extend to maps

$$M \times [0,1] \to B \times [0,1]$$

and

$$N \times [0,1] \to B \times [0,1]$$

via $p \times \mathrm{id}: M \times 1 \to B \times 1$ and $q \times \mathrm{id}: N \times 1 \to B \times 1$, respectively. When M is compact, this is equivalent to saying h defines a continuous family of homeomorphisms $h_s: M \to N$, $0 \leq s < 1$, such that qh_s converges to p and ph_s^{-1} converges to q as s converges to 1.

Recall from [6] that if $p: M \to B$ and $q: N \to B$ are manifold approximate fibrations (with M and N closed and of dimension greater than or equal to 5), then the following are equivalent.

(1) p and q are controlled homeomorphic.
(2) For every $\varepsilon > 0$, there exists a homeomorphism $h: M \to N$ such that p is ε-close to qh.
(3) There exists a 1-parameter family of manifold approximate fibrations connecting p and q in the sense that there is a map $f: E \to B \times [0, 1]$ such that the composition

$$\pi: E \xrightarrow{f} B \times [0, 1] \xrightarrow{\text{proj}} [0, 1]$$

is a locally trivial fiber bundle, $\pi^{-1}(0) = M$, $\pi^{-1}(1) = N$, $f = p$ over $B \times 0$, $f = q$ over $B \times 1$, and $f| : \pi^{-1}(t) \to B \times t$ is a manifold approximate fibration for each $t \in [0, 1]$.

For fibrations (or bundles) between closed manifolds without boundary there is a similar list of equivalences except that item (3) asserts the existence of a 1-parameter family of fibrations (or bundles, respectively).

If "rigidity" means homotopy information yields homeomorphism information, then we paraphrase the following statement by saying "manifold approximate fibrations over nonpositively curved manifolds are rigid".

THEOREM 1.2. *Let B be a closed manifold of nonpositive curvature and let M and N be closed manifolds of dimension ≥ 5. Then manifold approximate fibrations $p: M \to B$ and $q: N \to B$ are controlled homeomorphic if and only if there exists a homeomorphism $h: M \to N$ such that p is homotopic to qh.*

The following result is an immediate corollary of Theorem 1.2.

COROLLARY 1.3. *Let B be a closed manifold of nonpositive curvature. If $p, q: M \to B$ are homotopic manifold approximate fibrations and* dim $M \geq 5$, *then p and q are controlled homeomorphic.*

In fact, Theorem 1.2 follows from a more general result concerning the moduli space of all manifold approximate fibrations over B. Before discussing that result, we first mention an application to an old problem.

Soon after approximate fibrations were first defined, the following question was considered. Suppose $p: M \to B$ is a manifold approximate fibration which is homotopic to the projection map of a locally trivial fiber bundle. When is p the uniform limit of a sequence of projection maps of locally trivial fiber bundles? When $B = S^1$ and dim $M \geq 5$, Husch [12] showed that the answer is "always". On the other hand, Chapman and Ferry [1] gave counterexamples in the case $B = S^2$. The following result substantially generalizes Husch's theorem.

COROLLARY 1.4. *Let B be a closed manifold of nonpositive curvature and let $p: M \to B$ be a manifold approximate fibration where* dim $M \geq 5$. *Then p can be approximated arbitrarily closely by bundle projections if and only if p is homotopic to a bundle projection.*

Proof. The "only if" statement is obvious. For the converse, suppose p is homotopic to a bundle projection q. Then Corollary 1.3 implies that there exists a controlled homeomorphism $h: M \times [0, 1) \to M \times [0, 1)$ from p to q. It follows that $qh_s: M \to B$, $0 \leq s < 1$, are bundle projections converging to p as s converges to 1. ∎

Steve Ferry conjectured some 10 years ago that Corollary 1.4 should hold true for B any closed aspherical manifold. We strengthen that conjecture as follows.

MAF RIGIDITY CONJECTURE. *Let B be a closed aspherical manifold. Then manifold approximate fibrations $p: M \to B$ and $q: N \to B$ are controlled homeomorphic if and only if there is a homeomorphism $h: M \to N$ such that p is homotopic to qh.*

Thus, Theorem 1.2 verifies this conjecture in a special case.

In order to describe our main result about spaces of manifold approximate fibrations from which Theorem 1.2 follows, we give some definitions.

Fix a positive integer m and assume that B is a manifold without boundary, with a fixed metric. The simplicial set $\mathrm{MAF}(B)$ of *manifold approximate fibrations over B* was defined in [6]. A k-simplex is a map $p: M \to B \times \Delta^k$ such that the composition

$$M \xrightarrow{p} B \times \Delta^k \xrightarrow{\text{proj}} \Delta^k$$

is a fiber bundle projection with fibers m-dimensional manifolds without boundary, and for each t in Δ^k, $p| : p^{-1}(B \times t) \to B \times t$ is manifold approximate fibration. Actually, in [6] we require M to be embedded in $\ell_2 \times B \times \Delta^k$ (as a set of "small capacity") so that p is the restriction of the projection $\ell_2 \times B \times \Delta^k \to B \times \Delta^k$. We usually ignore this embedding property of the definition in this paper; the reader can easily supply the missing details.

A k-simplex of the simplicial set $\mathrm{Man}(B)$ of *m-manifolds over B* consists of a proper map $p: M \to B \times \Delta^k$ such that the composition

$$M \xrightarrow{p} B \times \Delta^k \xrightarrow{\text{proj}} \Delta^k$$

is a fiber bundle projection with fibers m-dimensional manifolds without boundary. (As above, M is embedded in $\ell_2 \times B \times \Delta^k$ as a set of "small capacity" so that p is the restriction of the projection. Also as above, this embedding data usually will be ignored.)

If B is a manifold, then $\varphi: \mathrm{MAF}(B) \to \mathrm{Man(B)}$ will denote the inclusion and we call φ the *forget control map.*

We can now state our main result.

THEOREM 1.5. *Let B be a closed manifold of nonpositive curvature and $m \geq 5$. Then the forget control map $\varphi: \mathrm{MAF}(B) \to \mathrm{Man}(B)$ is homotopy-split injective. That is, there exists a simplicial map $r: \mathrm{Man}(B) \to \mathrm{MAF}(B)$ such that $r \circ \varphi$ is homotopic to $\mathrm{id}_{\mathrm{MAF}(B)}$.*

Of course, the MAF Rigidity Conjecture can be globalized to state that Theorem 1.5 should hold for B any closed aspherical manifold.

In the special case $B = S^1$, Hughes and Prassidis [4] relate the splitting map r of Theorem 1.5 to Siebenmann's relaxation [13].

To see how Theorem 1.2 follows from Theorem 1.5 note that, under the hypothesis of Theorem 1.5, φ induces an injection $\varphi: \pi_0 \mathrm{MAF}(B) \to \pi_0 \mathrm{Man}(B)$ on components. Therefore, Theorem 1.2 is a consequence of the following description of π_0-equivalence in these two simplicial sets. In this proposition, no curvature assumption is needed on B.

PROPOSITION 1.6. *Let B be a closed manifold and let $p: M \to B$ and $q: N \to B$ be vertices of* $\mathrm{Man}(B)$ *(i.e. p and q are maps of closed m-manifolds to B).*

(1) *p and q are in the same component of* $\mathrm{Man}(B)$ *if and only if there exists a homeomorphism $h: M \to N$ such that p is homotopic to qh.*

(2) *If p and q are manifold approximate fibrations and $m \geq 5$, then p and q are in the same component of* $\mathrm{MAF}(B)$ *if and only if there exists a controlled homeomorphism from p to q.*

Proof. For (1) suppose first that p and q are in the same component of $\mathrm{Man}(B)$. Then there exists a map $f: E \to B \times [0, 1]$ such that the composition of f with the projection to $[0, 1]$ is a bundle projection $\pi: E \to [0, 1]$ such that $f|\pi^{-1}(0) = p$ and $f|\pi^{-1}(1) = q$. Let $H: M \times [0, 1] \to E$ be a trivialization of π such that $H|M \times 0 = \mathrm{id}$. Then $h = H_1: M \times 1 \to \pi^{-1}(1) = N$ is the desired homeomorphism.

For the converse, the 1-simplex in $\mathrm{Man}(B)$ connecting p and q comes from the mapping cylinder of h.

The proof of (2) is given in [6, 7.12]. ∎

2. BOUNDED AND APPROXIMATE FIBRATIONS OVER HADAMARD MANIFOLDS

In this section, we show that the space of manifold approximate fibrations over Euclidean space is homotopy-equivalent to the space of manifold *bounded* fibrations. Moreover, we show the same is true over an arbitrary Hadamard manifold. A *Hadamard manifold* is a simply connected manifold with a complete metric of nonpositive curvature. This is consistent with the "bounded equals controlled over Hadamard manifolds" philosophy of [8].

In the CW-complex (nonmanifold) case, there is a significant difference between approximate and bounded fibrations. This is discussed in [5] for $\mathbf{R}^1$.

For a metric space B with a fixed metric and for a fixed integer m, the simplicial set $\mathrm{MBF}(B)$ of *manifold bounded fibrations over B* is defined similarly to $\mathrm{MAF}(B)$ above, except now we only require

$$p|: p^{-1}(B \times t) \to B \times t$$

to be a proper c-fibration for some $c > 0$ which is independent of t (but does depend on p). Note that $\mathrm{MAF}(B)$ and $\mathrm{MBF}(B)$ are both Kan and $\mathrm{MBF}(B)$ contains $\mathrm{MAF}(B)$.

THEOREM 2.1. *If $m \geq 5$, then the inclusion* $\mathrm{MAF}(\mathbf{R}^i) \to \mathrm{MBF}(\mathbf{R}^i)$ *is a homotopy equivalence.*

Proof. We will show that the relative homotopy groups $\pi_k(\mathrm{MBF}(\mathbf{R}^i), \mathrm{MAF}(\mathbf{R}^i))$ vanish. Let $p: M \to \mathbf{R}^i \times \Delta^k$ represent a class in this group. For each t in Δ^k, let $M_t = p^{-1}(\mathbf{R}^i \times t)$. Then there exists a $c > 0$ so that $p_t: M_t \to \mathbf{R}^i$ is a c-fibration for each t in Δ^k. Moreover, p_t is an approximate fibration for each t in $\partial\Delta^k$.

Choose $K > 0$ large and define

$$\gamma_t: \mathbf{R}^i \times \Delta^k \to \mathbf{R}^i \times \Delta^k, \quad 0 \leq t \leq \tfrac{1}{2},$$

so that

(i) γ_t is a homeomorphism fibered over Δ^k,
(ii) $\gamma_0 = \mathrm{id}$,
(iii) $\gamma_t(x, y) = (x/(1 - t + tK), y)$ if y is not too close to $\partial\Delta^k$,
(iv) $\gamma_t|\mathbf{R}^i \times \partial\Delta^k = \mathrm{id}$. If this is done correctly, then

$$\gamma_{1/2}p: M \to \mathbf{R}^i \times \Delta^k$$

is an ε-fibration for some small $\varepsilon > 0$.

It follows from [2] that there is a homotopy

$$g_t: \gamma_{1/2}p \simeq g_1, \quad \tfrac{1}{2} \leq t \leq 1$$

so that g_t is fibered over Δ^k, g_1 is an approximate fibration and $g_t|p^{-1}(\mathbf{R}^i \times \partial\Delta^k) = p$. Then the concatenation of $\gamma_t p$ and g_t defines a map $\Gamma: M \times [0, 1] \to \mathbf{R}^i \times \Delta^k \times [0, 1]$ such that $\Gamma|M \times 1$ is a k-simplex in $\mathrm{MAF}(\mathbf{R}^i)$ which represents the same relative class as $p = \Gamma|M \times 0$. ■

Our next result is a bundle version of Theorem 2.1. This generalization (actually its Hadamard version; see Theorem 2.5 below) will be needed in the next section. For notation let $p: E \to B$ be a locally trivial fiber bundle with fiber F and structure group G. We assume that F is a metric space and that G acts on F by isometries.

A k-simplex of the simplicial set $\mathrm{MBF}(p: E \to B)$ consists of a subset M of $\ell_2 \times E \times \Delta^k$ (of "small capacity") such that

(i) the composition

$$M \xrightarrow{\text{proj} = f} E \times \Delta^k \xrightarrow{p \times \text{id}} B \times \Delta^k$$

is a fiber bundle projection with fibers m-manifolds without boundary,

(ii) the projection $f: M \to E \times \Delta^k$ has the property that for each (x, y) in $B \times \Delta^k$, the composition $f^{-1}(p^{-1}(x) \times y) \xrightarrow{f|} p^{-1}(x) \times y \xrightarrow{\approx} F$ is a manifold c-fibration for some $c > 0$ (independent of y). Here the homeomorphism $p^{-1}(x) \approx F$ comes from a local trivialization of $p: E \to B$. It does not matter which local trivialization is used because the transitions are isometries of F.

The simplicial set $\mathrm{MAF}(p: E \to B)$ is defined similarly, but now we require $f|: f^{-1}(p^{-1}(x) \times y) \to p^{-1}(x) \times y$ to be a manifold approximate fibration.

THEOREM 2.2. *If* $F = \mathbf{R}^i$ *(with the standard metric),* $m \geq 5$*, and* B *is a polyhedron, then the inclusion* $\mathrm{MAF}(p: E \to B) \to \mathrm{MBF}(p: E \to B)$ *is a homotopy equivalence.*

Proof. Again we show that the relative homotopy groups vanish. Let $f: M \to E \times \Delta^k$ be a k-simplex of $\mathrm{MBF}(p: E \to B)$ such that the part of f lying over $E \times \partial\Delta^k$ comes from $\mathrm{MAF}(p: E \to B)$. Assume that B is triangulated so fine that each simplex of B is contained in an open set over which p is trivial as a G-bundle.

It is easy to modify f as in the proof of Theorem 2.1, by working inductively up the skeleta of B, to get a representative of the relative class of f which is in $\mathrm{MAF}(p: E \to B)$. ■

For the remainder of this section, we let H be a Hadamard manifold of dimension i, we fix a point x_0 in H and we let $\exp \cdot \mathbf{R}^i \to H$ denote the exponential map at x_0. The proof of the next theorem is very similar to the proof of the Euclidean case (Theorem 2.1), so we only indicate the changes which need to be made.

THEOREM 2.3. *If* $m \geq 5$*, then the inclusion* $\mathrm{MAF}(H) \to \mathrm{MBF}(H)$ *is a homotopy equivalence.*

Proof. If $p: M \to H$ is a manifold c-fibration for some $c > 0$, then the key step is to see how to deform p, through bounded fibrations, to an approximate fibration. In the Euclidean case we first took a radial shrinking and turned p into an ε-fibration, and then we used a general result ("sucking") to deform the ε-fibration to an approximate fibration. The second part of this procedure fails in the non-Euclidean case. To apply "sucking" for an open non-Euclidean manifold like H we need to deform p, through bounded fibrations, to a map which is a $\mathscr{U}$-fibration for some prescribed open cover $\mathscr{U}$ of H. But if such a $\mathscr{U}$ is given, as opposed to some $\varepsilon > 0$, we need to be more careful about how we do the radial shrinking.

Given $\mathscr{U}$, then find a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ of positive numbers such that any map to H which is an ε_i-fibration over $B(x_0, i)\backslash \mathring{B}(x_0, i-2)$ for each $i = 1, 2, \ldots,$ is a $\mathscr{U}$-fibration over H. Moreover, assume that $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3 \geq \ldots$. Now inductively assume that p is an ε_i-fibration over $B(x_0, i)\backslash \mathring{B}(x_0, i-2)$ for each $i = 1, \ldots, k$. Then perform a radial shrink which leaves $B(x_0, k-1)$ fixed and takes $B(x_0, L)$ to $B(x_0, k)$ for some L much larger than c/ε_{k+1}. Of course, when we speak of a radial shrink of H, we mean that we first apply $\exp^{-1}$, then do a radial shrink of $\mathbf{R}^i$, and finally, apply exp. Continue this process until p has been deformed to a $\mathscr{U}$-fibration. ■

Since H is homeomorphic to $\mathbf{R}^i$, we have the following corollary of Theorems 2.1 and 2.3 saying that the homotopy type of $\mathrm{MBF}(H)$ is independent of the metric on H (as long as the metric is complete and of nonpositive curvature).

Corollary 2.4. *If $m \geq 5$, then* $\mathrm{MBF}(\mathbf{R}^i)$ *and* $\mathrm{MBF}(H)$ *are homotopy equivalent.*

The final result of this section is a Hadamard version of Theorem 2.2. The notation is as above so that $p: E \to B$ is a locally trivial fiber bundle whose structure group acts by isometries on the fiber.

Theorem 2.5. *If the fiber of $p: E \to B$ is a Hadamard manifold H, $m \geq 5$, and B is a polyhedron, then the inclusion* $\mathrm{MAF}(p: E \to B) \to \mathrm{MBF}(p: E \to B)$ *is a homotopy equivalence.*

The proof of Theorem 2.5 is not different from the proof of Theorem 2.2.

3. SPLITTING THE FORGET CONTROL MAP

Throughout this section, let B denote a closed i-manifold of nonpositive (sectional) curvature. Let $u: H \to B$ denote the universal covering of B and give H the (unique) Riemannian metric which makes u a Riemannian cover. Then H becomes a Hadamard manifold and the action of $\pi_1(B)$ on H is by a discrete group of isometries of H.

Now let $\pi: TB \to B$ denote the tangent bundle of B and let $\exp: TB \to B \times B$ denote the exponential map; i.e. $\exp(v) = (x, \exp_x(v))$ where $x = \pi(v)$ and $\exp_x: T_xB \to B$ is the standard exponential map at x. We assume that $\pi: TB \to B$ has the structure of a fiber bundle with fiber H and structure group the isometry group of H (in fact, $\pi_1(B)$). Moreover, we assume that if $h_x: H \to T_xB$ is a homeomorphism coming from a local trivialization, then

$$\begin{array}{ccc} H & \xrightarrow{h_x} & T_xB \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle \exp_x} & \\ & B & \end{array}$$

commutes.

From Section 2 we have a simplicial set $\mathrm{MBF}(\pi: TB \to B)$ associated with the bundle $\pi: TB \to B$. We emphasize that $\pi: TB \to B$ has fiber H and structure group $\pi_1(B)$ as discussed above.

We now want to discuss a *differential* δ coming out of Man(B) which, on the image of φ, takes values in MBF$(\pi: TB \to B)$. To this end let $M \subset \ell_2 \times B \times \Delta^k$ be a k-simplex of Man(B) and let $p: M \to B \times \Delta^k$ be the projection. Form the pull-back diagram

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{p}} & TB \times \Delta^k \\ e\downarrow & & \downarrow \exp \times \mathrm{id} \\ B \times M & \xrightarrow{\mathrm{id} \times p} & B \times B \times \Delta^k. \end{array}$$

There is a natural way to think of $\bar{M}$ as a subset of $\ell_2 \times TB \times \Delta^k$ (after replacing TB by the graph of exp). Moreover, the composition

$$\bar{M} \xrightarrow{\bar{p}} TB \times \Delta^k \xrightarrow{\pi \times \mathrm{id}} B \times \Delta^k$$

is equal to $(p_1 \times \mathrm{id})(\exp \times \mathrm{id})\bar{p} = (p_1 \times \mathrm{id})(\mathrm{id} \times p)e$ which is a bundle with m-manifold fibers (here $p_1: B \times B \to B$ is first coordinate projection). Thus, $\bar{p}: \bar{M} \to TB \times \Delta^k$ is a candidate for a k-simplex of MBF$(\pi: TB \to B)$. However, the bounded fibration condition will not be satisfied in general. At any rate define $\delta(M)$ to be $\bar{p}: \bar{M} \to TB \times \Delta^k$.

Let Man$(B)_\varphi$ denote the union of those components of Man(B) which meet the image of φ: MAF$(B) \to$ Man(B). By the Kan condition, every simplex in Man$(B)_\varphi$ is the face of some simplex in Man$(B)_\varphi$ which has a vertex in MAF(B).

PROPOSITION 3.1. *δ: Man$(B)_\varphi \to$ MBF$(\pi: TB \to B)$ is a simplicial map.*

Proof. We only need to check that δ, on Man$(B)_\varphi$, takes values in MBF$(\pi: TB \to B)$. To this end let $M \subset \ell_2 \times B \times \Delta^k$ be a k-simplex of Man$(B)_\varphi$. Let $p: M \to B \times \Delta^k$ be the projection. By the remarks above, we can assume that one of the vertices is in MAF(B). Let O be a vertex in Δ^k such that $p|: p^{-1}(B \times O) \to B$ is a manifold approximate fibration. Let $N = p^{-1}(B \times O)$ and $p_0 = p|: N \to B$.

Let $k: N \times \Delta^k \to M$ be a homeomorphism which trivializes the bundle $M \to \Delta^k$ so that $k| N \times O = \mathrm{id}$.

Consider the pull-back diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{p}} & TB \times \Delta^k \\ \tilde{e}\downarrow & & \downarrow \exp \times \mathrm{id} \\ B \times N \times \Delta^k & \xrightarrow{\mathrm{id} \times pk} & B \times B \times \Delta^k \end{array}$$

It suffices to show that there exists a $c > 0$ such that for each x in B and t in Δ^k, if $\tilde{M}_{(x,t)} = \tilde{e}^{-1}(x \times N \times t)$ then

$$\tilde{M}_{(x,t)} \xrightarrow{\tilde{p}|} T_x B \xrightarrow{h_x^{-1}} H$$

is a c-fibration. This will be accomplished by constructing a c-homotopy from $h_x^{-1}\tilde{p}|$ to an approximate fibration.

To this end let $h_s: B \times N \times \Delta^k \to B \times B \times \Delta^k$, $0 \le s \le 1$, be the obvious homotopy from $h_0 = \mathrm{id} \times pk$ to $h_1 = \mathrm{id} \times p_0 \times \mathrm{id}$ (i.e. $h_s(x, y, t) = (x, \mathrm{proj}_B pk(y, (1-s)t), t)$).

Let $\tilde{h}_s: \tilde{M} \to TB \times \Delta^k$, $0 \le s \le 1$, be the unique homotopy covering h_s such that $\tilde{h}_0 = \tilde{p}$. Since

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{h}_1} & TB \times \Delta^k \\ \tilde{e}\downarrow & & \downarrow \exp \times \mathrm{id} \\ B \times N \times \Delta^k & \xrightarrow{\mathrm{id} \times p_0 \times \mathrm{id}} & B \times B \times \Delta^k \end{array}$$

commutes, it follows that $\tilde{h}_1$ is a Δ^k-family of manifold approximate fibrations.

Thus, it remains to see that the homotopy

$$\tilde{M}_{(x,t)} \xrightarrow{\tilde{h}_s|} T_x B \xrightarrow{h_x^{-1}} H, \quad 0 \le s \le 1,$$

is a c-homotopy for some $c > 0$ independent of x and t. Observe that

$$\begin{array}{ccccccccc} \tilde{M}_{(x,t)} & \subset & \tilde{M} & \xrightarrow{\tilde{h}_s} & TB \times \Delta^k & \to & TB & \xleftarrow{h_x^{-1}} & T_x B \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow u \\ X \times N \times t & \subset & B \times N \times \Delta^k & \xrightarrow{h_s} & B \times B \times \Delta^k & \to & B \times B & \xrightarrow{p_2} & B \end{array}$$

commutes. Since N is compact the result will follow from the following lemma. ■

LEMMA 3.2. *Let X be compact and let $F: X \times [0, 1] \to B$ be a homotopy. Then there exists a $c > 0$ such that if $\omega: [0, 1] \to H$ is any path covering $F|x \times [0, 1]$ for some x in X, then* $\mathrm{diam}(\omega) < c$.

Proof. Choose $\delta > 0$ such that if $\omega: [0, 1] \to H$ is any path with $\mathrm{diam}(u\omega) < \delta$, then $\mathrm{diam}(\omega) < 1$. Choose $\eta > 0$ so that any two η-close maps into B are δ-homotopic. Choose a finite subset $\{x_1, \dots, x_n\}$ of X such that for each x in X there exists an i in $\{1, \dots, n\}$ so that for every t in $[0, 1]$, $d(F(x_i, t), F(x, t)) < \eta$.

For each i in $\{1, \dots, n\}$, let $\omega_i: [0, 1] \to H$ be some lift of $F|x_i \times [0, 1]$. Let $c = \max\{\mathrm{diam}(\omega_i) | i = 1, \dots, n\} + 2$.

To see that c works, let $\omega: [0, 1] \to H$ cover $F|x \times [0, 1]$. Find i such that $F|x_i \times [0, 1]$ is η-close to $F|x \times [0, 1]$. Then there is a δ-homotopy G from $F|x_i \times [0, 1]$ to $F|x \times [0, 1]$. Lift this homotopy to $\tilde{G}$, a homotopy from ω to some translate of ω_i, say $T\omega_i$, where $T \in \pi_1(B) \subset \mathrm{Isom}(H)$. Then $\tilde{G}$ is a 1-homotopy. Since $\mathrm{diam}(T\omega_i) = \mathrm{diam}(\omega_i) \le c - 2$, it follows that $\mathrm{diam}(\omega) \le c$. ■

We are now ready for the main result of this section.

Proof of Theorem 1.5. Recall that in [6] we constructed a differential

$$d: \mathrm{MAF}(B) \to \mathrm{MAF}(p_1: \tau B \to B)$$

where $p_1: \tau B \to B$ is a representative of the topological tangent microbundle, and proved that d is a homotopy equivalence. If one uses the exponential map to identify $p_1: \tau B \to B$ with $\pi: TB \to B$ and uses Theorem 2.5 above, then one can construct a homotopy equivalence $\gamma: \mathrm{MAF}(p_1: \tau B \to B) \to \mathrm{MBF}(\pi: TB \to B)$ which makes

$$\begin{array}{ccc} \mathrm{MAF}(B) & \xrightarrow{d} & \mathrm{MAF}(p_1: \tau B \to B) \\ \varphi \downarrow & & \downarrow \gamma \\ \mathrm{Man}(B)_\varphi & \xrightarrow{\delta} & \mathrm{MBF}(\pi: TB \to B) \end{array}$$

commute. The theorem follows. ■

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