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RELATIVE ROCHLIN INVARIANTS

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The Rochlin invariant of a compact 3-manifold with a fixed spin structure can be regarded as the signature (mod 16) of any solution to a certain surgery problem. This paper explores this remark in some detail. The relative Rochlin invariants arise from consideration of other surgery problems. We work out the general theory and apply it with RP^3 replacing S^3 to study free involutions on 3-manifolds. The Morgan–Sullivan linking cycle theory gives new insight into the relation between spin structures on the 3-manifold and how circles in the manifold link. From the algebra which expresses this relation one can calculate the relative Rochlin invariants mod 8, and can often recover the spin structure on the manifold.

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Introduction

The Rochlin invariant of a compact, oriented, spin 3-manifold is usually defined as the signature (mod 16) of an oriented spin 4-manifold which bounds the 3-manifold. This definition can also be considered as a relative surgery invariant: it is the signature of a normal bordism from the 3-manifold to S^3 over $S^3 \times I$. This point of view suggests that there is a formula for the Rochlin invariant mod 8, a point we develop in Sections 3 and 4. It also suggests the generalization of replacing S^3 by other 3-manifolds.

In this paper we will develop the case where S^3 is replaced by RP^3 in some detail. We will give the general definition in Section 1 and turn immediately to the RP^3 case in Section 2 where we will use it to classify free involutions on S^4 and S^5. This has been done: indeed the point of view expounded here arose from our attempt to understand the Fintushel and Stern proof that their involution is exotic [7] and how that proof relates to the Cappell and Shaneson proof that their involution is exotic [6]. However, our classification is in terms of invariant 3-manifolds and the passage from this sort of information to the classification seems new.

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After we develop the relation between linking in the 3-manifold and these invariants mod 8 in Sections 3 and 4, we will return to the RP^3 case in Section 5. We will relate the Rochlin invariant of a $\mathbb{Z}/2\mathbb{Z}$—homology 3-sphere with a free involution to its Browder—Livesay invariant, a problem of Neumann and Raymond [19]. These two numbers were shown to be equal by Yoshida [26] for the case of integral homology 3-spheres and we complete the story here.

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1. Conventions and the basic definition

Much of the ensuing discussion involves numbers such as linking numbers or signatures which depend on orientation. Hence we explicitly make:

Orientation convention. If W is an oriented manifold, ∂W is oriented so that the orientation for ∂W followed by the inward normal orients W.

It will also be useful to agree on a definition of spin structure. The most convenient for our purposes is the formulation of M. Hirsch (see [15]).

Spin structure. Let ν be an oriented bundle of dimension n, at least 2, and let $P\nu$ be the total space of the associated principal $SO(n)$-bundle. A spin structure on ν is a class $\phi \in H^1(P(\nu); \mathbb{Z}/2\mathbb{Z})$ which restricts non-zero on each fibre $SO(n)$.

Let us now turn to the definition of relative Rochlin invariants. The definition requires some initial data. We must first fix a reference manifold, Q^3, which is to be a compact, oriented 3-manifold without boundary. Normal bordism classes of degree 1 normal maps to Q^3 correspond to elements in $H^2(Q; \mathbb{Z}/2\mathbb{Z})$. This remark follows from Sullivan's thesis, which identifies normal bordism classes with $[Q, G/0]$ and identifies $[Q, G/PL]$ with $H^2(Q; \mathbb{Z}/2\mathbb{Z})$, plus smoothing theory (including Cerf's $\Gamma_4 = 0$) which identifies $[Q, G/0]$ with $[Q, G/PL]$. From Kirby–Siebenmann [11] we may also identify smooth and TOP normal bordism for 3-manifolds.

For each $x \in H^2(Q; \mathbb{Z}/2\mathbb{Z})$ choose a representative in the corresponding normal bordism class. This requires a choice of a manifold P_x^3 and a degree 1 map $f_x: P_x \to Q$ together with a bundle map $\hat{f}_x: \nu_{P_x} \to \nu_Q$ covering f. Here and hereafter ν_W denotes a bundle of dimension at least the dimension of W which is a normal bundle for W. We can normalize our choices a bit. In odd dimensions one can always do surgery to make ones map an $H_*(; \mathbb{Z})$ homology equivalence. For dimension at least five one can quote Cappell–Shaneson [5] 2.1: for dimension three just note that the proof continues to work. Hence we will assume all our $f_x: P_x \to Q$ are $H_*(; Z)$-equivalences.

We will define a relative Rochlin invariant whenever we have an oriented 3-manifold, M, without boundary; a spin structure, ϕ, on ν_M; and a degree 1 map $f: M \to Q^3$.

To make the definition requires two preliminary results:

Lemma 1.1. *Given a degree* 1 *map* $f: M \to Q^3$ *and spin structures* ϕ_M *on* ν_M *and* ϕ_Q *on* ν_Q *there exists a bundle map* $\hat{f}: \nu_M \to \nu_Q$ *covering* f *so that* $\hat{f}^*\phi_Q = \phi_M$.

Lemma 1.2. *Suppose given a degree* 1 *map* $f: M \to Q^3$ *together with a spin structure* ϕ_M *on* ν_M; *two spin structures* ϕ_1 *and* ϕ_2 *on* ν_Q; *and two bundle maps* $\hat{f}_1, \hat{f}_2: \nu_M \to \nu_Q$ *covering* f *with* $\hat{f}_i^*\phi_i = \phi_M$ *for* $i = 1, 2$. *Then there exists a bundle map* $\hat{h}: \nu_Q \to \nu_Q$ *covering the identity such that* $\hat{h} \circ \hat{f}_1 = \hat{f}_2$.

We defer the proofs a bit.

Given M, ϕ, and $f: M \to Q$ as above, use lemma 1.1 to cover f with a bundle map $\hat{f}$ by choosing any spin structure on ν_Q: since an oriented 3-manifold is parallelizable, ν_Q has a spin structure. Lemma 1.2 shows that the resulting normal bordism class, $x \in H^2(Q; \mathbb{Z}/2\mathbb{Z})$, is independent of all choices. Let W^4 denote the manifold which gives the normal bordism from $(M; f, \hat{f})$ to $(P_x; f_x, \hat{f}_x)$.

Definition 1.3. The relative Rochlin invariant, denoted $\mu(M; f, \phi)$, is defined by:

$$\mu(M; f, \phi) \equiv \sigma(W) \pmod{16},$$

where W is oriented so that M receives its given orientation and $\sigma(W)$ denotes the signature of W.

Remark 1.4. If $Q = S^3$, f is unique up to homotopy. If we use the identity, $S^3 \to S^3$, as the representative for $0 \in H^2(S^3; \mathbb{Z}/2\mathbb{Z})$ the relative Rochlin invariant is the usual Rochlin invariant (except for [10] where they take the negative of this definition). We use $\mu(M; \phi)$ to denote the usual Rochlin invariant.

We conclude this section by showing $\mu(M; f, \phi)$ is well-defined and then proving the lemmas.

To show $\mu(M; f, \phi)$ is well-defined suppose we have two normal bordisms, $(F_i, \hat{F}_i): W_i \to Q^3 \times I$, $i = 1, 2$, between M and P_x. We can glue W_1 to W_2 along the two boundary components M and P_x to get a 4-manifold V and a degree 1 map $G: V \to Q^3 \times S^1$. Moreover, $\sigma(V) = \sigma(W_1) - \sigma(W_2)$. Use Lemma 1.2 to construct a bundle, $\mathcal{S}$, over $Q^3 \times S^1$: $\mathcal{S}$ in general will not be $\nu_{Q \times S^1}$ because Lemma 1.2 says we must put in a twist by $\hat{h}$. Still, we have a bundle map $\hat{G}: \nu_V \to \mathcal{S}$ covering G. The theory of the Spivak normal fibration [23] shows that $\mathcal{S}$ has a spin structure and hence so does ν_V. By Rochlin's theorem [21], $\sigma(V) \equiv 0 \pmod{16}$ so $\mu(M; f, \phi)$ is well defined.

To prove Lemma 1.1, we first observe that there are bundle maps $g: \nu_M \to \nu_Q$ covering f since both bundles are trivial. Hence 1.1 will be shown if we can show

Lemma 1.5. *Let $\mathcal{S}$ be a trivial $SO(n)$-bundle over X, X a CW complex of dimension at most 3 with $n \geq 3$. If ϕ_1 and ϕ_2 are spin structures on $\mathcal{S}$ there exists a bundle map $\hat{h}: \mathcal{S} \to \mathcal{S}$ covering the identity on X with $\hat{h}^*\phi_1 = \phi_2$.*

Proof. Bundle maps covering the identity are given by $[X, SO(n)]$. There is an exact sequence

$$0 \to H^1(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p^*} H^1(P\mathcal{S}; \mathbb{Z}/2\mathbb{Z}) \to H^1(SO(n); \mathbb{Z}/2\mathbb{Z}) \to 0.$$

The map $SO(n) \xrightarrow{c} RP^\infty$ which classifies the double cover gives a map $c_*: [X, SO(n)] \to H^1(X; \mathbb{Z}/2\mathbb{Z})$. Finally, if $h \in [X, SO(n)]$ and $\phi \in H^1(P\mathcal{S}; \mathbb{Z}/2\mathbb{Z})$ is a spin structure,

$$\hat{h}^*\phi = \phi + p^*c_*(\phi).$$

Hence 1.5 is equivalent to the claim that c_* is onto. But $[X, SO(n)] \to H^1(X; \mathbb{Z}/2\mathbb{Z}) \to [X, \mathrm{BSpin}(n)]$ is exact and $\mathrm{BSpin}(n)$ is 3-connected for $n \geq 3$, so $[X, \mathrm{BSpin}(n)] \simeq *$. □

With 1.1 proved, let us turn to 1.2. Use 1.5 to find $\hat{h}_1: \nu_Q \to \nu_Q$ with $\hat{h}_1^*\phi_1 = \phi_2$. If we can find $\hat{h}_2: \nu_Q \to \nu_Q$ so that $\hat{h}_2 \circ (\hat{h}_1 \circ \hat{f}_1) = \hat{f}_2$ we are done so it is no loss of generality to assume $\phi_1 = \phi_2$.

One can always find $\hat{g}: \nu_M \to \nu_M$ so that $\hat{f}_1 \circ \hat{g} = \hat{f}_2$. Under our hypotheses, $\hat{g}^*\phi_M = \phi_M$, so $\hat{g} \in [M, SO(n)]$ comes from $\hat{g} \in [M, \mathrm{Spin}(n)]$.

For any 3-complex, X, $[X, \mathrm{Spin}(n)] \simeq H^3(X; \mathbb{Z})$ if $n \geq 3$ by a Postnikov argument. Hence we can find $\hat{h} \in [Q, \mathrm{Spin}(n)]$ so that $\hat{f}_1 \circ \hat{g} = \hat{h} \circ \hat{f}_1$. We are done.

Remark 1.6. If we change our choice of surgery problems (=normal bordism classes) $f_x, \hat{f}_x: P_x \to Q$ (still keeping the $H_*(;\mathbb{Z})$-equivalence property), the relative Rochlin invariant can change by only 0 or 8. Hence the relative Rochlin invariant mod 8 only depends on M, f, and ϕ, not on the choice of problems.

By taking connected sum with the Poincaré homology sphere one can effect this change. The relative Rochlin invariant can only change by 0 or 8 because, if we look at a normal bordism from P_x to P_x^1 over $Q \times I$, the signature of the bordism is the signature of an even, symmetric, non-singular (over $\mathbb{Z}$ since the boundaries are $H\ (;\mathbb{Z})$-equivalent) bilinear form.

2. $Q = RP^3$ and free involutions

We wish to apply the theory in Section 1 to the case $Q = RP^3$. In this case $H^2(Q; \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ so we need to choose two surgery problems. For $0 \in H^2(Q; \mathbb{Z}/2\mathbb{Z})$ let us choose the identity both for the degree 1 map and for the bundle map. There are many choices for the non-zero element and the following lemma will help us choose one.

Lemma 2.1. *Let M^3 be an oriented 3-manifold without boundary. Then M has a degree 1 map to RP^3 iff there exists a class $x \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ so that $x^3 \neq 0$. If $H^1(M; \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ then M, with either spin structure, is normally bordant to the identity surgery problem iff $H_1(\tilde{M}; \mathbb{Z})$ has order congruent to ± 1 (modulo 8). ($\tilde{M}$ is the double cover associated to x.)*

Remark 2.2. From 2.1 we see that we can choose the degree 1 map $L(6,1) \to RP^3$ as a representative for the non-zero normal bordism class over RP^3.

Proof of 2.1. The first part of 2.1 is follows easily from two facts: RP^3 is a 3-skeleton for $K(\mathbb{Z}/2\mathbb{Z}, 1) = RP^\infty$; and $S^3 \to RP^3$ has degree 2, so odd degree maps can be easily modified to get degree 1 maps.

The second part is more interesting. Under the hypotheses, the map $f: M \to RP^3$ induces a $\mathbb{Z}_{(2)}$-homology isomorphism (easy) and $\tilde{f}: \tilde{M} \to \tilde{RP}^3$ does the same (first show $\tilde{f}$ induces a $Z/2Z$-homology isomorphism via a spectral sequence argument).

A lemma of Wall's (see p. 267–268 of [25] where the result is embedded in a proof by contradiction) says that a symmetric non-singular matrix over $\mathbb{Z}_{(2)}[\mathbb{Z}/2\mathbb{Z}]$ has determinant $a + bT$ with a or b odd and the other congruent to 0 mod 4.

Let W be a normal bordism from M to RP^3 and do surgery if necessary to insure $\pi_1 W \simeq \mathbb{Z}/2\mathbb{Z}$. Let $\tilde{W}$ denote the non-trivial double cover. The order of the torsion in $H_1(\tilde{M}; \mathbb{Z})$ is the absolute value of the determinant of the map $H_2(\tilde{W}; \mathbb{Z}) \to H_2(\tilde{W}, \partial\tilde{W}; \mathbb{Z})$ (where we base $H_2(W, \partial W)$ using the dual basis $\mathrm{Hom}(H_2(\tilde{W}), \mathbb{Z}) \simeq H^2(\tilde{W}) \simeq H_2(W, \partial\tilde{W})$). Wall's lemma applies; the determinant over $\mathbb{Z}$ is just $a^2 - b^2$; and hence we have that if $M, f, \hat{f}$ is normally bordant to the identity, then $|H_1(\tilde{M}; \mathbb{Z})| \equiv \pm 1 \pmod 8$.

Notation. Hereafter, given any finite group, A, $|A|$ denotes the number of elements in A.

Since we have a degree 1 normal map $L(6,1) \to RP^3$ with $|H_1(M; \mathbb{Z})| = 3$, this normal map represents the non-zero element. If $|H_1(\tilde{M}; \mathbb{Z})| \equiv \pm 1 \pmod 8$ M can not be normally bordant to the $L(6,1)$ problem by another application of Wall's lemma and so $M, f, \hat{f}$ must represent the zero element in $H^2(RP^3; \mathbb{Z}/2\mathbb{Z})$. □

With 2.1 proved, let us proceed to study free involutions on 3-manifolds, S^4, and S^5.

Let us start with a connected, oriented, compact 3-manifold without boundary, denoted M, and suppose M has a free, orientation preserving, involution τ. The double cover $M \to M/\tau$ is classified by an element $x \in H^1(M/\tau; \mathbb{Z}/2\mathbb{Z})$. The involution τ induces an involution on $H_1(M; \mathbb{Z}/2\mathbb{Z})$ and we let $H_1(M; \mathbb{Z}/2\mathbb{Z})^\tau$ denote the subspace of classes fixed by τ.

Lemma 2.2. $x^3 \neq 0$ *iff* $\dim_{\mathbb{Z}/2\mathbb{Z}} H_1(M; \mathbb{Z}/2\mathbb{Z})^\tau = \frac{1}{2} \dim_{\mathbb{Z}/2\mathbb{Z}} H_1(M; \mathbb{Z}/2\mathbb{Z})$.

Proof. The dimension equality is equivalent to $H^*(Z/2Z; H^1(M; Z/2Z)) = 0$, for $* > 0$ which is a necessary condition for x^2 to survive to E_∞ in the Serre spectral sequence for the cover.

If $H^*(\mathbb{Z}/2\mathbb{Z}; H^1(M; \mathbb{Z}/2\mathbb{Z})) = 0$ for $* > 0$, but x^2 does not survive to E_∞ then there must be a non-zero d_3 from $H^0(\mathbb{Z}/2\mathbb{Z}; H^2(M; \mathbb{Z}/2\mathbb{Z})$ to x^3 in order to get the cup product structure right. But $Sq^1 x^3 = x^4$, so $Sq^1: H^2(M; Z/2Z) \to H^3(M; \mathbb{Z}/2\mathbb{Z})$ must be non-zero, a contradiction. □

Remark. It is an easy Gysin sequence argument to show $x^3 \neq 0$ if $H_1(M; \mathbb{Z}/2\mathbb{Z}) \simeq 0$.

Definition 2.3. Given a free involution, τ, on a connected, compact, oriented 3-manifold, M, with τ orientation preserving, and $x^3 \neq 0$, define

$$\rho(M, \tau) \varepsilon \mathbb{Z}/32\mathbb{Z}$$

by

$$\rho(M, \tau) \equiv \sigma(\tilde{W}) + \varepsilon \pmod{32},$$

where $\tilde{W}$ is the double cover of any normal bordism, W, from M/τ to RP^3 or $L(6, 1)$ and $\varepsilon = 0$ if M/τ is normally bordant over RP^3 to RP^3 and $\varepsilon = 10$ otherwise.

Remark 2.4. If $H_1(M; \mathbb{Z}/2\mathbb{Z}) = 0$, then $x^3 \neq 0$ and $\varepsilon = 0$ iff $|H_1(M; \mathbb{Z})| \equiv \pm 1 \pmod 8$.

The proof that ρ is well-defined is the same as the proof that the relative Rochlin invariant is well-defined once we recall that for a closed manifold (such as the V in the proof for the well-definedness of $\mu(M; f, \phi)$), $\sigma(\tilde{V}) = 2\sigma(V)$.

Remark 2.5. If we choose a spin structure on RP^3 we can use our normal maps to induce a spin structure on M/τ, and hence a Rochlin invariant. This Rochlin invariant, the Browder–Livesay invariant of τ, and $\rho(M, \tau)$ are all related.

Section 2 of [7] can be viewed as a calculation of ρ for M being the Brieskorn variety (3, 5, 19) with $\tau(z_0, z_1, z_2) = (-z_0, -z_1, -z_2)$. We will give a different calculation of this example later.

Let us next discuss free involutions on S^4 and S^5. The usual surgery calculations show that there are four smooth conjugacy classes of smooth involutions on S^5; on S^4 there are at most two smooth h-cobordism classes of smooth involutions [13]. Fintushel and Stern [7] show that there really are the two classes. Siebenmann [11] has shown the four classes on S^5 condense into two topological conjugacy classes, while Freedman [8] has shown all smooth involutions on S^4 are topologically conjugate.

Our aim is to describe this classification in terms of invariant 3-manifolds. Begin by letting τ denote a free involution on S^4 or S^5 and letting $\Sigma \subset S^n$ be a compact, connected, oriented submanifold which is invariant under τ. Assume τ preserves the orientation and that $x^3 \neq 0$.

For $n=4$ or 5, Σ bounds an oriented submanifold $V \subset S^n$. If $n=4$ this is obvious. If $n=5$ it is an example of a general fact: a codimension 2-submanifold of a sphere bounds orientably. Define $\sigma(\Sigma \subset S^n)$ as $\sigma(V)$ where we orient V so that Σ receives its given orientation. Note that for $n=4$, $\sigma(\Sigma \subset S^n)=0$.

Definition 2.6. Define

$$\delta(S^n, \tau) = \rho(\Sigma, \tau) - \sigma(\Sigma \subset S^n) \pmod{32}.$$

The surprise is that δ does not depend on the choice of Σ. We will show this when $n=5$: the case $n=4$ is similar but easier.

Suppose that we have $\Sigma_0, \Sigma_1 \subset S^5$. Let $P^5 = S^5/\tau$, and let $f: P^5 \to RP^5$ be a homotopy equivalence. Since $x^3 \neq 0$, we can homotop f to $f_i: P^5 \to RP^5$ with $f_i^{-1}(RR^3) = \Sigma_i/\tau$, $i=0$ and 1. Moreover we can make the homotopy transverse to RP^3 and we get a normal bordism $W \to RP^3 \times I$ between Σ_0/τ and Σ_1/τ.

Remark 2.7. Thus far we need only have assumed Σ_0 and Σ_1 are topological locally-flat submanifolds. If we assume they are smoothly embedded, we can assume W is smooth.

Hence $\rho(\Sigma_1, \tau) - \rho(\Sigma_0, \tau) = \sigma(\tilde{W})$ ($\tilde{W}$ orients Σ_1 correctly).

Let $V_i \subset S^5$ have boundary Σ_i, $i=0, 1$. Then $V_0 \cup \tilde{W} \cup (-V_1)$ is an oriented submanifold of $S^5 \times I \subset S^6$; hence is a boundary; hence has 0 signature: i.e. $\sigma(\Sigma_0 \subset S^5) + \rho(\Sigma_1, \tau) - \rho(\Sigma_0, \tau) - \sigma(\Sigma_1 \subset S^5) = 0$. Done.

We can now prove

Theorem 2.7. *The class of τ is determined by $\delta(S^n, \tau)$, $n=4$ or 5: $\delta(S^4, \tau)$ can be 0 or 16; $\delta(S^5, \tau)$ can be 0, 8, 16, or 24. The topological conjugacy class of τ is determined by $\delta(S^n, \tau)$ modulo 16.*

Proof. We begin with $n=5$. First we produce a construction which adds 16 to δ. Shaneson [22] (or even Browder [3]) has shown $L_5(\mathbb{Z}[\mathbb{Z}]) \simeq \mathbb{Z}$ with the codimension 1 signature as the invariant. If K^4 is the Kummer surface, $K^4 \times S^1 \to S^4 \times S^1$ is a problem representing twice the generator. It is easily observed that we can do surgery on this problem to $f: W^5 \to S^4 \times S^1$ with $\pi_1 W^5 = \mathbb{Z}$ and f an homology isomorphism. It is also easy to cut out a tube to get $f: X^5 \to D^4 \times S^1$ with $f|\partial$ a diffeomorphism.

Let $\Sigma/\tau \subset S^5/\tau$. Pick a circle in $S^5/\tau - \Sigma/\tau$ which maps to the circle in $RP^5 - RP^3$ under f. Cut out a tubular neighborhood of this circle and glue in X. The double cover is S^5; it has a free involution, $\tau_1 : \Sigma$ is still invariant in S^5 with $\tau_1|\Sigma$ still $\tau|\Sigma$; and $\sigma(\Sigma \subset S^5 \text{ for } \tau) + 16 = \sigma(\Sigma \subset S^5 \text{ for } \tau_1)$. Since ρ does not change we have shown

$$\delta(S^5, \tau_1) = \delta(S^5, \tau) + 16.$$

To finish the proof we discuss the Brieskorn involutions. If d is odd, $\{z_0^d+z_1^2+z_2^2+z_3^2=0\}\cap S^7=S_d$ is the standard 5 sphere: $\tau(z_0, z_1, z_2, z_3)=(z_0, -z_1, -z_2, -z_3)$ is a free involution. Let $\Sigma_d\subset S_d$ be the set of all $(z_0, z_1, z_2, z_3)\in S_d$ with $z_3=0$: Σ_d is invariant and $|H_1(\Sigma_d;\mathbb{Z})|=d$.

Let us compute $\sigma(\Sigma_d\subset S_d)$. First notice that if we embed S^5 in S^7 as $z_3=0$ we have $\Sigma_d=S^5\cap S_d$. Moreover, Brieskorn [2] has computed $\sigma(\Sigma_d\subset S^5)$. We recall the result.

Given the Brieskorn variety $(z_0^{a_0}+z_1^{a_1}+z_2^{a_2}=0)\cap S^5=\Sigma$, the signature $\sigma(\Sigma\subset S^5)=p^+-p^-$ where p^- is the number of timese $j_0/a_0+j_1/a_1+j_2/a_2$ is between 1 and 2 for $0<j_0<a_0$; $0<j_1<a_1$; $0<j_2<a_2$: $p^+=(a_0-1)(a_1-1)(a_2-1)-p^-$.

To compute $\sigma(\Sigma_d\subset S_d)$ we show $\sigma(\Sigma_d\subset S_d)=\sigma(\Sigma_d\subset S^5)=-(d-1)$. To this end consider D^8. In here we have $D^6(z_3=0)$ and $V^6=\{z_0^d+z_1^2+z_2^2+z_3^2=0\}$. In our case, V^6 is also a smooth disc and by a small isotopy rel$\|\partial$ we can assume $V^6\cap D^6=Y^4$ is transverse. Now looking in V^6 and arguing that codimension two submanifolds in spheres have 0 signature, we see $\sigma(\Sigma_d\subset S_d)=\sigma(Y^4)$. A similar result holds in D^6, so we are done.

Consider (S_3, τ). Since $|H_1(\Sigma_d;\mathbb{Z})|=3$, we see from remark 2.7 that this τ is not even topologically conjugate to the standard involution. Let us compute $\delta(S_3, \tau)$ modulo 16. It is $\rho(S_3, \tau)=(\mu(\Sigma_3)-\mu(L(3, 1)))$ minus $\sigma(\Sigma_3\subset S_3)=\mu(\Sigma_3)$ where μ denotes the Rochlin invariant. Hence $\delta(S_3, \tau)=-\mu(L(3, 1))+10=8$ mod 16, since $\mu(L(3, 1))=2$.

Hence $\delta(S^5, \tau)$ assumes four values, 0, 8, 16, and 24 and therefore $\delta(S^5, \tau)$ gives the complete classification.

For the case $n=4$, let us consider the Fintushel–Stern example S^4, T [7]. It has the Brieskorn manifold $\Sigma=\{z_0^3+z_1^5+z_2^{19}=0\}\cap S^5$ with $\tau(z_0, z_1, z_2)=(-z_0, -z_1, -z_2)$ as an invariant submanifold.

Since τ is the standard involution on S^5, $\rho(\Sigma, \tau)-\sigma(\Sigma\subset S^3)\equiv 0 \pmod{32}$ by our 5 dimensional result. Since $\sigma(\Sigma\subset S^5)=-80$, $\rho(\Sigma, \tau)\equiv 16 \pmod{32}$. Hence $\delta(S^4, T)=16$ and we are done in this case too. □

As a corollary of the proof of 2.7 we have

Proposition 2.8. *Let τ be a free smooth involution on S^5. If Σ is an invariant $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere in S^5, then τ is topologically conjugate to the standard action iff* $|H_1(\Sigma;\mathbb{Z})|\equiv\pm 1 \pmod 8$.

Remark. The Brieskorn involutions, (S_d, τ) on S^5, have an interesting history. Atiyah and Bott first stated the correct result

$$(S_{d_0}, \tau) \text{ is equivalent to } (S_{d_1}, \tau) \text{ iff } d_0\equiv\pm d_1 \pmod{16}$$

([1], Thm. 9.8). The proof is short a factor of 2 which Atiyah supplies in a note on p. 338 of [11]. The methods are operator theoretic and apply to any smooth free involution on S^5.

Meanwhile, Giffen [9] recovered the same result, although the reader will have to work to recover the above formulation. Browder [4] p. 222 has also recovered the result. Their methods also apply to any smooth free involution on S^5. Their methods are homotopy theoretic.

It is an instructive exercise to take Neumann's calculation of the Browder-Livesay invariant of (Σ_d, τ) [18] p. 70, together with his identification of S_d/τ with a lens space and compute $\delta(S_d, \tau)$ using 2.5. We have done this and again recovered the above result.

3. Quadratic linking forms

In this section we collect the algebraic results we will need later.

Definition 3.1. Let A denote a finite abelian group. A function $\gamma: A \to Q/Z$ is called a *quadratic linking form* iff it satisfies

i) if we define $l: A \times A \to Q/Z$ by

$$l(a_1, a_2) = \gamma(a_1) + \gamma(a_2) - \gamma(a_1 + a_2),$$

then l is a non-singular bilinear pairing;

ii) $\gamma(ra) = r^2\gamma(a)$ for all $a \in A$ and all $r \in Z$.

Remarks 3.2. Non-singular means that the adjoint homomorphism, $adl: A \to \operatorname{Hom}(A, Q/Z)$, is an isomorphism. We will call l the linking form associated to γ and we will say that γ is a quadratic enhancement of l.

If γ satisfies i) and if $\gamma(a) = \gamma(-a)$ for all $a \in A$, then γ satisfies ii).

We recall some standard constructions and results.

Orthogonal sum. Given (A_1, γ_1) and (A_2, γ_2), define $(A_1 \oplus A_2, \gamma_1 \perp \gamma_2)$ by $(\gamma_1 \perp \gamma_2)(a_1, a_2) = \gamma_1(a_1) + \gamma_2(a_2)$. This gives a quadratic linking form on $A_1 \oplus A_2$. Under the associated linking form, $A_1 \oplus 0$ and $0 \oplus A_2$ pair to 0.

Scalar multiplication. Given (A, γ) and $r \in Z$ with $(r, |A|) = 1$ define $(A, r\gamma)$ by $(r\gamma)(a) = r\gamma(a)$. This is a quadratic linking form. If $r \equiv \Delta^2 \pmod{|A|}$, then (A, γ) is isomorphic to $(A, r\gamma)$.

Isotropic subgroup. A subgroup $B \subset A$ for which $\gamma(b) = 0$ for all $b \in B$ is called an isotropic subgroup. A quadratic linking form (A, γ) is called *anisotropic* if A has no non-trivial isotropic subgroups.

Isotropic division. Let (A, γ) be a quadratic linking form with isotropic subgroup B. For any finite group, denote $\operatorname{Hom}(A, Q/Z)$ by A^*: let $l^*: A \to A^*$ denote the adjoint of our linking pairing. Then there exist maps l_1 and l_2 so that

$$\begin{array}{ccccccccc} 0 \to & B & \to & A & \overset{p}{\to} & A/B & \to 0 \\ & \downarrow{\scriptstyle l_1} & & \downarrow{\scriptstyle l^*} & & \downarrow{\scriptstyle l_2} & \\ 0 \to & (A/B)^* & \to & A^* & \to & B^* & \to 0 \end{array}$$

commutes. Let $K = \ker l_2$. Define $\hat{\gamma}: K \to Q/Z$ by $\hat{\gamma}(k) = \gamma(a)$ where $a \in A$ is any element with $p(a) = k$. One checks $\hat{\gamma}$ is well defined and a quadratic linking form on K. We say $(K, \hat{\gamma})$ is related to (A, γ) by isotropic division.

Remark. Note $|A| = |K||B||B^*| = |K||B|^2$. Hence an isotropic subgroup of A has order $\leq \sqrt{|A|}$.

The usual decomposition theorems for bilinear forms over fields are valid in our case with a little care. We begin by defining the basic pieces: p denotes a prime; Δ a non-negative integer, and a an integer prime to p.

3.3. $(C_p(\Delta); a)$ denotes the form on $C_p(\Delta) \simeq Z/p^\Delta Z$ with generator g satisfying

$$\gamma(g) = \begin{cases} a/p^\Delta, & p \text{ odd} \\ \dfrac{a}{2p^\Delta}, & p = 2. \end{cases}$$

3.4. $(E_2(\Delta); a_0, a_1)$ denotes the form on $E_2(\Delta) \simeq Z/2^\Delta Z \oplus Z/2^\Delta Z$ with generators g_0, g_1 which satisfy $\gamma(g_0) = a_0/2^\Delta$; $\gamma(g_1) = a_1/2^\Delta$; and $l(g_0, g_1) = 1/2^\Delta$. Neither a_0 nor a_1 need be odd.

Let us agree that $(C_p(0); a)$ and $(E_2(0); a_0, a_1)$ will denote the form on the trivial group.

Theorem 3.5. *Any quadratic linking form (A, γ) can be written as an orthogonal sum with each summand isomorphic to one of the forms* 3.3 *or* 3.4.

Proof. We induct on the order of A: the trivial group provides a trivial beginning for the induction.

Suppose that $x \in A$ generates a summand of order p^Δ and that

$$\gamma(x) = \begin{cases} a/p^\Delta, & p \text{ odd}, \\ a/2^{\Delta+1}, & p = 2, \end{cases}$$

with $(a, p) = 1$. The fundamental theorem of abelian groups and the Gram–Schmidt process can be combined to split orthogonally a $(C_p(\Delta); a)$ off of (A, γ).

With this case done we need only study the case in which each generator of a summand has a γ which is too small. Precisely, if x generates a $\mathbb{Z}/p^{\Delta}\mathbb{Z}$ summand of A then

$$\gamma(x)=\begin{cases} b/p^{\Delta-1}, & p \text{ odd}, \\ b/2^{\Delta}, & p=2, \end{cases}$$

(with no requirement that $(b,p)=1$).

Since x generates a summand, we can find a homomorphism $f: A \to \mathbb{Q}/\mathbb{Z}$ with $f(x)=1/p^{\Delta}$ and $p^{\Delta}\cdot f(a)=0$ for all $a \in A$. Since the linking form is non-degenerate, there exists a unique $y \in A$ such that $l(-,y)=f(\ )$. Notice that $l(x,y)=1/p^{\Delta}$ and that y has order p^{Δ}. Furthermore, $y \neq pz$ for any $z \in A$ since if it did $l(x,y)=pl(x,z)=b/p^{\Delta-1}$ where $b/p^{\Delta}=l(x,z)$. Hence y generates a summand.

Since $l(x,x)=c/p^{\Delta-1}$ for some c, $y \neq x$ and we claim $\langle x,y\rangle$ generates a $\mathbb{Z}/p^{\Delta}\mathbb{Z}\oplus\mathbb{Z}/p^{\Delta}\mathbb{Z}$ summand of A. It suffices to show that if $p^{i}(\alpha x+\beta y)=0$ with $(\alpha,\beta)=1$ then $i \geq \Delta$. By computing $0=l(x,p^{i}(\alpha x+\beta y))$ we see $p|\beta$ if $i<\Delta$: computing $l(y,p^{i}(\alpha x+\beta y))$ shows $p|\alpha$. This contradicts $(\alpha,\beta)=1$. If p is odd, compute $\gamma(x+y)$ and see that it has the form a/p^{Δ}, $(a,p)=1$. Hence $p=2$ and we have a summand of (A,γ) isomorphic to $(E_2(\Delta);\, a_0, a_1)$. Modify the Gram–Schmidt process to show that this summand can be made orthogonal. □

Next we discuss existence and uniqueness.

Theorem 3.6. *Let l be a non-degenerate linking form on a finite abelian group A. Then there exists a quadratic linking form γ on A whose associated linking form is l. The set of such quadratic linking forms is in bijective correspondence with* $\mathrm{Hom}(A,\mathbb{Z}/2\mathbb{Z})$.

Proof. We will never need the existence part of 3.6 so we merely sketch the proof. Just as in 3.5 write the linking form as an orthogonal sum of pieces like $(C_p(\Delta);\, a)$ and $(E_2(\Delta);\, a_0, a_1)$. Put a quadratic linking form on each one by hand.

Suppose γ_1, γ_2 are two quadratic linking forms. Then $\psi=\gamma_1-\gamma_2: A \to \mathbb{Q}/\mathbb{Z}$ is linear; $\psi(a)=\psi(-a)$; so ψ lands in $\pm 1 \subset \mathbb{Q}/\mathbb{Z}$. Conversely, it is easily checked that given $\psi: A \to \pm 1$, $\gamma+\psi$ is a quadratic linking form whose associated linking form is still l. □

The last project for this section is to discuss the Milgram Gauss sum formula. We begin with

Definition 3.7. Let (A,γ) denote a quadratic linking form and let

$$G(\gamma)=\frac{1}{\sqrt{|A|}}\sum_{a\in A}\exp(2\pi i\gamma(a)).$$

Theorem 3.8. $G(\gamma_1 \perp \gamma_2)=G(\gamma_1)G(\gamma_2)$: *if (A,γ) and $(K,\hat{\gamma})$ are related by isotropic division,* $G(\gamma)=G(\hat{\gamma})$.

Proof. The first result is straightforward, so we work out the second. To fix notation we have $B \subset A$ with $\gamma(B)=0$ and K is the kernel of the induced map $A/B \to B^*$. Then

$$\sum_{a\in A} \exp(2\pi i\gamma(a)) = \sum_{\alpha\in B^*}\sum_{k\in K}\sum_{b\in B} \exp(2\pi i\gamma(\hat\alpha+\hat k+b)),$$

where $\hat k$ denotes a choice of lift of k from A/B up to A and $\hat\alpha$ denotes first a choice of lift from B^* up to A/B and then from A/B on up to A. These lifts are fixed once and for all. Then $\hat\alpha+\hat k+b$ really does run once over each element in A so our equation is tautological.

But

$$\begin{aligned}\gamma(\hat\alpha+\hat k+b) &= \gamma(\hat\alpha)+\gamma(\hat k)+\gamma(b)+l(\hat\alpha,\hat k)+l(\hat\alpha,b)-l(\hat k,b)\\ &= \gamma(\hat\alpha)+\gamma(\hat k)+l(\hat\alpha,\hat k)+l(\hat\alpha,b).\end{aligned}$$

For $\hat\alpha$ fixed

$$\sum_{b\in B}\exp(2\pi i l(\hat\alpha,b)) = \begin{cases}0, & \hat\alpha \neq \text{lift of } 0\in B^*,\\ |B|, & \hat\alpha \text{ the lift of } 0,\end{cases}$$

$$G(\gamma) = \frac{|B|}{\sqrt{|A|}}\sum_{k\in K}\exp(2\pi i(\hat\gamma(k)) = G(\hat\gamma). \quad \square$$

Next we calculate $G(\gamma)$ on our indecomposable pieces.

Theorem 3.9. *$G(\gamma)=1$ if $(A,\gamma)=(C_p(\Delta);a)$ with Δ even and p odd or if $(A,\gamma)=(E_2(\Delta);a_0,a_1)$ with Δ even. Otherwise*

$$G(C_p(\Delta);a) = \begin{cases} i\left(\frac{a}{p}\right) & \text{if } \Delta \text{ odd}, p\equiv 3 \pmod 4,\\ \left(\frac{a}{p}\right) & \text{if } \Delta \text{ odd}, p\equiv 1 \pmod 4,\\ \rho^{\pm 1} & \text{if } \Delta \text{ odd}, p=2, a\equiv \pm 1 \pmod 4,\\ \rho^{a} & \text{if } \Delta \text{ even}, p=2,\end{cases}$$

where $\rho=\exp(2\pi i/8)$, and $(\frac{a}{p})$ is ± 1 as a is a quadratic residue mod p *or not.*

$$G(E_2(\Delta);a_0,a_1) = \begin{cases} 1 & \text{if } \Delta \text{ odd but } a_0a_1 \text{ is even},\\ -1 & \text{if } \Delta \text{ odd but } a_0a_1 \text{ is odd.}\end{cases}$$

Proof. First apply isotropic division. If p is odd, $(C_p(\Delta);a)$ divides out to the 0-form if Δ is even or to $(C_p(1);a)$ if Δ is odd. The remaining calculation is classical (see e.g. Lang [12] IV, Section 3 for this and the remaining calculational claims).

If $p=2$ and Δ is odd proceed as above. If Δ is even we divide out to get the form $(C_2(2);a)$. Here we just do the calculation directly.

The form $(E_2(\Delta); a_0, a_1)$ divides out to the 0-form if Δ is even and to $(E_2(1); a_0, a_1)$ if Δ is odd. □

Corollary 3.10. *If $|A|$ is odd and $(r, |A|) = 1$, then*

$$G(\gamma) = \left(\frac{r}{|A|}\right) G(r\gamma) \quad \textit{where } \left(\frac{r}{|A|}\right)$$

is the Jacobi symbol.

The proof follows from 3.5 and 3.9.

We notice also from 3.5 and 3.9 that $G(\gamma)$ is always an eighth root of unity. The Milgram Gauss sum formula provides another description of this number.

Now we assume that we have a vector space V over $\mathbb{Q}$ and a non-degenerate, symmetric bilinear pairing $\beta: V \times V \to \mathbb{Q}$. We denote the signature of this form by $\sigma(\beta)$. Let L be a $\mathbb{Z}$-lattice inside V and assume β restricted to L takes values in $2\mathbb{Z} \subset \mathbb{Q}$. (There always are such lattices.) Let $L^{\#} = \{v \in V | \beta(v, l) \in \mathbb{Z} \text{ for all } l \in L\}$ denote the dual lattice. Clearly $L \subset L^{\#}$ and $L^{\#}/L$ is a finite abelian group.

Define a function $\hat{\beta}: L^{\#}/L \to \mathbb{Q}/\mathbb{Z}$ by

$$\hat{\beta}(x) = 1/2\beta(y, y) \in \mathbb{Q}/\mathbb{Z},$$

where $y \in L^{\#}$ is any element which hits $x \in L^{\#}/L$.

Theorem 3.11. (The Milgram Gauss sum formula.) *The function $\hat{\beta}$ is a quadratic linking form on $L^{\#}/L$, and*

$$G(\hat{\beta}) = \exp\left(2\pi i \frac{\sigma(\beta)}{8}\right).$$

See [16] Appendix 4 for a proof and discussion.

Corollary 3.12. $\sigma(\beta) \equiv \dim_{\mathbb{Z}/2\mathbb{Z}} L^{\#}/L \otimes \mathbb{Z}/2\mathbb{Z} \pmod 2$. *If $|L^{\#}/L|$ is odd,* $\sigma(\beta) \equiv 1 - |L^{\#}/L| \pmod 4$.

Proof. These results follow easily by using 3.5 to reduce to indecomposable case and then applying 3.9 and 3.11. □

Finally, we convert Theorem 3.11 into the justification for a definition.

Definition 3.13. If (A, γ) is any quadratic linking form, define the signature of γ, denoted $\sigma(\gamma)$, as the integer mod 8 which satisfies

$$\exp\left(2\pi i \frac{\sigma(\gamma)}{8}\right) = G(\gamma).$$

4. Spin structures and quadratic linking forms

The primary goal of this section is to use a spin structure on M^3 to put a quadratic linking form on $H_1(M)$ whose associated linking form is the linking form on $H_1(M)$. We can then use Section 3 to compute relative Rochlin invariants modulo 8.

So let us fix a spin structure, ϕ, on a compact, oriented spin manifold, M^3, without boundary. We will define a function

$$\gamma: T_1(M) \to \mathbb{Q}/\mathbb{Z},$$

where $T_1(M)$ = torsion subgroup of $H_1(M; \mathbb{Z})$.

Given $x \in T_1(M)$, pick an embedding $\hat{x}: S^1 \to M$ so that, after we choose a fundamental class, $\hat{x}_*[S^1] = x$. The normal bundle of this embedding is trivial and framings correspond bijectively to non-zero sections. Pick a section, Δ, and use Δ to get another embedding $\Delta_*\hat{x}: S^1 \to M - \hat{x}(S^1)$ by pushing $\hat{x}(S^1)$ out by the section. Since $x \in T_1(M)$ there exists an r so that $r \cdot x = 0$. We can then find an oriented surface F and a map $f: F \to M^3$ so that ∂F is the disjoint union of r copies of S^1 and so that f takes each S^1 in ∂F to $\Delta_*\hat{x}(S^1)$ by an orientation preserving homeomorphism. Define $l_\Delta(x) = [F \cdot \hat{x}(S^1)/r] \in \mathbb{Q}$ where $\cdot$ denotes intersection number. It is a standard argument that $l_\Delta(x)$ is well-defined.

Having picked a section, the set of all sections corresponds bijectively to $\mathbb{Z}$ and $l_{\Delta+a}(x) = l_\Delta(x) + a$ for all $a \in \mathbb{Z}$.

So far we have not used the spin structure. The spin structure on M induces one on $TM|\hat{x}(S^1)$. There are two spin structures on S^1 but only one of them makes S^1 into a spin boundary [15]. With this spin structure on S^1, pick a section of the normal bundle for $\hat{x}$ so that the spin structure on $TM|\hat{x}(S^1)$ is given by the spin structure on S^1 plus the spin structure induced by Δ. Any two sections with this property, say Δ_1 and Δ_2, satisfy $\Delta_1 + a = \Delta_2$ with a even.

$$\gamma(x) = \tfrac{1}{2} l_\Delta(x) \in \mathbb{Q}/Z.$$

Theorem 4.1. *With the above definition, γ is a quadratic linking form on $T_1(M)$ whose associated linking form is the usual linking form on $T_1(M)$.*

Proof. We have argued above that once the embedding $\hat{x}$ is fixed, the resulting number in $\mathbb{Q}/\mathbb{Z}$ is well-defined. Our first step is to show $\gamma(x)$ is well-defined. We show a bit more for later convenience.

Suppose $\coprod_u S^1 \to M$ is an embedding of u circles representing x. We can go through the above procedure and calculate another number in $\mathbb{Q}/\mathbb{Z}$. We show this agrees with the first number we calculated.

We have codimension two submanifolds again and hence an oriented surface E with $\partial E = S^1 \amalg \coprod_u S^1$ and an embedding $E \subset M \times I$ with $E \subset M \times 0$ being $\hat{x}$ and $E \subset M \times 1$ being $\coprod_u S^1 \to M$. The normal bundle to E is trivial and it is easy to see that once we fix the sections for the normal bundles for $\coprod_u S^1 \to M$ then there is only one section for the normal bundle to $\hat{x}$ which comes from a section of the normal bundle to $E \to M \times I$.

It is now a standard argument to show that the two numbers one calculates for $\gamma(x)$ are equal: hence $\gamma(x)$ is well-defined.

To show $\gamma(x+y) = \gamma(x) + \gamma(y) + l(x, y)$ where l is the linking form on $T_1(M)$, choose disjoint embeddings for x and y and use the disjoint union to calculate $\gamma(x+y)$.

Furthermore $\gamma(x) = \gamma(-x)$. To see this contemplate the result of reversing the orientation on S^1. If we use the same pushout the intersection numbers don't change. What we need to see is that this pushout is still given by a section with the correct spin structure. We can reverse our procedure and use the spin structure on TM and the given section to induce a spin structure on S^1 with both of its orientations. Since it bounds a spin manifold with one orientation we can just reverse the orientation on the boundary surface to see that it bounds with the other orientation.

A remark in 3.2 finishes the proof. □

Theorem 4.1 says that a spin structure determines a quadratic enhancement of the linking form. We consider the relation between these two concepts in more detail. We have seen that $\mathrm{Hom}(T_1(M), \mathbb{Z}/2\mathbb{Z})$ acts on quadratic enhancements (3.6) and it is well-known that $H^1(M; \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Hom}(H_1(M); \mathbb{Z}/2\mathbb{Z})$ acts on spin structures. There is a natural epimorphism $\theta: H^1(M; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(T_1(M), \mathbb{Z}/2\mathbb{Z})$.

Theorem 4.2. *Let ϕ_1, ϕ_2 be two spin structures on a compact, oriented 3-manifold, M, without boundary and let γ_1, γ_2 denote the associated quadratic enhancements of the linking form on $T_1(M)$. Then*

$$\theta(\phi_1 - \phi_2) = \gamma_1 - \gamma_2.$$

Proof. Recall that the only role the spin structure plays in the definition of $\gamma(x)$ is to determine a section of the normal bundle to $\hat{x}: S^1 \to M$. Hence if ϕ_1 and ϕ_2 determine the same spin structure in a neighborhood of $\hat{x}(S^1)$, $\gamma_1(x) = \gamma_2(x)$.

But ϕ_1 and ϕ_2 determine the same spin structure near $\hat{x}(S^1)$ iff $(\phi_1 - \phi_2)(x) = 0$. Hence $(\phi_1 - \phi_2)(x) = 0$ implies $(\gamma_1 - \gamma_2)(x) = 0$.

Conversely, if $(\phi_1 - \phi_2)(x) \neq 0$, we have different spin structures near $\hat{x}(S^1)$. Since the spin structure we put on S^1 is fixed, we must switch spin structures in the normal bundle for $\hat{x}$. We saw above that this makes a difference of $1/2$ so $\gamma_1(x) = \gamma_2(x) + 1/2$.

A check of the definition of $\gamma_1 - \gamma_2$ shows that we have shown $\theta(\phi_1 - \phi_2) = \gamma_1 - \gamma_2$. □

Corollary 4.3. *Any quadratic enhancement of the linking form on $T_1(M)$ comes from a spin structure by our procedure.*

Corollary 4.4. *If $H_1(M; \mathbb{Q}) = 0$, the quadratic enhancement determines the spin structure canonically.*

Next we relate the Rochlin invariant, the quadratic enhancement, and the Milgram Gauss sum formula.

Theorem 4.5. *Let M be a compact, oriented, 3-manifold without boundary. Let ϕ be a spin structure on M and let γ denote the resulting quadratic linking form on $T_1(M)$. Then*

$$\mu(M;\phi) \equiv -\sigma(\gamma) \quad (\mathrm{mod}\ 8).$$

Proof. The first step is to show we can assume $H_1(M;\mathbb{Q})=0$. If this is not the case choose a basis for the free part of $H_1(M)$ and do surgery to kill the circles, producing a spin bordism, W, from M to N with $H_1(N;\mathbb{Q})=0$. If $\cup$ denotes a tubular neighborhood of the surgered circles, $(M-\cup)\times I \subset W$, and $T_1(M) \simeq T_1(M-\cup) \simeq H_1(N)$. Since the signature of W, is $=0$, and since $(T_1(M), \gamma)$ and $(H_1(N), \gamma)$ are isomorphic, we can assume $H_1(M;\mathbb{Q})=0$ without loss of generality.

Let V be a simply connected spin manifold with $\partial V = M$ and so that the unique spin structure on V restricts to ϕ on M. To do this requires $\Omega_3^{\mathrm{Spin}}=0$ and a little surgery.

Let β denote the intersection pairing $H_2(V,\partial V;\mathbb{Q})\otimes H_2(V,\partial V;\mathbb{Q})\to\mathbb{Q}$. It is defined since $H_1(\partial V;\mathbb{Q})=0$. Inside $H_2(V,\partial V;\mathbb{Q})$ we have the image of $H_2(V;Z)$, which we denote L. Poincaré duality says that the image of $H_2(V,\partial V;\mathbb{Z})$ is the dual lattice $L^{\#}$. It is easy to show $L^{\#}/L \simeq H_1(M)$ since $0\to H_2(V)\to H_2(V,\partial V)\to H_1(M)\to 0$ is exact.

Theorem 4.5 will follow from Theorem 3.11 once we show $-\gamma=\hat{\beta}$, since $\sigma(-\gamma) = -\sigma(\gamma)$. To define $\hat{\beta}(x)$ lift $x\in H_1(M)$ to $\bar{x}\in H_2(V,\partial V)$ and calculate $\bar{x}\cdot\bar{x}\in\mathbb{Q}$. To do this, multiply $\bar{x}$ by r so that $r\bar{x}$ comes from $y\in H_2(V)$. Then $\bar{x}\cdot\bar{x}=1/r\bar{x}\cdot y$.

Geometrically, to lift x to $\bar{x}$ we can find an oriented surface $F\subset V$ with $\partial F\to M$ being a representative for x. Since V is spin, the normal bundle to F is trivial so we can choose a section and get another copy of F, say $\hat{F}\subset V$ which is disjoint from F. Pick a spin structure on F (any oriented surface has one) and choose a section for the normal bundle of F in V for which the spin structure on $TV|F$ agrees with the one we get from our chosen spin structure on TF plus the one our section chose on the normal bundle of F in V.

Now take r parallel copies of $\hat{F}$. In the definition of $\gamma(x)$ we had to choose an embedded surface, $K\subset M$, with ∂K being r parallel copies of $\partial\hat{F}$. Glue K and $r\hat{F}$ together to get a closed surface which represents the class $y=r\bar{x}$.

Then $\bar{x}\cdot\bar{x}=-l_\Delta(x)$. The minus sign comes in because to compute $\bar{x}\cdot\bar{x}$ we need to have $\hat{F}$ correctly orient the circles in M whereas to compute $l_\Delta(x)$ we need K to correctly orient the same circles. But the orientations given by $\hat{F}$ and by K must be opposite because we want $r\hat{F}\cup K$ to be orientable. □

We list some corollaries. The first is an unpublished result of A. Casson's.

Corollary 4.6. *If $H_1(M;\mathbb{Z}/2\mathbb{Z})=0$, the linking form on $H_1(M;\mathbb{Z})$ determines the Rochlin invariant* mod 8.

Proof. Since $H_1(M;\mathbb{Z})$ is an odd order torsion group, the linking form determines the quadratic enhancement. □

In the same vein we have

Corollary 4.7. *If $H_1(M;\mathbb{Z}/2\mathbb{Z})=0$ then*

$$\mu(M)\equiv|H_1(M)|-1 \pmod 4.$$

Proof. Combine 4.5 and 3.12. □

Corollary 4.8. *If $H_1(M;\mathbb{Q})=0$ then*

$$\mu(M;\phi)\equiv\dim_{\mathbb{Z}/2\mathbb{Z}} H_1(M;\mathbb{Z}/2\mathbb{Z}) \pmod 2.$$

Proof. Combine 4.5 and 3.12. □

Finally, we can compute relative Rochlin invariants mod 8. Given a degree 1 map $f: M\to P^3$ let $T_1(f)$ denote the torsion subgroup of the kernel of $H_1(M)\to H_1(P)$. Then $T_1(M)\simeq T_1(f)\oplus T_1(P)$ and $T_1(f)$ is paired orthogonally to $T_1(P)$ by the linking pairing on $T_1(M)$. Moreover, the induced linking form on $T_1(P)$ is the same as the one coming from P. Choosing a spin structure, ϕ, on M gives a decomposition

$$(T_1(M),\gamma)\simeq(T_1(f),\gamma)\perp(T_1(P),\gamma).$$

Theorem 4.9. *Let $f: M\to P$ be degree one and let ϕ be a spin structure on M. Then*

$$\mu(M;f,\phi)\equiv-\sigma(\gamma|T_1(f)) \pmod 8.$$

Proof. Recall notation: we have $F: W\to P\times I$ with $F:\partial^- W\to P\times 0$ being $f: M\to P$ and $F:\partial^+ W\to P\times 1$ being an integral homology equivalence. Furthermore, F is covered by a bundle map between normal bundles. Hence W is a spin manifold so $\mu(M;f,\phi)=\sigma(W)=-\sigma(T_1(M),\gamma)+\sigma(T_1(\partial_+ W),\gamma)$ mod 8 by Theorem 4.5. From Theorem 3.8 we see that it will suffice to prove $(T_1(P),\gamma)$ is isomorphic to $(T_1(\partial_+ W),\gamma)$, so we will do this.

Before beginning, do surgery if necessary on $W \operatorname{rel}|\partial W$ to make F induce an isomorphism on π_1. Unfortunately, $H_1(W)$ may not be 0 in this case so we cannot recover all of the quadratic linking form on $H_1(\partial W)$ from the intersection form on W, but if $x\in T_1(\partial W)$ and x goes to 0 in $H_1(W)$ we can recover $\gamma(x)$ by our usual procedure: first lift x to $\bar{x}\in H_2(W,\partial W)$ and find $y\in H_2(W)$ with y hitting $r\bar{x}$ in $H_2(W,\partial W)$; $\gamma(x)=-x\cdot y/2r$.

The fact that F is degree 1 provides an orthogonal sum decomposition $H_2(W) \simeq H_2(F) \oplus H_2(P)$; $H_2(W, \partial W) \simeq H_2(F, \partial F) \oplus H_2(P \times I, \partial)$; $H_1(M) \simeq H_1(f) \oplus H_1(P)$; and $H_1(\partial_+ W) \simeq H_1(P)$. In particular, there is a specific isomorphism between $H_1(\partial_+ W)$ and the $H_1(P)$ summand of $H_1(M)$. With these identifications we have a commutative diagram of exact sequences

$$\begin{array}{ccccc} H_2(P \times I) & \to & H_2(P \times I, \partial) & \to & H_1(P \amalg P) \\ \downarrow & & \downarrow & & \downarrow \\ H_2(W) & \to & H_2(W, \partial W) & \to & H_1(\partial W) \end{array}$$

and the vertical maps on the H_2's preserve intersections. The top row shows that the two quadratic linking forms on $T_1(P)$ are isomorphic, so we have shown $(T_1(\partial_+ W), \gamma)$ is isomorphic to the $(T_1(P), \gamma)$ summand of $(T_1(M), \gamma)$. □

Remark. The idea of using geometry to pass from a linking form to a quadratic linking form can be found in the Milgram [14] and Morgan–Sullivan [17] papers on surgery theory where they use the geometry of the normal map to enhance the linking form on a surgery kernel. In the presence of more geometry (e.g. a framing) one can enhance the whole linking form. We have worked through the three-dimensional case and acquired an unexpected bonus in the close connection between enhancements of the linking form and spin structures. It also seemed easier to show the Morgan–Sullivan proofs worked directly in our case than to prove that our enhancement on the surgery kernel agreed with theirs, a prerequisite to just quoting their results.

5. Free involutions on $\mathbb{Z}/2\mathbb{Z}$-homology 3-spheres

In [19] W. Neumann and F. Raymond discovered that in many cases the Browder–Livesay invariant of a free involution on a $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere is equal to the Rochlin invariant of the $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere. This equality was demonstrated for all integral homology 3-spheres by Yoshida in [26]. Moreover, he pointed out that there is a counterexample to this being true in general.

To describe the general situation, let Σ be a $\mathbb{Z}/2\mathbb{Z}$-homology 3-sphere with a free involution, denoted τ. Since $H_1(\Sigma)$ has odd order, τ_* acting on $H_1(\Sigma)$ decomposes it as $H_1(\Sigma) = T^+ \oplus T^-$ where $T^{\pm} = \{x \in H_1(\Sigma) | \tau_* x = \pm x\}$. Moreover, T^+ is orthogonal to T^- under the linking form. Since T has odd order, the linking form, l, can be viewed as a quadratic linking form.

Definition 5.1. Let $\sigma_-(\tau)$ denote the mod 8 integer obtained by applying Definition 3.13 to the quadratic linking form (T_1^-, l). Let $\alpha(\Sigma, \tau)$ denote the Browder–Livesay invariant for this involution. Then we have

Theorem 5.2. *We notate as above. We have*

$$\alpha(\Sigma,\tau)\equiv\mu(\Sigma)+2\sigma_-(\tau)\pmod{16}.$$

Corollary 5.3. *If* $\tau_*=\mathrm{Id}$, *then* $\alpha(\Sigma,\tau)\equiv\mu(\Sigma)\pmod{16}$.

Remark 5.4. Since all of the Neumann–Raymond involutions are homotopic to the identity, we recover their result. We also recover Yoshida's result.

Proof of 5.2. First calculate that the result is true for the standard involutions on S^3 and $L(3,1)$ with quotients RP^3 and $L(6,1)$ respectively. Of course this also follows from the Neumann–Raymond calculations.

Now let W be a normal bordism over $RP^3\times I$ between Σ/τ and RP^3 or $L(6,1)$. Let Σ/τ be denoted by M and let Q denote either RP^3 or $L(6,1)$ so $\partial W=M\coprod Q$. Then $\alpha(\Sigma,\tau)-\alpha(\tilde{Q},\tau)=2\sigma(W)-\sigma(\tilde{W})$ and $\mu(\Sigma)-\mu(\tilde{Q})=\sigma(\tilde{W})$. Thus we have $\alpha(\Sigma,\tau)-\mu(\Sigma)=2(\sigma(W)-\sigma(\tilde{W}))$.

From Theorem 4.9 we see that $\sigma(W)=\sigma(-\gamma|T_1(f))$ and $\sigma(\tilde{W})=\sigma(-\gamma|T_1(\tilde{f}))$ where $f:M\coprod Q\to RP^3\coprod RP^3$ and $\tilde{f}$ is the cover $\Sigma\coprod\tilde{Q}\to S^3\coprod S^3$.

Since $T_1(\tilde{f})$ has odd order, so does $T_1(f)$ and the covering projection $\pi:\partial\tilde{W}\to\partial W$ gives an isomorphism of $T_1(\tilde{f})^+$ onto $T_1(f)$. Moreover, π sends $T_1(\tilde{f})^-$ to 0.

Let λ denote the linking form in $H_1(\partial W)$ and let l denote the linking form in $H_1(\partial\tilde{W})$. There is a transfer map $\mathrm{tr}:H_1(\partial W)\to H_1(\partial\tilde{W})$ which almost splits π in that $\pi_*\circ\mathrm{tr}$ is multiplication by 2. Moreover, if $x\in H_1(\partial W)$

$$l(\mathrm{tr}\,x,\mathrm{tr}\,x)=2\lambda(x,x).$$

This equation follows from the intersection number equation $y\cdot\mathrm{tr}\,x=\pi_*y\cdot x$ for y a 2-chain in $\partial\tilde{W}$ and x a 1-chain in ∂W. Since $T_1(f)$ and $T_1(\tilde{f})$ have odd order, the quadratic linking forms on these groups are determined by the associated linking forms. Hence $(T_1(f),\gamma)$ is isomorphic to $(T_1^+(\tilde{f}),2\gamma)$.

Since $|H_1(\partial\tilde{W})|\equiv\pm1\pmod 8$ (Lemma 2.1) we can use Corollary 3.10 and the fact that

$$\left(\frac{2}{|H_1(\partial\tilde{W})|}\right)=1$$

to see that $\sigma(-\gamma|T_1(\tilde{f}))=\sigma(-2\gamma|T_1(\tilde{f}))$. Hence

$$\begin{aligned}\sigma(-\gamma|T_1(f))-\sigma(-\gamma|T_1(\tilde{f}))&=-\sigma(-2\gamma|T_1^-(f))=\sigma(2\gamma|T_1^-(\tilde{f}))\\&=\sigma(l|T_1^-(f))=\sigma_-(\tau).\end{aligned}$$ □

The proof of 5.2 leads to the proof of an amusing formula. Note Σ/τ has two spin structures distinguished by their quadratic enhancements of the linking form. Let $\gamma_\pm$ denote the enhancement distinguished by $\gamma_\pm(x)=\pm\frac{1}{4}$ where x is the unique element of order 2 in H_1. Let $\mu_\pm(\Sigma/\tau)$ denote the corresponding Rochlin invariants.

The proof of 5.2 also shows:

Theorem 5.5. $\mu_+(\Sigma/\tau)+\mu_-(\Sigma/\tau)\equiv 2\mu(\Sigma)+2\sigma_-(\tau)$ (mod 16).

Corollary 5.6. *Let m be odd. Then*

$$\mu_+(L(2m,q))+\mu_-(L(2m,q))\equiv 2\mu(L(m,q)) \pmod{16}.$$

Corollary 5.6 leads to the following question. Von Randow [24] has shown $L(p,q)$ is the boundary of an explicit simply-connected, spin 4 manifold. If p is even, which spin structure does $L(p,q)$ receive?

6. Change of spin structure

Let M be a compact, oriented, 3-manifold without boundary and let ϕ be a spin structure on M with associated enhancement of the linking form denoted by γ. The problem we begin to study in this section is to determine how the Rochlin invariant $\mu(M;\phi)$ changes if we change ϕ by $x\in H^1(M;\mathbb{Z}/2\mathbb{Z})$.

The first case we will do is the case $x^3\neq 0$. In this case we have a degree 1 normal map $f: M\to RP^3$.

Theorem 6.1. *Let $y\in H_1(M)$ be the Poincaré dual to $\delta x\in H^2(M)$ where δ denotes integral Bockstein. Then $\gamma(y)=\pm\frac{1}{4}$. Let $\tilde{\gamma}$ denote the enhancement associated to $\gamma+x$. Then $\tilde{\gamma}(y)=-\gamma(y)$. Let $\mu_+(M)$ denote the Rochlin invariant for the spin structure whose quadratic linking form has value $\frac{1}{4}$ on y: let $\mu_-(M)$ denote the Rochlin invariant for the other spin structure. Then, modulo* 16,

$$\mu_+(M)-\mu_-(M)\equiv\begin{cases}-2 & \text{iff is normally bordant to } 1_{RP^3},\\ +2 & \text{iff is normally bordant to } L(6,1)\to RP^3.\end{cases}$$

Proof. Since $x^3\neq 0$ and $\delta: H^1(RP^3;\mathbb{Z}/2\mathbb{Z})\to H^2(RP^3,\mathbb{Z})$ is an isomorphism, y is an element of order 2 which is orthogonal to the torsion in the kernel of $f_*: H_1(M)\to H_1(RP^3)$. Hence $\gamma(y)=\pm\frac{1}{4}$. The result on $\tilde{\gamma}$ follows from this and 4.2.

Let W be the normal bordism over $RP^3\times I$ from M to either RP^3 or $L(6,1)$. If we choose the spin structure on M with $\gamma(y)=\frac{1}{4}$ and extend it across W we get the spin structure at the other end which evaluates $\frac{1}{4}$ on the unique element of order 2. Hence $\mu_+(M)-\mu_-(M)$ has the same value as $\mu_+(Q)-\mu_-(Q)$ where $Q=RP^3$ or $L(6,1)$. Compute this number directly: it is -2 ($Q=RP^3$) or $+2$ ($Q=L(6,1)$). □

The other case we will do here is the case $\delta x=0$. Then x comes from an integral class $\bar{x}\in H^1(M;\mathbb{Z})$ and we can assume $\bar{x}$ generates a summand. It is not hard to construct a degree 1 map $f: M\to S^2\times S^1$ corresponding to $\bar{x}$. Since $H^2(S^2\times$

$S^1, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, f is either normally cobordant to the identity or to one other problem.

Theorem 6.2. *With notation as above, modulo* 16,

$$\mu(M;\phi) = \mu(M;\phi + x) + \begin{cases} 0 & \textit{if } f \textit{ is normally bordant to } \text{id.}, \\ 8 & \textit{otherwise.} \end{cases}$$

Proof. Since $S^2 \times S^1$ bounds $D^3 \times S^1$ with either spin structure, the result if f is normally bordant so the identity is clear.

Given any knot $k: S^1 \to S^3$ we can do 0 framed surgery on it to get a M^3 and an $H_*(;\mathbb{Z})$ isomorphism $f: M^3 \to S^2 \times S^1$. In one spin structure, M^3 bounds a homology $S^2 \times D^2$ and hence had Rochlin invariant 0. If the *arf* invariant of the knot is not zero, M^3 also bounds a spin manifold of index 8 [20]. Hence any such 0 framed surgery represents the non-trivial normal bordism class and the result follows. □

Remark 6.3. To actually compute relative Rochlin invariants for $P = RP^3$ we need to choose a representative, $M^3 \to RP^3$, of the non-zero class which is an $H_*(;\mathbb{Z})$-equivalence. It is not hard to show that we can take the +2 surgery on the trefoil knot for M. Indeed one can take the ± 2 surgery on any knot of *arf* invariant 1. If we do take the +2 surgery on the trefoil as our representative, $\mu(L(6,1); f, \phi) = 0$ for either spin structure ϕ.

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